One- and Two-Variable p-adic Measures in Iwasawa Theory

by

Scott Michael Zinzer

A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Approved April 2015 by the
Graduate Supervisory Committee:

Nancy Childress, Chair
Andrew Bremner
Susanna Fishel
John Jones
John Spielberg

ARIZONA STATE UNIVERSITY

May 2015

ABSTRACT

In 1984, Sinnott used p-adic measures on $\mathbb{Z}_p$ to give a new proof of the Ferrero-Washington Theorem for abelian number fields by realizing p-adic L-functions as (essentially) the Γ-transform of certain p-adic rational function measures. Shortly afterward, Gillard and Schneps independently adapted Sinnott's techniques to the case of p-adic L-functions associated to elliptic curves with complex multiplication (CM) by realizing these p-adic L-functions as Γ-transforms of certain p-adic rational function measures. The results in the CM case give the vanishing of the Iwasawa μ-invariant for certain $\mathbb{Z}_p$-extensions of imaginary quadratic fields constructed from torsion points of CM elliptic curves.

In this thesis, I develop the theory of p-adic measures on $\mathbb{Z}_p^d$, with particular interest given to the case of $d > 1$. Although I introduce these measures within the context of p-adic integration, this study includes a strong emphasis on the interpretation of p-adic measures as p-adic power series. With this dual perspective, I describe p-adic analytic operations as maps on power series; the most important of these operations is the multivariate Γ-transform on p-adic measures.

This thesis gives new significance to product measures, and in particular to the use of product measures to construct measures on $\mathbb{Z}_p^2$ from measures on $\mathbb{Z}_p$. I introduce a subring of pseudo-polynomial measures on $\mathbb{Z}_p^2$ which is closed under the standard operations on measures, including the Γ-transform. I obtain results on the Iwasawa-invariants of such pseudo-polynomial measures, and use these results to deduce certain continuity results for the Γ-transform. As an application, I establish the vanishing of the Iwasawa μ-invariant of Yager's two-variable p-adic L-function from measure theoretic considerations.

To Elizabeth Ashley Shain, for the years that have been and the years still yet to come.

And to my parents, for a job well done.

ACKNOWLEDGEMENTS

A special thanks is due to the faculty at Aurora University who recognized my potential as an undergraduate and directed me towards a graduate education. Without the encouragement and support I received during my undergraduate years, I certainly would have never unlocked my great passion for mathematics. Thank you to Geoffrey Apel, Saib Othman, and Ariel Ramirez for introducing me to the world of mathematics. Thank you to Daniel Hipp and Mark Walter for nurturing my literary and philosophical musings and queries within and without mathematics. Thank you to Renae Franiuk; your mentorship and continued friendship are the very backbone of this culminating work.

Thank you to my primary and secondary school mathematics teachers, and especially to those who challenged me as a student. It is hard to imagine reaching this point without so many memorable and enjoyable experiences with mathematics in my years before college.

Thanks to my family and friends both near and far for their continued support and love. My decision to pursue graduate studies represented a significant shift from the future I once had in mind for myself, and I am most thankful for the encouragement and enthusiasm offered by my closest friends and family.

I give many thanks also to Arizona State University and the School of Mathematical and Statistical Sciences for the financial support which allowed me to pursue this study. SoMSS also provided an abundance of enjoyable teaching experiences that helped me grow as a mathematics educator and research appointments which ultimately led to this thesis.

Thank you to my colleagues within the number theory group at ASU. I enjoyed many enlightening conversations with Shawn Elledge, Donald Adams, and Zachary

Harrison during my graduate years. Thanks also to my dissertation committee members; it is my hope that the scope of topics contained in this thesis is some indication of the great impact you have had in my career.

Lastly, I would like to extend my thanks to my advisor, Dr. Nancy Childress. Her influence is all-pervasive in the work that follows. Nancy first introduced me to the theory of p-adic measures early in my graduate career and guided me toward the problems that form the foundation of this thesis. Her careful guidance and seemingly infinite patience made this project a reality. I look forward to continuing this line of research and exploring the many new problems arising from this thesis.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS/NOMENCLATURE

Symbol		Page
$C_{\boldsymbol{n}}$	$C'_{\boldsymbol{n}} \cap \mathcal{E}_{\boldsymbol{n}}$	151
$\bar{\mathcal{E}}_{\boldsymbol{n}}$	The Closure of $\mathcal{E}_{\boldsymbol{n}}$ in $\boldsymbol{U}_{\boldsymbol{n}}$	151
$\bar{C}_{\boldsymbol{n}}$	The Closure of $C_{\boldsymbol{n}}$ in $\boldsymbol{U}_{\boldsymbol{n}}$	151
$\mathcal{E}_{\infty}$	The Projective Limit of the $\bar{\mathcal{E}}_{\boldsymbol{n}}$	151
C_{∞}	The Projective Limit of the $\bar{C}_{\boldsymbol{n}}$	151
A_n	The p-Part of the Class Group of $K_{n\mathbf{1}}$	151
A_{∞}	The Projective Limit of the A_n	151
M_{∞}	An Abelian p-Extension of K_{∞}	151
X_{∞}	$\mathrm{Gal}(M_{\infty}/K_{\infty})$	151
P_i	A Height One Prime Ideal of Λ	152
r_i	A Positive Integer	152
$\mathrm{char}(M)$	The Characteristic Ideal of the Λ-Module M	152
$K_{\mathbb{Z}_p}$	The Composite of All $\mathbb{Z}_p$-Extensions of K	153
K_n	$K_{n\mathbf{1}} \cap K_{\mathbb{Z}_p}$	153
B_n	The p-Part of the Ideal Class Group of K_n	153
$e_n(K_{\infty}/K_{\mathbf{0}})$	The Integer Satisfying $p^{e_n(K_{\infty}/K_{\mathbf{0}})} = \lvert A_n \rvert$	153
$e_n(K)$	The Integer Satisfying $p^{e_n(K)} = \lvert B_n \rvert$	153
$m_0(K_{\infty}/K_{\mathbf{0}})$	The m_0-Invariant of the $\mathbb{Z}_p^2$-Extension $K_{\infty}/K_{\mathbf{0}}$	154
$l_0(K_{\infty}/K_{\mathbf{0}})$	The l_0-Invariant of the $\mathbb{Z}_p^2$-Extension $K_{\infty}/K_{\mathbf{0}}$	154
$m_0(K)$	The m_0-Invariant of the $\mathbb{Z}_p^2$-Extension $K_{\mathbb{Z}_p}/K$	154
$l_0(K)$	The l_0-Invariant of the $\mathbb{Z}_p^2$-Extension $K_{\mathbb{Z}_p}/K$	154
APPENDIX		
$\mathbb{R}_+$	The Set of Non-Negative Real Numbers	162
d	A Metric	163

PREFACE

> *"Voici mon secret. Il est très simple: on ne voit bien qu'avec le cœur. L'essentiel est invisible pour les yeux."*
>
> – Antoine de Saint Exupéry, *Le Petit Prince*

The purpose of this dissertation is to assemble in one place various results on "multivariate" p-adic measures and to apply these results to the Iwasawa theory of imaginary quadratic fields. Except where explicit references indicate otherwise, the content of this dissertation is my independent intellectual product.

In Chapter 1, I introduce the multivariate notation that will be used throughout this dissertation. I then provide some background in the theory of partially ordered sets for the sake of applications to p-adic power series in Chapter 2. Particular attention is given to the product order on the d-fold product of the natural numbers with their usual order and linear extensions of this order. Although the results contained in Chapter 1 are well-known, they are included here for completeness and to help familiarize the reader with some of the technical results which lead to the definition of multivariate Iwasawa λ-invariants in Chapter 2.

Chapter 2 contains the bulk of the work of this dissertation. I begin by developing the theory of p-adic measures and integration from a purely p-adic analytic viewpoint. I consider the case of p-adic valued measures on $\mathbb{Z}_p^d$, where d is a positive integer. The results contained in the first sections of Chapter 2 are almost certainly well-known to experts in the field, but I have not found detailed proofs of these results in the existing literature. Although the development of the theory of multivariate p-adic measures closely follows the theory of measures on $\mathbb{Z}_p$, this dissertation appears to include the first detailed account for the case $d > 1$. Furthermore, the measures under consideration take values in more general p-adic rings than those considered in the existing literature. This generality is developed for the sake of later applications, and leads to some nuanced differences between the theory developed in this dissertation and the theory of p-adic integral valued measures on $\mathbb{Z}_p$.

After building the theory of measures from a p-adic analytic foundation, I discuss the viewpoints of measures as elements of a certain profinite completed group ring and as elements of a p-adic power series ring. The power series viewpoint is highly favored in the last half of Chapter 2. There I introduce several operations on measures and provide descriptions of each of these operations in terms of associated power series. This study culminates in the definition of the Γ-transform on p-adic measures, which may be also be viewed as a map on rings of p-adic power series. Once again, these results are likely well-known to the experts, but have not appeared together within the literature. In Section 2.5, I introduce the multivariate analogues of the Iwasawa invariants of p-adic power series. In particular, the multivariate λ-invariants under consideration were first introduced and studied in Childress and Zinzer (2015), and I present a slight strengthening of one of the main theorems from that paper.

In Sections 2.6 and 2.7 I explore the product measure construction as a means of producing useful measures on $\mathbb{Z}_p^2$ from measures on $\mathbb{Z}_p$. The product measure construction for p-adic measures has not received much emphasis in the literature. Once again, the analytic viewpoint is developed in tandem with the power series viewpoint. I then define a new notion of two-variable pseudo-polynomials in terms of the product measure construction. Multivariate pseudo-polynomials were first considered in Childress and Zinzer (2015) as a generalization of the objects first introduced in Rosenberg (1996). Using methods similar to those used in Childress and Zinzer (2015), I describe the Iwasawa invariants of these generalized two-variable pseudo-polynomials. Using these results, I further obtain continuity results for the Γ-transform on these pseudo-polynomials. Such continuity results permit a detailed study of a measure associated to the limit of a sequence of pseudo-polynomials, which is the main contribution in Section 2.7.

Finally, Chapter 3 contains the main application of the measure theoretic developments appearing in Chapter 2. Yager's two-variable p-adic L-function arises in part from the Γ-transform of certain measures on $\mathbb{Z}_p^2$. Each of these measures is associated to a limit of a sequence of generalized two-variable pseudo-polynomials; for this reason, the μ-invariant of the two-variable Γ-transforms are given by μ-invariants of certain one-variable Γ-transforms. I then illustrate how the methods in Schneps (1987) can be applied to Yager's setting to obtain the vanishing of the μ-invariants of the relevant Γ-transforms. In the final section of Chapter 3, I explore applications to the Iwasawa theory of imaginary quadratic fields, and in particular to the question of class group growth in the unique $\mathbb{Z}_p^2$-extension of an imaginary quadratic field. The main result of this thesis is a new method of proof establishing the vanishing of the Cuoco-Monsky m_0-invariant for an imaginary quadratic field K of class number one.

The Appendices are included to collect several preliminary results needed in Chapter 2. Appendix A focuses on the topology of $\mathbb{Z}_p$ and $\mathbb{Z}_p^d$, Appendix B focuses on rings of formal power series, and Appendix C includes a sample of useful combinatorial identities. The reader with limited knowledge of elementary p-adic analysis is strongly encouraged to consult Appendix A before beginning Chapter 2.

Scott Zinzer

Tempe, AZ

April 24, 2015

CHAPTER 1

MULTIVARIATE NOTATION AND ORDER THEORY

Let $\mathfrak{X}$ be any set. For a positive integer d, we denote by $\mathfrak{X}^d$ the d-fold Cartesian product of $\mathfrak{X}$ with itself. Elements of $\mathfrak{X}^d$ will be written in boldfaced letters. We may equivalently view $\boldsymbol{x} \in \mathfrak{X}^d$ as a d-tuple, that is, a function $\boldsymbol{x} : \{1, \ldots, d\} \to \mathfrak{X}$. We often adopt this dual view; given $\boldsymbol{x} \in \mathfrak{X}^d$, we may also write $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$, or $\boldsymbol{x} = (x_i)_{1 \leq i \leq d}$, where $\boldsymbol{x}(i) = x_i \in \mathfrak{X}$ is the "ith component" of $\boldsymbol{x}$. With this notation, $\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{X}^d$ if and only if $x_i = y_i$ for all $1 \leq i \leq d$.

Whenever $\mathfrak{X}$ is an object in some category $\mathcal{C}$ in which products exist, we may view $\mathfrak{X}^d$ as the d-fold product in $\mathcal{C}$ of $\mathfrak{X}$ with itself. Of particular importance for this thesis are the cases where $\mathfrak{X}$ is an algebraic object, in which case $\mathfrak{X}^d$ is the d-fold direct product of $\mathfrak{X}$ with itself with component-wise operations, and where $\mathfrak{X}$ is a topological space, in which case $\mathfrak{X}^d$ is endowed with the product topology.

As usual, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, and $\mathbb{C}$ the field of complex numbers. We denote the set of positive integers by $\mathbb{Z}_+$. For a prime p, let $\mathbb{Z}_p$ denote the ring of p-adic integers.

Let $\mathbb{N}$ denote the set of non-negative integers. The set $\mathbb{N}^d$ will play a significant role in the work that follows. $\mathbb{N}$ is a commutative monoid under addition, so $\mathbb{N}^d$ is also a commutative monoid under termwise addition. For $k \in \mathbb{N}$ and $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$, we define

$$k\boldsymbol{n} := (kn_1, \ldots, kn_d) \in \mathbb{N}^d.$$

For each $1 \le i \le d$, we let $\boldsymbol{e}_i \in \mathbb{N}^d$ denote the d-tuple with

$$\boldsymbol{e}_i(j) = \begin{cases} 1 & : j = i \\ 0 & : j \neq i \end{cases}$$

($\boldsymbol{e}_i$ is the "ith standard basis vector"). Then for $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$, we may uniquely write

$$\boldsymbol{n} = \sum_{i=1}^{d} n_i \boldsymbol{e}_i.$$

In $\mathbb{N}^d$, we set $\mathbf{0} = (0, 0, \ldots, 0)$, $\mathbf{1} = (1, 1, \ldots, 1)$, $\mathbf{2} = (2, 2, \ldots, 2)$ and $\boldsymbol{p} = (p, p, \ldots, p)$, when p is a fixed prime.

For $\boldsymbol{n} \in \mathbb{N}^d$, we define

$$|\boldsymbol{n}| := \sum_{i=1}^{d} n_i \in \mathbb{N}$$
$$\boldsymbol{n}! := \prod_{i=1}^{d} n_i! \in \mathbb{N},$$

where in the second line, $n_i!$ denotes the usual factorial of $n_i \in \mathbb{N}$:

$$n_i! = \prod_{k=1}^{n_i} k,$$

together with the convention $0! = 1$.

Suppose $\mathfrak{X}$ is a multiplicative semi-group and $\boldsymbol{a} \in \mathfrak{X}^d$. If $\boldsymbol{b}$ is a d-tuple from some set $\mathfrak{Y}$ for which $a_i^{b_i}$ is a well-defined element of $\mathfrak{X}$ for all $1 \le i \le d$, then we define

$$\boldsymbol{a}^{\boldsymbol{b}} := \prod_{i=1}^{d} a_i^{b_i} \in \mathfrak{X}$$
$$\boldsymbol{a}^{\wedge \boldsymbol{b}} := (a_1^{b_1}, a_2^{b_2}, \ldots, a_d^{b_d}) \in \mathfrak{X}^d.$$

If $\mathfrak{X}$ is a $\mathbb{Z}$-algebra, then for $\boldsymbol{x} \in \mathfrak{X}^d$ and $\boldsymbol{n} \in \mathbb{N}^d$, we define

$$\binom{\boldsymbol{x}}{\boldsymbol{n}} := \prod_{i=1}^{d} \binom{x_i}{n_i} \in \mathfrak{X} \otimes_{\mathbb{Z}} \mathbb{Q},$$

where

$$\binom{x}{n} = \begin{cases} \frac{x(x-1)\cdots(x-n+1)}{n!} & : n > 0 \\ 1 & : n = 0 \end{cases}$$

is the usual binominal coefficient function for $x \in \mathfrak{X}$ and $n \in \mathbb{N}$.

If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^d$ and $\boldsymbol{m} \in (\mathbb{Z}_+)^d$, we will write

$$\boldsymbol{x} \equiv \boldsymbol{y} \pmod{\boldsymbol{m}}$$

to denote that

$$x_i \equiv y_i \pmod{m_i}$$

for all $1 \leq i \leq d$. Similarly, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}_p^d$, we will write

$$\boldsymbol{x} \equiv \boldsymbol{y} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{m}} \mathbb{Z}_p^d}$$

to denote that

$$x_i \equiv y_i \pmod{p^{m_i} \mathbb{Z}_p}$$

for all $1 \leq i \leq d$.

For $\boldsymbol{q} \in \mathbb{Q}^d$, we put

$$\lfloor \boldsymbol{q} \rfloor = (\lfloor q_1 \rfloor, \ldots, \lfloor q_d \rfloor) \in \mathbb{Z}^d,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x \in \mathbb{Q}$.

1.1 The Product Order on $\mathbb{N}^d$

Much of what follows (and many further topics) can be found in Chapter 3 of Stanley (2012).

Definition 1. *A partially ordered set (poset) is a set $\mathfrak{X}$ together with a binary relation $\preccurlyeq$ such that for all $a, b, c \in \mathfrak{X}$,*

1. *$a \preccurlyeq a$.*
2. *If $a \preccurlyeq b$ and $b \preccurlyeq a$, then $a = b$.*
3. *If $a \preccurlyeq b$ and $b \preccurlyeq c$, then $a \preccurlyeq c$.*

If $(\mathfrak{X}, \preccurlyeq)$ is a partially ordered set, we write $a \prec b$ if $a \preccurlyeq b$ but $a \neq b$.

Definition 2. *Let $(\mathfrak{X}, \preccurlyeq)$ be a partially ordered set.*

1. *Two elements $a, b \in \mathfrak{X}$ are said to be incomparable with respect to $\preccurlyeq$ if neither $a \preccurlyeq b$ nor $b \preccurlyeq a$. a and b are called comparable if $a \preccurlyeq b$ or $b \preccurlyeq a$.*
2. *Let $S \subseteq \mathfrak{X}$. An element $a \in S$ is said to be minimal with respect to $\preccurlyeq$ if there does not exist $b \in S$ with $b \prec a$.*

Definition 3. *Let $(\mathfrak{X}, \preccurlyeq)$ be a partially ordered set.*

1. *$\preccurlyeq$ is a total order (or $(\mathfrak{X}, \preccurlyeq)$ is totally ordered) if for all $a, b \in \mathfrak{X}$, $a \preccurlyeq b$ or $b \preccurlyeq a$.*
2. *$\preccurlyeq$ is a well order (or $(\mathfrak{X}, \preccurlyeq)$ is well ordered) if $\preccurlyeq$ is a total order and every non-empty subset of $\mathfrak{X}$ has a unique minimal element with respect to $\preccurlyeq$.*

Definition 4. *Let $(\mathfrak{X}, \preccurlyeq)$ be a partially ordered set and S a non-empty subset of $\mathfrak{X}$.*

1. *An element $x \in \mathfrak{X}$ is an upper bound for S if $y \preccurlyeq x$ for all $y \in S$. $x \in \mathfrak{X}$ is a least upper bound (supremum) of S if x is an upper bound for S and $x \preccurlyeq y$ for all upper bounds y for S. A supremum of S is unique, if it exists.*
2. *An element $x \in \mathfrak{X}$ is a lower bound for S if $x \preccurlyeq y$ for all $y \in S$. $x \in \mathfrak{X}$ is a greatest lower bound (infimum) of S if x is a lower bound for S and $y \preccurlyeq x$ for all lower bounds y for S. An infimum of S is unique, if it exists.*

3. *$(\mathfrak{X}, \preccurlyeq)$ is a lattice if for every $x, y \in \mathfrak{X}$, the set $\{x, y\}$ has an infimum and supremum. If $(\mathfrak{X}, \preccurlyeq)$ is a lattice, we write $\inf(x, y)$ and $\sup(x, y)$ for the infimum and supremum of $\{x, y\}$, respectively.*

4. *If $(\mathfrak{X}, \preccurlyeq)$ is a lattice, then $(\mathfrak{X}, \preccurlyeq)$ is distributive if for all $x, y, z \in \mathfrak{X}$,*

$$\sup(x, \inf(y, z)) = \inf(\sup(x, y), \sup(x, z))$$
$$\inf(x, \sup(y, z)) = \sup(\inf(x, y), \inf(x, z)).$$

5. *If $(\mathfrak{X}, \preccurlyeq)$ is a lattice, then the bottom element of $\mathfrak{X}$ is the infimum of $\mathfrak{X}$, if it exists. The top element of $\mathfrak{X}$ is the supremum of $\mathfrak{X}$, if it exists.*

Let $\leq$ denote the usual well-order on $\mathbb{N}$.

Definition 5. *The product order on $\mathbb{N}^d$, denoted $\preccurlyeq_d$, is given by $\boldsymbol{a} \preccurlyeq_d \boldsymbol{b}$ if and only if $a_i \leq b_i$ for all $1 \leq i \leq d$.*

$(\mathbb{N}^d, \preccurlyeq_d)$ is a partially ordered set and will be the backdrop for much of the work that follows.

Lemma 1. *Every non-empty subset of $\mathbb{N}^d$ contains a minimal element with respect to $\preccurlyeq_d$.*

Proof. Let $S \subseteq \mathbb{N}^d$ be non-empty. Let

$$\boldsymbol{x}(1) \succ_d \boldsymbol{x}(2) \succ_d \boldsymbol{x}(3) \succ_d \cdots$$

be a maximal decreasing sequence of distinct elements of S comparable with respect to $\preccurlyeq_d$ (such a sequence exists because S is non-empty). For each $1 \leq i \leq d$, we obtain a non-increasing sequence $(x(n)_i)_n$ in $\mathbb{N}$. Thus, each $(x(n)_i)_n$ is eventually constant. Since the $\boldsymbol{x}(n)$ are distinct elements of S, it must be that the sequence

$$\boldsymbol{x}(1) \succ_d \boldsymbol{x}(2) \succ_d \boldsymbol{x}(3) \succ_d \cdots$$

terminates, say with $\boldsymbol{x}(n)$. Then $\boldsymbol{x}(n) \in S$ is minimal with respect to $\preccurlyeq_d$ by the maximality of the given sequence. □

The following observations will be quite useful.

Lemma 2.

1. *$(\mathbb{N}^d, \preccurlyeq_d)$ is a distributive lattice with*

$$\sup(\boldsymbol{m}, \boldsymbol{n}) = (\sup\{m_i, n_i\})_{1 \le i \le d} \quad \textit{and} \quad \inf(\boldsymbol{m}, \boldsymbol{n}) = (\inf\{m_i, n_i\})_{1 \le i \le d}.$$

   $\mathbf{0}$ is the bottom element of $(\mathbb{N}^d, \preccurlyeq_d)$, and $(\mathbb{N}^d, \preccurlyeq_d)$ has no top element.

2. *$(\mathbb{N}^d, \preccurlyeq_d)$ is interval finite: Given $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^d$, the set*

$$[\boldsymbol{m}, \boldsymbol{n}] = \{\boldsymbol{x} \in \mathbb{N}^d : \boldsymbol{m} \preccurlyeq_d \boldsymbol{x} \textit{ and } \boldsymbol{x} \preccurlyeq_d \boldsymbol{n}\}$$

   is finite.

Proof.

1. It suffices to check for each $1 \le i \le d$ that

$$\inf\{x_i, \sup\{y_i, z_i\}\} = \sup\{\inf\{x_i, y_i\}, \inf\{x_i, z_i\}\}$$

   and

$$\sup\{x_i, \inf\{y_i, z_i\}\} = \inf\{\sup\{x_i, y_i\}, \sup\{x_i, z_i\}\}$$

   for $x_i, y_i, z_i \in \mathbb{N}$. But these hold because $(\mathbb{N}, \le)$ itself is a distributive lattice. As 0 is the bottom element of $(\mathbb{N}, \le)$, it is clear that $\mathbf{0}$ is the bottom element of $(\mathbb{N}^d, \preccurlyeq_d)$. Since $\boldsymbol{n} \prec_d \boldsymbol{n} + \boldsymbol{e}_i$ for all $\boldsymbol{n} \in \mathbb{N}^d$ and all $1 \le i \le d$, $(\mathbb{N}^d, \preccurlyeq_d)$ has no top element.

2. We have

$$[\boldsymbol{m}, \boldsymbol{n}] = \{\boldsymbol{x} \in \mathbb{N}^d : m_i \le x_i \le n_i \text{ for all } i\},$$

   where this latter set is obviously finite.

□

Given $\boldsymbol{n} \in \mathbb{N}^d$, we will denote the finite set $[\mathbf{0}, \boldsymbol{n}]$ simply by $[\boldsymbol{n}]$.

Corollary 1. *Let $S \subseteq \mathbb{N}^d$ be nonempty. Every $\boldsymbol{x} \in S$ is comparable to some element of S which is minimal with respect to $\preccurlyeq_d$.*

Proof. Let $M \subseteq S$ be the set of elements of S which are minimal in S with respect to $\preccurlyeq_d$. By Lemma 1, M is non-empty. Let $\boldsymbol{x} \in S$. The set $[\boldsymbol{x}] \cap S$ is non-empty, so contains a minimal element $\boldsymbol{m}$ with respect to $\preccurlyeq_d$. Then $\boldsymbol{m} \preccurlyeq_d \boldsymbol{x}$. Suppose $\boldsymbol{a} \in S$ is such that $\boldsymbol{a} \prec_d \boldsymbol{m}$ Then $\boldsymbol{a} \preccurlyeq_d \boldsymbol{x}$, so $\boldsymbol{a} \in [\boldsymbol{x}] \cap S$. But this is impossible since $\boldsymbol{m}$ is minimal in $[\boldsymbol{x}] \cap S$ with respect to $\preccurlyeq_d$. Therefore, $\boldsymbol{m} \in M$ and $\boldsymbol{x}$ is comparable to $\boldsymbol{m}$. □

We will use the following lemma in several key instances below (see Dickson (1913)).

Lemma 3 (Dickson's lemma)**.** *Let $S \subseteq \mathbb{N}^d$ be non-empty. Then S contains finitely many elements which are minimal with respect to $\preccurlyeq_d$.*

Proof. Let $M \subseteq S$ be the subset of minimal elements with respect to the product order on $\mathbb{N}^d$. By Lemma 1, M is non-empty. Moreover, the elements of M are pairwise incomparable with respect to $\preccurlyeq_d$. Suppose M is infinite. Since M is countable, we can list the elements of M as a sequence $\boldsymbol{x}(1), \boldsymbol{x}(2), \ldots$ of distinct elements.

Fix $1 \leq i \leq d$. We thus obtain a sequence of ith components $x_i(1), x_i(2), \ldots$. If this sequence is bounded, then it has a constant subsequence. If the sequence is unbounded, then it has a strictly increasing subsequence. In any case, we may replace the sequence $(\boldsymbol{x}(n))_n$ with a subsequence (which we continue to denote $(\boldsymbol{x}(n))_n$) in which the ith components form a non-decreasing sequence.

We can carry out the above construction inductively component by component to obtain a sequence $(\boldsymbol{x}(n))_n$ of elements of M such that the sequence of ith components $(x_i(n))_n$ is non-decreasing for all i. But then $\boldsymbol{x}(1) \preccurlyeq_d \boldsymbol{x}(n)$ for all n with $\boldsymbol{x}(1) \neq \boldsymbol{x}(n)$, contradicting incomparability with respect to $\preccurlyeq_d$. Therefore M is a finite subset of S. □

Definition 6. *An extension of $\preccurlyeq_d$ is any partial order $\preccurlyeq$ on $\mathbb{N}^d$ such that $\boldsymbol{m} \preccurlyeq_d \boldsymbol{n}$ implies $\boldsymbol{m} \preccurlyeq \boldsymbol{n}$. A linear extension of $\preccurlyeq_d$ is an extension of $\preccurlyeq_d$ which is a total-order.*

The following is an important theorem in order theory which we will use several times in the sequel (see Szpilrajn (1930)).

Theorem 1 (Extension Theorem)**.** *Every partial order may be extended to a total order.*

Consequently, linear extensions of $\preccurlyeq_d$ exist.

Lemma 4. *Let $M \subseteq \mathbb{N}^d$ be a finite set of pairwise incomparable elements of $\mathbb{N}^d$. For each $\boldsymbol{x} \in M$, there exists a linear extension $\preccurlyeq$ of $\preccurlyeq_d$ for which $\boldsymbol{x}$ is minimal in M with respect to $\preccurlyeq$.*

Proof. Let $\boldsymbol{x} \in M$. Define a new order $\preccurlyeq_{\boldsymbol{x}}$ on $\mathbb{N}^d$ as follows: For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^d$, $\boldsymbol{a} \preccurlyeq_{\boldsymbol{x}} \boldsymbol{b}$ if and only if one of the following occurs

1. $\boldsymbol{a} \preccurlyeq_d \boldsymbol{b}$.
2. $\boldsymbol{a} \preccurlyeq_d \boldsymbol{x}$ and $\boldsymbol{y} \preccurlyeq_d \boldsymbol{b}$ for some $\boldsymbol{y} \in M \setminus \{\boldsymbol{x}\}$.

By condition 1, $\boldsymbol{a} \preccurlyeq_{\boldsymbol{x}} \boldsymbol{a}$ for all $\boldsymbol{a} \in \mathbb{N}^d$.

Suppose now that $\boldsymbol{a} \preccurlyeq_{\boldsymbol{x}} \boldsymbol{b}$ and $\boldsymbol{b} \preccurlyeq_{\boldsymbol{x}} \boldsymbol{a}$ for some $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^d$. If both $\boldsymbol{a} \preccurlyeq_d \boldsymbol{b}$ and $\boldsymbol{b} \preccurlyeq_d \boldsymbol{a}$, then $\boldsymbol{a} = \boldsymbol{b}$. If $\boldsymbol{a} \preccurlyeq_d \boldsymbol{b}$, and $\boldsymbol{b} \preccurlyeq_d \boldsymbol{x}$ and $\boldsymbol{y} \preccurlyeq_d \boldsymbol{a}$ for some $\boldsymbol{y} \in M \setminus \{\boldsymbol{x}\}$, then

$$\boldsymbol{y} \preccurlyeq_d \boldsymbol{a} \preccurlyeq_d \boldsymbol{b} \preccurlyeq_d \boldsymbol{x},$$

which contradicts that $\boldsymbol{x}$ are $\boldsymbol{y}$ are incomparable with respect to $\preccurlyeq_d$. If $\boldsymbol{b} \preccurlyeq_d \boldsymbol{a}$, and $\boldsymbol{a} \preccurlyeq_d \boldsymbol{x}$ and $\boldsymbol{y} \preccurlyeq_d \boldsymbol{b}$ for some $\boldsymbol{y} \in M \setminus \{\boldsymbol{x}\}$, then

$$\boldsymbol{y} \preccurlyeq_d \boldsymbol{b} \preccurlyeq_d \boldsymbol{a} \preccurlyeq_d \boldsymbol{x},$$

which contradicts that $\boldsymbol{x}$ are $\boldsymbol{y}$ are incomparable with respect to $\preccurlyeq_d$. If $\boldsymbol{a} \preccurlyeq_d \boldsymbol{x}$ and $\boldsymbol{y} \preccurlyeq_d \boldsymbol{b}$ for some $\boldsymbol{y} \in M \setminus \{\boldsymbol{x}\}$, and $\boldsymbol{b} \preccurlyeq_d \boldsymbol{x}$ and $\boldsymbol{z} \preccurlyeq_d \boldsymbol{a}$ for some $\boldsymbol{z} \in M \setminus \{\boldsymbol{x}\}$ then

$$\boldsymbol{z} \preccurlyeq_d \boldsymbol{a} \preccurlyeq_d \boldsymbol{x},$$

which contradicts that $\boldsymbol{x}$ are $\boldsymbol{z}$ are incomparable with respect to $\preccurlyeq_d$.

Finally, suppose $\boldsymbol{a} \preccurlyeq_{\boldsymbol{x}} \boldsymbol{b}$ and $\boldsymbol{b} \preccurlyeq_{\boldsymbol{x}} \boldsymbol{c}$ for some $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{N}^d$. If $\boldsymbol{a} \preccurlyeq_d \boldsymbol{b}$ and $\boldsymbol{b} \preccurlyeq_d \boldsymbol{c}$, then $\boldsymbol{a} \preccurlyeq_d \boldsymbol{c}$, whence $\boldsymbol{a} \preccurlyeq_{\boldsymbol{x}} \boldsymbol{c}$. If $\boldsymbol{a} \preccurlyeq_d \boldsymbol{b}$ and $\boldsymbol{b} \preccurlyeq_d \boldsymbol{x}$ and $\boldsymbol{y} \preccurlyeq_d \boldsymbol{c}$ for some $\boldsymbol{y} \in M \setminus \{\boldsymbol{x}\}$, then $\boldsymbol{a} \preccurlyeq_d \boldsymbol{x}$, so $\boldsymbol{a} \preccurlyeq_{\boldsymbol{x}} \boldsymbol{c}$. If $\boldsymbol{a} \preccurlyeq_d \boldsymbol{x}$ and $\boldsymbol{y} \preccurlyeq_d \boldsymbol{b}$ for some $\boldsymbol{y} \in M \setminus \{\boldsymbol{x}\}$ and $\boldsymbol{b} \preccurlyeq_d \boldsymbol{c}$, then $\boldsymbol{y} \preccurlyeq_d \boldsymbol{b} \preccurlyeq_d \boldsymbol{c}$, so $\boldsymbol{a} \preccurlyeq_{\boldsymbol{x}} \boldsymbol{c}$. If $\boldsymbol{a} \preccurlyeq_d \boldsymbol{x}$ and $\boldsymbol{y} \preccurlyeq_d \boldsymbol{b}$ for some $\boldsymbol{y} \in M \setminus \{\boldsymbol{x}\}$ and $\boldsymbol{b} \preccurlyeq_d \boldsymbol{x}$ and $\boldsymbol{z} \preccurlyeq_d \boldsymbol{c}$ for some $\boldsymbol{z} \in M \setminus \{\boldsymbol{x}\}$, then $\boldsymbol{a} \preccurlyeq_{\boldsymbol{x}} \boldsymbol{c}$.

Thus, $\preccurlyeq_{\boldsymbol{x}}$ is a partial order on $\mathbb{N}^d$. By condition 1, $\preccurlyeq_{\boldsymbol{x}}$ extends $\preccurlyeq_d$, and by condition 2, $\boldsymbol{x} \preccurlyeq_{\boldsymbol{x}} \boldsymbol{y}$ for all $\boldsymbol{y} \in M \setminus \{\boldsymbol{x}\}$. By Theorem 1, there is a linear extension $\preccurlyeq$ of $\preccurlyeq_{\boldsymbol{x}}$. Since $\preccurlyeq_{\boldsymbol{x}}$ extends $\preccurlyeq_d$, $\preccurlyeq$ is also a linear extension of $\preccurlyeq_d$. Because $\boldsymbol{x} \prec_{\boldsymbol{x}} \boldsymbol{y}$ for all $\boldsymbol{y} \in M \setminus \{\boldsymbol{x}\}$, also $\boldsymbol{x} \prec \boldsymbol{y}$ for all $\boldsymbol{y} \in M \setminus \{\boldsymbol{x}\}$, so $\boldsymbol{x}$ is minimal in M with respect to $\preccurlyeq$. □

Corollary 2. *Let S be a non-empty subset of $\mathbb{N}^d$. For each $\boldsymbol{x} \in S$ which is minimal with respect to $\preccurlyeq_d$, there is a linear extension $\preccurlyeq$ of $\preccurlyeq_d$ for which $\boldsymbol{x}$ is the minimal element of S with respect to $\preccurlyeq$.*

Proof. Apply Lemma 4 and then Theorem 1. □

Proposition 1. *If $\preccurlyeq$ is a linear extension of $\preccurlyeq_d$, then $\preccurlyeq$ is a well-ordering on $\mathbb{N}^d$.*

Proof. Let $S \subseteq \mathbb{N}^d$ be non-empty. For each $\boldsymbol{a} \in S$, the set $[\boldsymbol{a}] \cap S$ is finite and nonempty, so there exists a unique element $\boldsymbol{m}(\boldsymbol{a}) \in [\boldsymbol{a}] \cap S$ which is minimal with respect to the total order $\preccurlyeq$.

Now let $S' = \{\boldsymbol{m}(\boldsymbol{a}) : \boldsymbol{a} \in S\}$. Then $S' \subseteq S$, and by Lemma 3 (Dickson's lemma), S' contains finitely many elements which are minimal with respect to $\preccurlyeq_d$. Let M be the finite subset of S' consisting of the elements of S' which are minimal with respect to $\preccurlyeq_d$. Let $\boldsymbol{m} \in M$ be the minimal element of M with respect to the total order $\preccurlyeq$. Transitivity then gives that $\boldsymbol{m}$ is the minimal element of S with respect to $\preccurlyeq$. □

Definition 7. *A monomial order on $\mathbb{N}^d$ is a total order $\preccurlyeq$ satisfying the following two conditions.*

1. $\mathbf{0} \preccurlyeq \boldsymbol{a}$ *for all* $\boldsymbol{a} \in \mathbb{N}^d$.
2. *For all* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{N}^d$, *if* $\boldsymbol{a} \preccurlyeq \boldsymbol{b}$, *then* $\boldsymbol{a} + \boldsymbol{c} \preccurlyeq \boldsymbol{b} + \boldsymbol{c}$.

The following are some examples of monomial orders on $\mathbb{N}^d$.

- Lexicographical order, $\preccurlyeq_{\text{lex}}$, on $\mathbb{N}^d$ is a monomial order. Here $\boldsymbol{a} \prec_{\text{lex}} \boldsymbol{b}$ if and only if the first nonzero entry of $\boldsymbol{b} - \boldsymbol{a}$ is positive (where the subtraction is taking place in $\mathbb{Z}^d$).
- Let S_d denote the symmetric group on d letters, and let S_d act on $\mathbb{N}^d$ by permuting subscripts. Let $\sigma \in S_d$ be any permutation. The lexicogrphical order with respect to σ, $\preccurlyeq_{\text{lex}}^{\sigma}$, on $\mathbb{N}^d$ is a monomial order. Here $\boldsymbol{a} \prec_{\text{lex}}^{\sigma} \boldsymbol{b}$ if and only if $\sigma(\boldsymbol{a}) \prec_{\text{lex}} \sigma(\boldsymbol{b})$.
- Graded lexicographical order, $\preccurlyeq_{\text{glex}}$, on $\mathbb{N}^d$ is a monomial order. Here $\boldsymbol{a} \prec_{\text{glex}} \boldsymbol{b}$ if and only if $|\boldsymbol{b}| > |\boldsymbol{a}|$ or $|\boldsymbol{b}| = |\boldsymbol{a}|$ and $\boldsymbol{a} \prec_{\text{lex}} \boldsymbol{b}$.

- Graded reverse lexicographical order, $\preccurlyeq_{\text{grlex}}$, on $\mathbb{N}^d$ is a monomial order. Here $\boldsymbol{a} \prec_{\text{grlex}} \boldsymbol{b}$ if and only if $|\boldsymbol{b}| > |\boldsymbol{a}|$ or $|\boldsymbol{b}| = |\boldsymbol{a}|$ and $\boldsymbol{b} \prec_{\text{lex}}^{\sigma} \boldsymbol{a}$, where $\sigma(i) = d-(i-1)$ for $1 \leq i \leq d$.
- The ith univariate order, $\preccurlyeq_{\text{univ}}^{i}$, on $\mathbb{N}^d$ is a monomial order. Given $\boldsymbol{a} \in \mathbb{N}^d$, let $\boldsymbol{a}'$ be the $(d-1)$-tuple in $\mathbb{N}^{d-1}$ obtained by deleting the ith component a_i of $\boldsymbol{a}$. Then $\boldsymbol{a} \prec_{\text{univ}}^{i} \boldsymbol{b}$ if and only if $\boldsymbol{a}' \prec_{\text{grlex}} \boldsymbol{b}'$ (here $\prec_{\text{grlex}}$ is the monomial order on $\mathbb{N}^{d-1}$) or $\boldsymbol{a}' = \boldsymbol{b}'$ and $a_i < b_i$.
- We let $\cdot : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ denote the Euclidean inner product. Let $\preccurlyeq$ be any fixed monomial order on $\mathbb{N}^d$ and fix $\boldsymbol{w} \in \mathbb{R}^d$ with $w_i \geq 0$ for all i. Then the weighting of $\preccurlyeq$ by $\boldsymbol{w}$, $\preccurlyeq_{\boldsymbol{w}}$, on $\mathbb{N}^d$ is a monomial order. Here $\boldsymbol{a} \prec_{\boldsymbol{w}} \boldsymbol{b}$ if and only if $\boldsymbol{w} \cdot (\boldsymbol{b} - \boldsymbol{a}) > 0$ or $\boldsymbol{w} \cdot (\boldsymbol{b} - \boldsymbol{a}) = 0$ and $\boldsymbol{a} \prec \boldsymbol{b}$.

Lemma 5. *Let $\preccurlyeq$ be a monomial order on $\mathbb{N}^d$, and let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^d$ with $\boldsymbol{a} \preccurlyeq \boldsymbol{b}$. Then for all $n \in \mathbb{N}$, $n\boldsymbol{a} \preccurlyeq n\boldsymbol{b}$.*

Proof. Certainly $0\boldsymbol{a} = 0\boldsymbol{b}$, and by assumption, $\boldsymbol{a} \preccurlyeq \boldsymbol{b}$. Suppose now that $n\boldsymbol{a} \preccurlyeq n\boldsymbol{b}$ for some $n \geq 1$. Condition 2 in Definition 7 implies

$$(n+1)\boldsymbol{a} = \boldsymbol{a} + n\boldsymbol{a} \preccurlyeq \boldsymbol{b} + n\boldsymbol{a} \preccurlyeq \boldsymbol{b} + n\boldsymbol{b} = (n+1)\boldsymbol{b}.$$

□

Lemma 6. *Let $\preccurlyeq$ be a monomial order on $\mathbb{N}^d$. Then $\preccurlyeq$ is a linear extension of $\preccurlyeq_d$.*

Proof. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^d$ with $\boldsymbol{b} \preccurlyeq_d \boldsymbol{a}$. Then $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b} \in \mathbb{N}^d$. If $\boldsymbol{a} \preccurlyeq \boldsymbol{b}$, we have

$$\boldsymbol{a} + \boldsymbol{c} \preccurlyeq \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{a}.$$

But Definition 7 implies

$$\boldsymbol{a} = \boldsymbol{a} + \boldsymbol{0} \preccurlyeq \boldsymbol{a} + \boldsymbol{c}.$$

It follows that $\boldsymbol{a} = \boldsymbol{a} + \boldsymbol{c}$, forcing $\boldsymbol{c} = \boldsymbol{0}$ and $\boldsymbol{b} = \boldsymbol{a}$. □

Example 1. *There are linear extensions of* $\preccurlyeq_d$ *which are not monomial orders. In* $\mathbb{N}^2$*, consider the set* $X = \{(3,0),(2,1),(0,3)\}$*. The elements of* X *are pairwise incomparable with respect to* $\preccurlyeq_2$*. By Lemma 4, there is a linear extension* $\preccurlyeq$ *of* $\preccurlyeq_d$ *for which* $(2,1)$ *is the minimal element of* X*. Suppose* $\preccurlyeq$ *is a monomial order. There are two possibilities to consider.*

If $\boldsymbol{e}_1 \prec \boldsymbol{e}_2$*, then*

$$(3,0) = 2\boldsymbol{e}_1 + \boldsymbol{e_1} \prec 2\boldsymbol{e}_1 + \boldsymbol{e_2} = (2,1).$$

On the other hand, if $\boldsymbol{e}_2 \prec \boldsymbol{e}_1$*, then*

$$(0,3) = 2\boldsymbol{e}_2 + \boldsymbol{e_2} \prec 2\boldsymbol{e}_1 + \boldsymbol{e_2} = (2,1).$$

Thus, $(2,1)$ *is not minimal in* X *with respect to* $\preccurlyeq$*.* $\Diamond$

CHAPTER 2

p-ADIC MEASURES

Fix an odd rational prime p, and let $\mathbb{Z}_p^d$ denote the d-fold direct product of the topological ring $\mathbb{Z}_p$. In this chapter, we develop a theory of measures on the additive group $\mathbb{Z}_p^d$ which take values in a non-Archimedean normed ring. There are at least three different ways to view such measures, and each perspective has its unique advantages in different settings.

2.1 The Analytic Viewpoint

Much of the material that follows was adapted from Childress (2012), but aspects of what follows can be found scattered throughout the literature. For the analytic viewpoint, we mention in particular the works of Amice (1965), Amice (1978), Chapitre VI of Monna (1970), and Chapter 7 of van Rooij (1978). Additional references include Chapter 2 of Koblitz (1984), Appendices A.5 and A.6 of Schikhof (2007), Chapter 12 of Washington (1997), and Chapters 4 and 12 in Lang (1990).

As usual, $\mathbb{Q}_p$ will denote the field of p-adic numbers, the fraction field of $\mathbb{Z}_p$. We refer to Appendix A for all relevant topological properties of $\mathbb{Z}_p^d$. We denote by $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$. Let ord denote the p-adic valuation on $\mathbb{C}_p$, normalized so that $\operatorname{ord}(p) = 1$, and $|\cdot|_p$ the p-adic absolute value on $\mathbb{C}_p$, normalized so that $|p|_p = p^{-1}$. Throughout, all rings are commutative with identity element $1 \neq 0$. For a ring $\mathcal{A}$, we denote the group of units of $\mathcal{A}$ by $\mathcal{A}^\times$.

Definition 8. *Let $\mathcal{A}$ be a ring. A norm on $\mathcal{A}$ is a function $\|\cdot\| : \mathcal{A} \to \mathbb{R}$ satisfying*

1. $\|a\| \geq 0$ *for all* $a \in \mathcal{A}$.

2. $\|a\| = 0$ *if and only if* $a = 0$.

3. $\|a + b\| \leq \|a\| + \|b\|$ *for all* $a, b \in \mathcal{A}$.

4. $\|ab\| \leq \|a\|\|b\|$ *for all* $a, b \in \mathcal{A}$.

5. $\|1\| = 1$.

In condition 3, if instead $\|a+b\| \leq \max\{\|a\|, \|b\|\}$ *for all* $a, b \in \mathcal{A}$, *then* $\|\cdot\|$ *is called non-archimedean. In condition 4, if instead* $\|ab\| = \|a\|\|b\|$ *for all* $a, b \in \mathcal{A}$, *then* $\|\cdot\|$ *is called multiplicative. If* $\|\cdot\|$ *is a norm on* $\mathcal{A}$, *then the pair* $(\mathcal{A}, \|\cdot\|)$ *will be called a normed ring.*

If $(\mathcal{A}, \|\cdot\|)$ is a normed ring, then the function $\mathsf{d} : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ given by

$$\mathsf{d}(x, y) = \|x - y\|$$

is a metric. We will always endow $(\mathcal{A}, \|\cdot\|)$ with this metric topology. Note that conditions 2 and 3 in Definition 8 give that addition and multiplication are continuous as functions $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$. Thus, $(\mathcal{A}, \|\cdot\|)$ is a topological ring.

Definition 9. *Let* $(R, \|\cdot\|_R)$ *be a normed ring. A normed* R*-algebra is a normed ring* $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *such that* $\mathcal{A}$ *is an* R*-algebra and* $\|ra\|_{\mathcal{A}} = \|r\|_R\|a\|_{\mathcal{A}}$ *for all* $r \in R$ *and* $a \in \mathcal{A}$.

If $(R, \|\cdot\|_R)$ is a normed ring and $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a normed R-algebra, then Definition 9 gives that scalar multiplication is continuous as a function $R \times \mathcal{A} \to \mathcal{A}$. Consequently, $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a topological R-algebra. The structure map $R \to \mathcal{A}$ given by $r \mapsto r1$ is thus a continuous ring homomorphism. When this map is injective, we identify R with its image in $\mathcal{A}$ and view R as a subring of $\mathcal{A}$. Note that the subspace topology on R when viewed as a subring of $\mathcal{A}$ agrees with the topology on R induced by $\|\cdot\|_R$ since the norm on $\mathcal{A}$ extends the norm on R in this case. Consequently, R is a topological subring of $\mathcal{A}$.

For the following, we fix a normed $\mathbb{Z}_p$-algebra $(R, \|\cdot\|_R)$ such that the structure map $\mathbb{Z}_p \to R$ is injective, $\|\cdot\|_R$ is non-archimedean, and R is complete with respect to the topology induced by $\|\cdot\|_R$.

We may now proceed to the theory of measures and integration. As in Appendix A, let CO_d denote the set of compact-open subsets of $\mathbb{Z}_p^d$.

Definition 10. *An R-valued measure on $\mathbb{Z}_p^d$ is a function $\alpha : CO_d \to R$ such that*

1. *$\alpha(A \cup B) = \alpha(A) + \alpha(B)$ for all $A, B \in CO_d$ with $A \cap B = \emptyset$.*
2. *There exists a real number $C_\alpha \geq 0$ with $\|\alpha(A)\|_R \leq C_\alpha$ for all $A \in CO_d$.*

Let $\mathcal{M}_d = \mathcal{M}_d(R)$ denote the set of R-valued measures on $\mathbb{Z}_p^d$.

Note that the condition 2 in Definition 10 above is automatically satisfied if $\|\cdot\|_R$ is bounded as a function $R \to \mathbb{R}$. The additivity in condition 1 of Definition 10 may be extended to finite disjoint unions by a standard induction argument.

We recall that every set in CO_d can be written as a finite disjoint union of sets of the form

$$\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d = \prod_{i=1}^{d} a_i + p^{n_i}\mathbb{Z}_p,$$

with $\boldsymbol{a} \in \mathbb{Z}_p^d$ and $\boldsymbol{n} \in \mathbb{N}^d$, which we refer to as "polyballs" in $\mathbb{Z}_p^d$. In light of condition 1 of Definition 10, we may specify a measure α by its values $\alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p)$ for $\boldsymbol{a} \in \mathbb{Z}_p^d$ and $\boldsymbol{n} \in \mathbb{N}^d$.

Lemma 7. *Let $\alpha, \beta \in \mathcal{M}_d$ and suppose $\alpha(A) = \beta(A)$ for all polyballs A in $\mathbb{Z}_p^d$. Then $\alpha = \beta$.*

Proof. Let $A \in CO_d$. Write

$$A = \bigsqcup_{i=1}^{n} B_i,$$

with the B_i polyballs in $\mathbb{Z}_p^d$. Then

$$\alpha(A) = \sum_{i=1}^{n} \alpha(B_i) = \sum_{i=1}^{n} \beta(B_i) = \beta(A).$$

Thus, $\alpha = \beta$ as functions $CO_d \to R$. $\square$

Furthermore, since the norm on R is non-archimedean, a function $\alpha : CO_d \to R$ satisfying condition 1 of Definition 10 is a measure if and only if α is bounded with respect to $\|\cdot\|_R$ on the set of polyballs in $\mathbb{Z}_p^d$.

Example 2. *A trivial example of an R-valued measure on $\mathbb{Z}_p^d$ is the "zero measure" α_0 defined by $\alpha_0(A) = 0$ for all $A \in CO_d$. We will always denote the zero measure by α_0.* $\Diamond$

Example 3. *Fix $\boldsymbol{s} \in \mathbb{Z}_p^d$. The Dirac measure of mass 1 centered at $\boldsymbol{s}$ is given by*

$$\delta_{\boldsymbol{s}}(A) = \begin{cases} 1 & : \boldsymbol{s} \in A \\ 0 & : \boldsymbol{s} \notin A \end{cases}$$

for $A \in CO_d$. In terms of polyballs in $\mathbb{Z}_p^d$, $\delta_{\boldsymbol{s}}$ is given simply by

$$\delta_{\boldsymbol{s}}(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d) = \begin{cases} 1 & : \boldsymbol{s} \equiv \boldsymbol{a} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d} \\ 0 & : else \end{cases}$$

It is routine to see that $\delta_{\boldsymbol{s}} \in \mathcal{M}_d$ .The Dirac measures will be some of the most important examples of measures in the sequel. $\Diamond$

We may endow $\mathcal{M}_d$ with the supremum norm as a set of R-valued functions:

$$\|\alpha\|_u = \sup\{\|\alpha(A)\|_R : A \in CO_d\}$$

for each $\alpha \in \mathcal{M}_d$. In light of condition 2 in Definition 10, $\|\alpha\|_u$ exists for each $\alpha \in \mathcal{M}_d$. As a collection of R-valued functions, $\mathcal{M}_d$ is an R-module under pointwise addition of measures and scalar multiplication by R:

$$(\alpha + \beta)(A) = \alpha(A) + \beta(A)$$
$$(c\alpha)(A) = c\alpha(A)$$

for $\alpha, \beta \in \mathcal{M}_d$, $c \in R$, and all $A \in CO_d$. Indeed, since $\|\cdot\|_R$ is non-archimedean, we have

$$\|\alpha + \beta\|_u \leq \max\{\|\alpha\|_u, \|\beta\|_u\}$$
$$\|c\alpha\|_u \leq \|c\|_R \|\alpha\|_u,$$

so that $\alpha + \beta, c\alpha \in \mathcal{M}_d$ for all $\alpha, \beta \in \mathcal{M}_d$ and all $c \in R$. Further, $\|\cdot\|_u$ is non-archimedean. As usual, $\mathcal{M}_d$ becomes a non-archimedean normed R-module under $\|\cdot\|_u$.

We now develop a theory of integration for continuous functions $\mathbb{Z}_p^d \to R$. Let $C(\mathbb{Z}_p^d, R)$ denote the set of all continuous functions $\mathbb{Z}_p^d \to R$. Under pointwise addition and multiplication and pointwise scalar multiplication by R, $C(\mathbb{Z}_p^d, R)$ forms an R-algebra (e.g., van Rooij (1978), Chapter 6). We endow $C(\mathbb{Z}_p^d, R)$ with the supremum norm:

$$\|f\|_\infty = \sup\{\|f(\boldsymbol{x})\|_R : \boldsymbol{x} \in \mathbb{Z}_p^d\}.$$

When endowed with the supremum norm, $C(\mathbb{Z}_p^d, R)$ becomes a complete non-archimedean normed topological R-algebra. Recall that the subset of locally constant functions $\mathbb{Z}_p^d \to R$ forms a dense normed R-subalgebra of $C(\mathbb{Z}_p^d, R)$, denoted $LC(\mathbb{Z}_p^d, R)$ (see Appendix A). Recall further that for $\boldsymbol{n} \in \mathbb{N}^d$, the $\boldsymbol{n}$th level of $\mathbb{Z}_p^d$ is the following collection of polyballs:

$$L_{\boldsymbol{n}} = \{\boldsymbol{x} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d : \boldsymbol{x} \in \mathbb{Z}_p^d\}.$$

We begin by defining integration on $LC(\mathbb{Z}_p^d, R)$. Toward that end, let $f \in LC(\mathbb{Z}_p^d, R)$. Then f factors through $L_{\boldsymbol{n}}$ for some $\boldsymbol{n} \in \mathbb{N}^d$. We may write

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x})$$

where, as in Appendix A, $g_{\boldsymbol{a},\boldsymbol{n}}$ is the characteristic function of the polyball $\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d$. For $\alpha \in \mathcal{M}_d$, we define

$$\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f(\boldsymbol{a}) \alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d).$$

For any $\boldsymbol{m} \in \mathbb{N}^d$ with $\boldsymbol{n} \preccurlyeq_d \boldsymbol{m}$, also

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1}} f(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{m}}(\boldsymbol{x}).$$

We can write each polyball $\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d$ as a disjoint union of all polyballs of the form $\boldsymbol{b} + \boldsymbol{p}^{\wedge \boldsymbol{m}} \mathbb{Z}_p^d$ where $\boldsymbol{b} \equiv \boldsymbol{a} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d}$, and for which $f(\boldsymbol{b}) = f(\boldsymbol{a})$ since f is locally constant. The finite additivity of the measure α gives that

$$\sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1}} f(\boldsymbol{a}) \alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{m}} \mathbb{Z}_p^d) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f(\boldsymbol{a}) \alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d).$$

Thus, if f factors through both $L_{\boldsymbol{n}}$ and $L_{\boldsymbol{m}}$, then f factors through $L_{\sup(\boldsymbol{n},\boldsymbol{m})}$, and the above gives that

$$\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x})$$

is well-defined.

Lemma 8. *Fix $\alpha \in \mathcal{M}_d$. Define $I_\alpha : LC(\mathbb{Z}_p^d, R) \to R$ by*

$$I_\alpha(f) = \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}).$$

Then I_α is a continuous R-linear map.

Proof. Let $f_1, f_2 \in LC(\mathbb{Z}_p^d, R)$. Then there is some $\boldsymbol{n} \in \mathbb{N}^d$ for which we may write

$$f_1(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f_1(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x})$$
$$f_2(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f_2(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}).$$

Then $f_1 + f_2$ is locally constant with

$$(f_1 + f_2)(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} (f_1 + f_2)(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}).$$

Additionally, if $a \in R$, then af_1 is locally constant with

$$(af_1)(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} af_1(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}) = a \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f_1(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}).$$

Then clearly

$$\int_{\mathbb{Z}_p^d} (f_1 + f_2)(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = \int_{\mathbb{Z}_p^d} f_1(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) + \int_{\mathbb{Z}_p^d} f_2(\boldsymbol{x})\, d\alpha(\boldsymbol{x})$$

and

$$\int_{\mathbb{Z}_p^d} (af_1)(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = a \int_{\mathbb{Z}_p^d} f_1(\boldsymbol{x})\, d\alpha(\boldsymbol{x}).$$

This gives the R-linearity of I_α on $LC(\mathbb{Z}_p^d, R)$.

For continuity, again take $f_1, f_2 \in LC(\mathbb{Z}_p^d, R)$ and write

$$f_1(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f_1(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x})$$
$$f_2(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f_2(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x})$$

for some $\boldsymbol{n} \in \mathbb{N}^d$. Then

$$(f_1 - f_2)(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} (f_1 - f_2)(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}).$$

Since I_α is R-linear, we have

$$\begin{aligned}\|I_\alpha(f_1) - I_\alpha(f_2)\|_R &= \|I_\alpha(f_1 - f_2)\|_R \\ &= \left\| \int_{\mathbb{Z}_p^d} (f_1 - f_2)(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \right\|_R \\ &= \left\| \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} (f_1 - f_2)(\boldsymbol{a}) \alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d) \right\|_R \\ &\leq \max_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \left\{ \|(f - g)(\boldsymbol{a})\|_R \|\alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d)\|_R \right\} \\ &\leq \|\alpha\|_u \|f - g\|_\infty.\end{aligned}$$

Thus, I_α is continuous on $LC(\mathbb{Z}_p^d, R)$. □

Corollary 3. *Let $\alpha \in \mathcal{M}_d$. For any $A \in CO_d$, we have*

$$\int_{\mathbb{Z}_p^d} g_A(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = \alpha(A),$$

where $g_A(\boldsymbol{x})$ is the characteristic function of A.

Proof. Suppose first that A is a polyball. Then g_A is locally constant, and by definition,

$$\int_{\mathbb{Z}_p^d} g_A(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = \alpha(A).$$

In general, A may be written as a disjoint union of polyballs in $\mathbb{Z}_p^d$, whence g_A is expressible as a finite sum of characteristic functions of polyballs in $\mathbb{Z}_p^d$. The corollary now follows from the linearity in Lemma 8 and the additivity of α. □

In Appendix A, we define a d-net to be a net indexed by $(\mathbb{N}^d, \preccurlyeq_d)$. Any continuous function $f : \mathbb{Z}_p^d \to R$ is a uniform limit of a d-net of locally constant functions $(f_{\boldsymbol{n}})$, where $f_{\boldsymbol{n}}$ factors through $L_{\boldsymbol{n}}$ (see Appendix A). For each $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}^d$ with $\boldsymbol{n} \preccurlyeq_d \boldsymbol{m}$, Lemma 8 gives

$$\|I_\alpha(f_{\boldsymbol{m}}) - I_\alpha(f_{\boldsymbol{n}})\|_R \leq \|\alpha\|_u \|f_{\boldsymbol{m}} - f_{\boldsymbol{n}}\|_\infty.$$

Thus, the d-net of values $(I_\alpha(f_{\boldsymbol{n}}))$ is Cauchy in R, so converges to an element of R.

In the event that $\lim f_{\boldsymbol{n}} = \lim g_{\boldsymbol{n}}$ for d-nets $(f_{\boldsymbol{n}})$ and $(g_{\boldsymbol{n}})$ of locally constant functions with the $\boldsymbol{n}$th term factoring through $L_{\boldsymbol{n}}$, then $(f_{\boldsymbol{n}} - g_{\boldsymbol{n}})$ is also a d-net of locally constant functions with

$$f_{\boldsymbol{n}} - g_{\boldsymbol{n}} \to 0 \in LC(\mathbb{Z}_p^d, R).$$

In this case, the continuity and linearity of I_α on $LC(\mathbb{Z}_p^d, R)$ gives

$$I_\alpha(f_{\boldsymbol{n}}) - I_\alpha(g_{\boldsymbol{n}}) = I_\alpha(f_{\boldsymbol{n}} - g_{\boldsymbol{n}}) \to I_\alpha(0) = 0.$$

In light of the above considerations, for $f \in C(\mathbb{Z}_p^d, R)$ with $f = \lim f_{\boldsymbol{n}}$, where $f_{\boldsymbol{n}}$ is locally constant factoring through $L_{\boldsymbol{n}}$, we may put

$$\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = I_\alpha(f) := \lim_{\boldsymbol{n}} I_\alpha(f_{\boldsymbol{n}}) = \lim_{\boldsymbol{n}} \int_{\mathbb{Z}_p^d} f_{\boldsymbol{n}}(\boldsymbol{x})\, d\alpha(\boldsymbol{x}).$$

We have extended the map I_α from the dense R-subalgebra $LC(\mathbb{Z}_p^d, R)$ to all of $C(\mathbb{Z}_p^d, R)$ by continuity.

Proposition 2. *Fix $\alpha \in \mathcal{M}_d$. The map $I_\alpha : C(\mathbb{Z}_p^d, R) \to R$ is a continuous R-linear map and $\|I_\alpha(f)\|_R \leq \|\alpha\|_u \|f\|_\infty$ for all $f \in C(\mathbb{Z}_p^d, R)$.*

Proof. The R-linearity of I_α on $C(\mathbb{Z}_p^d, R)$ is immediate from the linearity of I_α on $LC(\mathbb{Z}_p^d, R)$. Let $f \in C(\mathbb{Z}_p^d, R)$. We can write f as the uniform limit of a d-net of locally constant functions $f_{\boldsymbol{n}}$ with $f_{\boldsymbol{n}}$ factoring through $L_{\boldsymbol{n}}$ and satisfying $\|f_{\boldsymbol{n}}\|_\infty \leq \|f\|_\infty$ for each $\boldsymbol{n} \in \mathbb{N}^d$ (as in the proof of the density of the locally constant functions in

$C(\mathbb{Z}_p^d, R)$ in Appendix A). Therefore,

$$\begin{aligned}\|I_\alpha(f)\|_R &= \left\|\lim_{\boldsymbol{n}} I_\alpha(f_{\boldsymbol{n}})\right\|_R \\ &= \lim_{\boldsymbol{n}} \|I_\alpha(f_{\boldsymbol{n}})\|_R \\ &\leq \lim_{\boldsymbol{n}} \|f_{\boldsymbol{n}}\|_\infty \|\alpha\|_u \\ &\leq \|f\|_\infty \|\alpha\|_u\end{aligned}$$

where the second to last line above follows from the proof of the continuity of I_α on $LC(\mathbb{Z}_p^d, R)$ in Lemma 8. This bound, together with the linearity of I_α on $C(\mathbb{Z}_p^d, R)$, gives the continuity of I_α on $C(\mathbb{Z}_p^d, R)$. For $f_1, f_2 \in C(\mathbb{Z}_p^d, R)$, we have

$$\|I_\alpha(f_1) - I_\alpha(f_2)\|_R = \|I_\alpha(f_1 - f_2)\|_R \leq \|\alpha\|_u \|f_1 - f_2\|_\infty.$$

□

Let X be any normed topological $\mathbb{Z}_p$-module with norm $\|\cdot\|_X$, and let $\mathrm{Hom}_{\mathrm{cont}}(X, R)$ denote the R-module of continuous R-linear maps $X \to R$. An R-linear map $f : X \to R$ is called bounded if there exists a real number $C \geq 0$ such that

$$\|f(x)\|_R \leq C\|x\|_X$$

for all $x \in X$. We recall the following important fact (see, e.g., Chapter 1 of Schneider (2002)).

Lemma 9. *Let X be any normed topological $\mathbb{Z}_p$-module with norm $\|\cdot\|_X$. An R-linear map $f : X \to R$ is continuous if and only if f is bounded.*

Proof. Suppose first that $f : X \to R$ is R-linear and bounded. Let $C \geq 0$ be such that $\|f(x)\|_R \leq C\|x\|_X$ for all $x \in X$. Then for $x, y \in X$, we have

$$\|f(x) - f(y)\|_R = \|f(x - y)\|_R \leq C\|x - y\|_X,$$

so that f is Lipschitz continuous. Conversely, suppose f is continuous. Then f is continuous at $0 \in X$, and there exists $\delta > 0$ such that $\|x\|_X < \delta$ implies $\|f(x)\|_R < 1$. Let $x \in X \setminus \{0\}$. Since

$$\left\| \frac{\delta x}{2\|x\|_X} \right\|_X < \delta,$$

we have

$$\|f(x)\|_R = \left\| \frac{2\|x\|_X}{\delta} f\left(\frac{\delta x}{2\|x\|_X} \right) \right\|_R < \frac{2\|x\|_X}{\delta}.$$

Since $\|f(0)\|_R = 0 = \|0\|_X$, f is bounded. □

Now for $I \in \mathrm{Hom}_{\mathrm{cont}}(C(\mathbb{Z}_p^d, R), R)$, we define

$$\|I\|_u = \sup\left\{ \frac{\|I(f)\|_R}{\|f\|_\infty} : f \in C(\mathbb{Z}_p^d, R) \setminus \{0\} \right\},$$

which exists by Lemma 9.

Theorem 2. *If $I : C(\mathbb{Z}_p^d, R) \to R$ is any continuous R-linear map, then there exists $\alpha \in \mathcal{M}_d$ such that for any $f \in C(\mathbb{Z}_p^d, R)$,*

$$I(f) = I_\alpha(f) = \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}).$$

Proof. For $A \in CO_d$, put $\alpha(A) = I(g_A)$, where $g_A : \mathbb{Z}_p^d \to R$ denotes the characteristic function of A. Let $A, B \in CO_d$ with $A \cap B = \emptyset$ and let $g_A, g_B \in C(\mathbb{Z}_p^d, R)$ denote the characteristic functions of A and B, respectively. The characteristic function of $A \cup B$ is $g_A + g_B$, and the linearity of I gives

$$\alpha(A \cup B) = I(g_A + g_B) = I(g_A) + I(g_B) = \alpha(A) + \alpha(B),$$

as needed. Since the norm on R is non-archimedean, it suffices to show that α is bounded on polyballs in $\mathbb{Z}_p^d$. To that end, let A be a polyball in $\mathbb{Z}_p^d$ and let $g_A \in C(\mathbb{Z}_p^d, R)$ be its characteristic function. By the continuity of I, there is $\delta > 0$ so that $\|I(f)\|_R < 1$ whenever $\|f\|_\infty < \delta$. Since $\mathbb{Z}_p \subseteq R$ and $\|\cdot\|_R$ extends $|\cdot|_p$, we may choose an element $c \in \mathbb{Z}_p$ with $0 < \|c\|_R < \delta$. Now

$$\|cg_A\|_\infty = \|c\|_R \|g_A\|_\infty = \|c\|_R < \delta,$$

so that

$$\|I(cg_A)\|_R < 1.$$

By the R-linearity of I,

$$I(cg_A) = cI(g_A) = c\alpha(A).$$

Thus,

$$\|\alpha(A)\|_R < \|c\|_R^{-1}.$$

Consequently, $\alpha \in \mathcal{M}_d$.

Finally, if $f \in C(\mathbb{Z}_p^d, R)$ is locally constant, say with

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}),$$

then

$$\begin{aligned} I_\alpha(f) &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\ &= \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f(\boldsymbol{a}) \alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d) \\ &= \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f(\boldsymbol{a}) I(g_{\boldsymbol{a},\boldsymbol{n}}) \\ &= I\left(\sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}} \right) \\ &= I(f) \end{aligned}$$

From this, we readily obtain

$$I(f) = I_\alpha(f) = \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x})$$

for any $f \in C(\mathbb{Z}_p^d, R)$ by realizing f as a uniform limit of a d-net of locally constant functions and employing the continuity of I and I_α on $LC(\mathbb{Z}_p^d, R)$. $\square$

We have thus established a bijection between R-valued measures on $\mathbb{Z}_p^d$ and continuous (bounded) R-valued linear maps on $C(\mathbb{Z}_p^d, R)$ via $\alpha \leftrightarrow I_\alpha$, where

$$I_\alpha(f) = \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}).$$

We have seen that $\|I_\alpha\|_u \leq \|\alpha\|_u$. On the other hand, If A is a polyball in $\mathbb{Z}_p^d$ and g_A is its characteristic function, then $\|g_A\|_\infty = 1$, and we have

$$\|\alpha(A)\|_R = \|I_\alpha(g_A)\|_R \leq \|I_\alpha\|_u \|g_A\|_\infty = \|I_\alpha\|_u,$$

so that $\|\alpha\|_u \leq \|I_\alpha\|_u$. Consequently, the bijection $\alpha \mapsto I_\alpha$ is norm-preserving.

We will now consider certain spaces of d-nets in R, which are the obvious analogues of the usual sequence spaces (see Appendix A for the definition of d-net). Since $\mathbb{N}$ and $\mathbb{N}^d$ are of the same cardinality, these spaces of d-nets are in fact topologically isomorphic to the usual sequence spaces (see Chapter II of Schneider (2002)).

Definition 11. *A d-net $\bar{a} = (a_{\boldsymbol{n}})$ in R is bounded if*

$$\sup_{\boldsymbol{n} \in \mathbb{N}^d} \{\|a_{\boldsymbol{n}}\|_R\} < \infty.$$

Note that if $(a_{\boldsymbol{n}})$ is a d-net in R converging to 0, then $(a_{\boldsymbol{n}})$ is bounded and we have

$$\sup_{\boldsymbol{n} \in \mathbb{N}^d} \{\|a_{\boldsymbol{n}}\|_R\} = \max_{\boldsymbol{n} \in \mathbb{N}^d} \{\|a_{\boldsymbol{n}}\|_R\}.$$

(This follows because $\mathbb{N}^d$ is an interval-finite poset; see Lemma 2).

Let $c_0^{(d)}(R)$ be the set of all d-nets in R which converge to 0. For a d-net $\bar{a} = (a_{\boldsymbol{n}}) \in c_0^{(d)}(R)$, we put

$$\|\bar{a}\|_u = \sup_{\boldsymbol{n} \in \mathbb{N}^d} \{\|a_{\boldsymbol{n}}\|_R\} = \max_{\boldsymbol{n} \in \mathbb{N}^d} \{\|a_{\boldsymbol{n}}\|_R\}.$$

$c_0^{(d)}(R)$ is a R-module under termwise addition of d-nets and termwise scalar multiplication by R:

$$(a_{\boldsymbol{n}})_{\boldsymbol{n}} + (b_{\boldsymbol{n}})_{\boldsymbol{n}} = (a_{\boldsymbol{n}} + b_{\boldsymbol{n}})_{\boldsymbol{n}}$$

$$\alpha(a_{\boldsymbol{n}})_{\boldsymbol{n}} = (\alpha a_{\boldsymbol{n}})_{\boldsymbol{n}}$$

$c_0^{(d)}(R)$ becomes an R-algebra under "Cauchy multiplication":

$$(a_{\boldsymbol{n}})_{\boldsymbol{n}}(b_{\boldsymbol{n}})_{\boldsymbol{n}} = \left(\sum_{\boldsymbol{k} \preccurlyeq_d \boldsymbol{n}} a_{\boldsymbol{k}} b_{\boldsymbol{n}-\boldsymbol{k}}\right)_{\boldsymbol{n}}.$$

With respect to the norm $\| \cdot \|_u$, $c_0^{(d)}(R)$ is a complete non-archimedean normed R-algebra (e.g., Chapitre III of Monna (1970), Chapter 6 of van Rooij (1978)).

Now let $f \in C(\mathbb{Z}_p^d, R)$. Then the net of Mahler coefficients $(a_{\boldsymbol{n}}(f))_{\boldsymbol{n}}$ is an element of $c_0^{(d)}(R)$ (see Appendix A).

Proposition 3. *The map* $\Psi : C(\mathbb{Z}_p^d, R) \to c_0^{(d)}(R)$ *given by* $\Psi : f \mapsto (a_{\boldsymbol{n}}(f))$ *is a normed* R*-module isomorphism.*

Proof. In Appendix A we showed that for $f, g \in C(\mathbb{Z}_p^d, R)$ and $c \in R$,

$$a_{\boldsymbol{n}}(f+g) = a_{\boldsymbol{n}}(f) + a_{\boldsymbol{n}}(g)$$

$$a_{\boldsymbol{n}}(cf) = c a_{\boldsymbol{n}}(f)$$

(these follow from the uniqueness of the Mahler expansion). Thus, the map Ψ is an R-module homomorphism. We also showed that Ψ is bijective in Appendix A. Finally, we have also recorded

$$\|f\|_\infty = \sup_{n \in \mathbb{N}^d} \{\|a_{\boldsymbol{n}}(f)\|_R\} = \|(a_{\boldsymbol{n}}(f))\|_u,$$

which gives continuity of Ψ and Ψ^{-1}. □

With Ψ as in Proposition 3, we use the notation $\Psi(f) = (a_{\boldsymbol{n}}(f))_{\boldsymbol{n}} \in c_0^{(d)}(R)$ and $\Psi^{-1}(\bar{a}) = f_{\bar{a}} \in C(\mathbb{Z}_p^d, R)$.

Corollary 4. *There is a bijection between R-valued measures on $\mathbb{Z}_p^d$ and continuous R-linear maps $c_0^{(d)}(R) \to R$ via $\alpha \leftrightarrow J_\alpha$, where*

$$J_\alpha(\bar{a}) = I_\alpha(f_{\bar{a}}).$$

Let $\ell^\infty_{(d)}(R)$ be the set of all bounded d-nets in R, so that $c_0^{(d)}(R) \subseteq \ell^\infty_{(d)}(R)$. $\ell^\infty_{(d)}(R)$ is likewise a normed R-algebra under termwise addition of d-nets, Cauchy multiplication, and termwise scalar multiplication by R. When endowed with the norm $\|\cdot\|_u$, $\ell^\infty_{(d)}(R)$ is a complete non-archimedean normed R-algebra and $c_0^{(d)}(R)$ is a closed normed R-subalgebra of $\ell^\infty_{(d)}(R)$.

Proposition 4. *Let J be a continuous R-linear map $c_0^{(d)}(R) \to R$. There exists a d-net $(b_{\boldsymbol{n}}) \in \ell^\infty_{(d)}(R)$ depending only on J, such that for any $\bar{a} \in c_0^{(d)}(R)$,*

$$J(\bar{a}) = \sum_{\boldsymbol{n}\in\mathbb{N}^d} a_{\boldsymbol{n}} b_{\boldsymbol{n}}.$$

Conversely, any such d-net $\bar{b} \in \ell^\infty_{(d)}(R)$ gives a continuous R-linear map $J_{\bar{b}} : c_0^{(d)}(R) \to R$ given by

$$J_{\bar{b}}(\bar{a}) = \sum_{\boldsymbol{n}\in\mathbb{N}^d} a_{\boldsymbol{n}} b_{\boldsymbol{n}}.$$

Proof. For each $\boldsymbol{k} \in \mathbb{N}^d$, define a d-net $\bar{e}^{(\boldsymbol{k})} = (e_{\boldsymbol{n}}^{(\boldsymbol{k})})_{\boldsymbol{n}} \in c_0^{(d)}(R)$ by

$$e_{\boldsymbol{n}}^{(\boldsymbol{k})} = \begin{cases} 1 & : \boldsymbol{n} = \boldsymbol{k} \\ 0 & : \text{else} \end{cases}$$

Put $b_{\boldsymbol{k}} = J(\bar{e}^{(\boldsymbol{k})}) \in R$ and consider the d-net $\bar{b} = (b_{\boldsymbol{n}})_{\boldsymbol{n}}$ in R. Then $\bar{b}$ depends only on J. Let $\alpha \in \mathcal{M}_d$ be the measure associated to J by Corollary 4, so that $J = J_\alpha$. We have

$$b_{\boldsymbol{n}} = J_\alpha(\bar{e}^{(\boldsymbol{n})}) = I_\alpha(f_{\bar{e}^{(\boldsymbol{n})}}) = \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{n}} \, d\alpha(\boldsymbol{x}).$$

By Proposition 2,

$$\|b_{\boldsymbol{n}}\|_R \leq \|\alpha\|_u \left\| \binom{\boldsymbol{x}}{\boldsymbol{n}} \right\|_\infty = \|\alpha\|_u,$$

so $\bar{b}$ is bounded.

Now let $\bar{a} \in c_0^{(d)}(R)$ be arbitrary. For each $\boldsymbol{N} \in \mathbb{N}^d$, let $\bar{a}^{(\boldsymbol{N})} = (a_{\boldsymbol{n}}^{(\boldsymbol{N})})_{\boldsymbol{n}} \in c_0^{(d)}(R)$ be given by

$$a_{\boldsymbol{n}}^{(\boldsymbol{N})} = \begin{cases} a_{\boldsymbol{n}} & : \boldsymbol{n} \preccurlyeq_d \boldsymbol{N} \\ 0 & : \text{else} \end{cases}$$

In other words,

$$\bar{a}^{(\boldsymbol{N})} = \sum_{\boldsymbol{n} \preccurlyeq_d \boldsymbol{N}} a_{\boldsymbol{n}} \bar{e}^{(\boldsymbol{n})}.$$

By R-linearity,

$$J(\bar{a}^{(\boldsymbol{N})}) = \sum_{\boldsymbol{n} \preccurlyeq_d \boldsymbol{N}} a_{\boldsymbol{n}} J(\bar{e}^{(\boldsymbol{n})}) = \sum_{\boldsymbol{n} \preccurlyeq_d \boldsymbol{N}} a_{\boldsymbol{n}} b_{\boldsymbol{n}}$$

Now $(\bar{a}^{(\boldsymbol{N})})_{\boldsymbol{N}}$ is a d-net in $c_0^{(d)}(R)$ converging to $\bar{a} \in c_0^{(d)}(R)$ since for all $\boldsymbol{N} \in \mathbb{N}^d$,

$$\|\bar{a}^{(\boldsymbol{N})} - \bar{a}\|_u = \sup\{\|a_{\boldsymbol{n}}\|_R : \boldsymbol{n} \not\preccurlyeq_d \boldsymbol{N}\}.$$

By continuity, we have

$$J(\bar{a}) = \lim_{\boldsymbol{N}} J((\bar{a}^{(\boldsymbol{N})})_{\boldsymbol{N}}) = \lim_{\boldsymbol{N}} \sum_{\boldsymbol{n} \preccurlyeq_d \boldsymbol{N}} a_{\boldsymbol{n}} b_{\boldsymbol{n}} = \sum_{\boldsymbol{n} \in \mathbb{N}^d} a_{\boldsymbol{n}} b_{\boldsymbol{n}}.$$

For the converse, let $\bar{b} \in \ell^\infty_{(d)}(R)$. For any $\bar{a} \in c_0^{(d)}(R)$, we have that $(a_{\boldsymbol{n}} b_{\boldsymbol{n}})_{\boldsymbol{n}}$ converges to 0, so the series

$$\sum_{\boldsymbol{n} \in \mathbb{N}^d} a_{\boldsymbol{n}} b_{\boldsymbol{n}}$$

converges to a well-defined element of R. Thus, $J_{\bar{b}}$ is a well-defined map $c_0^{(d)}(R) \to R$. $J_{\bar{b}}$ is clearly R-linear. For continuity, note that if $\bar{a}, \bar{c} \in c_0^{(d)}(R)$, then

$$\begin{aligned}
\|J_{\bar{b}}(\bar{a}) - J_{\bar{b}}(\bar{c})\|_R &= \|J_{\bar{b}}(\bar{a} - \bar{c})\|_R \\
&= \left\| \sum_{\boldsymbol{n} \in \mathbb{N}^d} (a_{\boldsymbol{n}} - c_{\boldsymbol{n}}) b_{\boldsymbol{n}} \right\|_R \\
&\leq \sup_{\boldsymbol{n} \in \mathbb{N}^d} \{\|(a_{\boldsymbol{n}} - c_{\boldsymbol{n}}) b_{\boldsymbol{n}}\|_R\} \\
&\leq \|\bar{a} - \bar{c}\|_u \|\bar{b}\|_u.
\end{aligned}$$

□

By combining all that we have done so far, we have the following.

Corollary 5. *There is a bijection between R-valued measures on $\mathbb{Z}_p^d$ and bounded d-nets in R given by*

$$\alpha \leftrightarrow \left(\int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{n}} \, d\alpha(\boldsymbol{x}) \right)_{\boldsymbol{n}}.$$

Definition 12. *Let $\alpha \in \mathcal{M}_d(R)$ and $\boldsymbol{n} \in \mathbb{N}^d$. The $\boldsymbol{n}$th Mahler moment of α is*

$$m_{\boldsymbol{n}}(\alpha) = \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{n}} \, d\alpha(\boldsymbol{x}) \in R.$$

In light of Corollary 5, a measure α is uniquely determined by its (bounded) d-net of Mahler moments.

Definition 13. *Let $\alpha \in \mathcal{M}_d(R)$ and $\boldsymbol{n} \in \mathbb{N}^d$. The $\boldsymbol{n}$th moment of α is*

$$M_{\boldsymbol{n}}(\alpha) = \int_{\mathbb{Z}_p^d} \boldsymbol{x}^{\boldsymbol{n}} \, d\alpha(\boldsymbol{x}) \in R.$$

A measure α is uniquely determined by its bounded d-net of moments (see the identities in Appendix C).

Example 4. *Fix $\boldsymbol{s} \in \mathbb{Z}_p^d$ and let $\delta_{\boldsymbol{s}}$ be Dirac measure of mass 1 centered at $\boldsymbol{s}$ (see Example 3). Let $I_{\boldsymbol{s}}$ be the associated map $C(\mathbb{Z}_p^d, R) \to R$, $J_{\boldsymbol{s}}$ the associated map $c_0^{(d)}(R) \to R$, and $\bar{b}_{\boldsymbol{s}}$ the associated bounded d-net in R. If $f : \mathbb{Z}_p^d \to R$ is locally constant, say with*

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}),$$

then

$$\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\delta_{\boldsymbol{s}}(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f(\boldsymbol{a}) \delta_{\boldsymbol{s}}(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d) = f(\boldsymbol{a}_{\boldsymbol{s}})$$

where $\boldsymbol{a}_{\boldsymbol{s}} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}$ is the unique element with $\boldsymbol{a}_{\boldsymbol{s}} \equiv \boldsymbol{s} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d}$. Since f is locally constant, we have

$$I_{\boldsymbol{s}}(f) = \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\delta_{\boldsymbol{s}}(\boldsymbol{x}) = f(\boldsymbol{s}).$$

If now $f \in C(\mathbb{Z}_p^d, R)$, then f is the uniform limit of locally constant functions, and we have

$$I_{\boldsymbol{s}}(f) = \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\delta_{\boldsymbol{s}}(\boldsymbol{x}) = f(\boldsymbol{s}),$$

i.e., $I_{\boldsymbol{s}} : C(\mathbb{Z}_p^d, R) \to R$ is simply evaluation at $\boldsymbol{s}$. From the proof of Proposition 4, we have that if $\bar{b}_{\boldsymbol{s}} = (b_{\boldsymbol{n}}(\boldsymbol{s}))_{\boldsymbol{n}}$, then

$$b_{\boldsymbol{n}}(\boldsymbol{s}) = \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{n}} d\delta_{\boldsymbol{s}}(\boldsymbol{x}) = \binom{\boldsymbol{s}}{\boldsymbol{n}}.$$

Of course, $\bar{b}_{\boldsymbol{s}}$ is simply the d-net of Mahler moments of $\delta_{\boldsymbol{s}}$:

$$\bar{b}_s = (m_{\boldsymbol{n}}(\delta_{\boldsymbol{s}}))_{\boldsymbol{n}}.$$

Thus, for $\bar{a} \in c_0^{(d)}(R)$,

$$J_{\boldsymbol{s}}(\bar{a}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} a_{\boldsymbol{n}} \binom{\boldsymbol{s}}{\boldsymbol{n}}.$$

For $f \in C(\mathbb{Z}_p^d, R)$, we have

$$I_{\boldsymbol{s}}(f) = f(\boldsymbol{s}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} a_{\boldsymbol{n}}(f) \binom{\boldsymbol{s}}{\boldsymbol{n}} = \sum_{\boldsymbol{n} \in \mathbb{N}^d} a_{\boldsymbol{n}} m_{\boldsymbol{n}}(\delta_{\boldsymbol{s}}) = J_{\boldsymbol{s}}(a_{\boldsymbol{n}}(f)).$$

$\Diamond$

Finally, let $R \subseteq S$, where S is a complete normed ring whose norm extends the norm on R, so that S is an R-algebra. Then the canonical inclusion $R \to S$ is R-linear and continuous, and there are natural inclusions

$$\mathcal{M}_d(R) \subseteq \mathcal{M}_d(S)$$
$$C(\mathbb{Z}_p^d, R) \subseteq C(\mathbb{Z}_p^d, S)$$
$$c_0^{(d)}(R) \subseteq c_0^{(d)}(S)$$
$$\ell^\infty_{(d)}(R) \subseteq \ell^\infty_{(d)}(S)$$

all of which are R-linear and continuous. For $\alpha \in \mathcal{M}_d(S)$, we have by definition that $\alpha \in \mathcal{M}_d(R)$ if and only if the image of α is contained in R. In terms of what we have done above, this translates into the following equivalent characterizations.

Corollary 6. *Let $\alpha \in \mathcal{M}_d(S)$. Then α lies in $\mathcal{M}_d(R)$ if and only if the continuous S-linear maps*

$$I_\alpha : C(\mathbb{Z}_p^d, R) \to S$$
$$J_\alpha : c_0^{(d)}(R) \to S$$

actually have their images in R. Equivalently, α lies in $\mathcal{M}_d(R)$ if and only if its bounded d-net of Mahler moments and its bounded d-net of moments are elements of $\ell^\infty_{(d)}(R)$. $\square$

2.1.1 Addition and Convolution of Measures

As above, let $\mathrm{Hom}_{\mathrm{cont}}(C(\mathbb{Z}_p^d, R), R)$ denote the R-algebra of continuous (bounded) R-linear maps $C(\mathbb{Z}_p^d, R) \to R$ and $\mathrm{Hom}_{\mathrm{cont}}(c_0^{(d)}(R), R)$ denote the R-algebra of continuous (bounded) R-linear maps $c_0^{(d)}(R) \to R$.

Recall for $\alpha, \beta \in \mathcal{M}_d$, $\alpha + \beta \in \mathcal{M}_d$ is the pointwise sum of α and β as functions $CO_d \to R$. By definition, if $f \in C(\mathbb{Z}_p^d, R)$ is the characteristic function of a polyball A in $\mathbb{Z}_p^d$, then

$$\begin{aligned} I_{\alpha+\beta}(f) &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d(\alpha+\beta)(\boldsymbol{x}) \\ &= (\alpha+\beta)(A) \\ &= \alpha(A) + \beta(A) \\ &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) + \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\beta(\boldsymbol{x}) \\ &= I_\alpha(f) + I_\beta(f). \end{aligned}$$

By linearity of the maps $I_{\alpha+\beta}$, I_α, and I_β, we have

$$I_{\alpha+\beta}(f) = I_\alpha(f) + I_\beta(f)$$

for all $f \in LC(\mathbb{Z}_p^d, R)$. By continuity of these maps, we have

$$I_{\alpha+\beta}(f) = I_\alpha(f) + I_\beta(f)$$

for all $f \in C(\mathbb{Z}_p^d, R)$. Thus, the bijection $\mathcal{M}_d \to \mathrm{Hom}_{\mathrm{cont}}(C(\mathbb{Z}_p^d, R), R)$ given by $\alpha \mapsto I_\alpha$ is an additive map.

Recall from Corollary 4 that for $\alpha \in \mathcal{M}_d$, the map $J_\alpha \in \mathrm{Hom}_{\mathrm{cont}}(c_0^{(d)}(R), R)$ is defined by

$$J_\alpha(\bar{a}) = I_\alpha(f_{\bar{a}}).$$

Consequently, the bijection $\mathcal{M}_d \to \mathrm{Hom}_{\mathrm{cont}}(c_0^{(d)}(R), R)$ given by $\alpha \mapsto J_\alpha$ is also additive.

Finally, the bijection $\mathcal{M}_d \to \ell^\infty_{(d)}(R)$ given by $\alpha \mapsto (m_{\boldsymbol{n}}(\alpha))_{\boldsymbol{n}}$ is also additive, as

$$\begin{aligned} m_{\boldsymbol{n}}(\alpha+\beta) &= \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{n}}\, d(\alpha+\beta)(\boldsymbol{x}) \\ &= \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{n}}\, d\alpha(\boldsymbol{x}) + \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{n}}\, d\beta(\boldsymbol{x}) \\ &= m_{\boldsymbol{n}}(\alpha) + m_{\boldsymbol{n}}(\beta) \end{aligned}$$

for each $\boldsymbol{n} \in \mathbb{N}^d$.

Similarly, for $\alpha \in \mathcal{M}_d$, and $c \in R$, $c\alpha \in \mathcal{M}_d(R)$ is the pointwise scalar product of c and α. As for additivity above, we find that each of the maps

$$\begin{aligned} &\mathcal{M}_d \to \mathrm{Hom}_{\mathrm{cont}}(C(\mathbb{Z}_p^d, R), R) \\ &\mathcal{M}_d \to \mathrm{Hom}_{\mathrm{cont}}(c_0^{(d)}(R), R) \\ &\mathcal{M}_d \to \ell^\infty_{(d)}(R) \end{aligned}$$

are R-module isomorphisms.

Definition 14. *Let $\alpha, \beta \in \mathcal{M}_d$. Define $\alpha * \beta$ by*

$$\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d(\alpha*\beta)(\boldsymbol{x}) = \int_{\mathbb{Z}_p^d} \left(\int_{\mathbb{Z}_p^d} f(\boldsymbol{x}+\boldsymbol{y})\, d\alpha(\boldsymbol{x}) \right) d\beta(\boldsymbol{y})$$

for $f \in C(\mathbb{Z}_p^d, R)$. $\alpha * \beta$ is called the convolution of the measures α and β.

Proposition 5. *For each $\alpha, \beta \in \mathcal{M}_d$, $\alpha * \beta \in \mathcal{M}_d$ and $\|\alpha*\beta\|_u \le \|\alpha\|_u \|\beta\|_u$.*

Proof. Let $f \in C(\mathbb{Z}_p^d, R)$; then f is uniformly continuous since $\mathbb{Z}_p^d$ is compact. For $\boldsymbol{y} \in \mathbb{Z}_p^d$, recall that translation by $\boldsymbol{y}$ is an isometric isomorphism $\mathbb{Z}_p^d \to \mathbb{Z}_p^d$. We define $f_{\boldsymbol{y}} \in C(\mathbb{Z}_p^d, R)$ by

$$f_{\boldsymbol{y}}(\boldsymbol{x}) = f(\boldsymbol{x}+\boldsymbol{y}).$$

We first show that the function $g_f : \boldsymbol{y} \mapsto I_\alpha(f_{\boldsymbol{y}})$ is an element of $C(\mathbb{Z}_p^d, R)$. Fix $\boldsymbol{y}_1 \in \mathbb{Z}_p^d$ and let $\varepsilon > 0$. Choose $\delta > 0$ such that $\|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_d < \delta$ implies

$$\|f(\boldsymbol{z_1}) - f(\boldsymbol{z}_2)\|_R < \frac{\varepsilon}{1 + \|\alpha\|_u}.$$

Let $\boldsymbol{y}_2 \in \mathbb{Z}_p^d$ with $\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|_d < \delta$; then also $\|(\boldsymbol{x} + \boldsymbol{y}_1) - (\boldsymbol{x} + \boldsymbol{y}_2)\|_d < \delta$ for all $\boldsymbol{x} \in \mathbb{Z}_p^d$. By the choice of δ,

$$\sup\{\|f(\boldsymbol{x} + \boldsymbol{y}_1) - f(\boldsymbol{x} + \boldsymbol{y}_2)\|_R : \boldsymbol{x} \in \mathbb{Z}_p^d\} \leq \frac{\varepsilon}{1 + \|\alpha\|_u}.$$

But now by Proposition 2,

$$\begin{aligned}
\|I_\alpha(f_{\boldsymbol{y}_1}) - I_\alpha(f_{\boldsymbol{y}_2})\|_R &= \|I_\alpha(f_{\boldsymbol{y}_1} - f_{\boldsymbol{y}_2})\|_R \\
&\leq \|\alpha\|_u \|f_{\boldsymbol{y}_1} - f_{\boldsymbol{y}_2}\|_\infty \\
&= \|\alpha\|_u \sup\{\|f_{\boldsymbol{y}_1}(\boldsymbol{x}) - f_{\boldsymbol{y}_2}(\boldsymbol{x})\|_R : \boldsymbol{x} \in \mathbb{Z}_p^d\} \\
&= \|\alpha\|_u \sup\{\|f(\boldsymbol{x} + \boldsymbol{y}_1) - f(\boldsymbol{x} + \boldsymbol{y}_2)\|_R : \boldsymbol{x} \in \mathbb{Z}_p^d\} \\
&< \varepsilon.
\end{aligned}$$

This gives continuity of the function $g_f : \boldsymbol{y} \mapsto I_\alpha(f_{\boldsymbol{y}})$, and therefore

$$\int_{\mathbb{Z}_p^d} \left(\int_{\mathbb{Z}_p^d} f(\boldsymbol{x} + \boldsymbol{y}) \, d\alpha(\boldsymbol{x}) \right) d\beta(\boldsymbol{y}) = \int_{\mathbb{Z}_p^d} g_f(\boldsymbol{y}) \, d\beta(\boldsymbol{y})$$

is defined. Since

$$\begin{aligned}
(f_1 + f_2)_{\boldsymbol{y}} &= (f_1)_{\boldsymbol{y}} + (f_2)_{\boldsymbol{y}} \\
(cf_1)_{\boldsymbol{y}} &= c(f_1)_{\boldsymbol{y}}
\end{aligned}$$

for all $f_1, f_2 \in C(\mathbb{Z}_p^d, R)$ and $c \in R$, the map $C(\mathbb{Z}_p^d, R) \to R$ given by

$$f \mapsto \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) \, d(\alpha * \beta)(\boldsymbol{x}) = \int_{\mathbb{Z}_p^d} g_f(\boldsymbol{y}) \, d\beta(\boldsymbol{y})$$

is R-linear. Now let $f \in C(\mathbb{Z}_p^d, R)$. For each $\boldsymbol{y} \in \mathbb{Z}_p^d$, we have

$$\|f\|_\infty = \|f_{\boldsymbol{y}}\|_\infty.$$

Thus, for each $\boldsymbol{y} \in \mathbb{Z}_p^d$,

$$\|g_f(\boldsymbol{y})\|_R = \|I_\alpha(f_{\boldsymbol{y}})\|_R \leq \|\alpha\|_u \|f_{\boldsymbol{y}}\|_\infty = \|\alpha\|_u \|f\|_\infty.$$

This last fact gives

$$\|g_f(\boldsymbol{y})\|_\infty = \sup\{\|g_f(\boldsymbol{y})\|_R : \boldsymbol{y} \in \mathbb{Z}_p^d\} \leq \|\alpha\|_u \|f\|_\infty.$$

We have

$$\begin{aligned} \left\| \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d(\alpha * \beta)(\boldsymbol{x}) \right\|_R &= \left\| \int_{\mathbb{Z}_p^d} g_f(\boldsymbol{y})\, d\beta(\boldsymbol{y}) \right\|_R \\ &\leq \|\beta\|_u \, \|g_f(\boldsymbol{y})\|_\infty \\ &\leq \|\alpha\|_u \|\beta\|_u \|f\|_\infty. \end{aligned}$$

By Proposition 2, $\alpha * \beta \in \mathcal{M}_d$, as claimed, and we have seen that $\|\alpha * \beta\|_u \leq \|\alpha\|_u \|\beta\|_u$. □

Next, we wish to describe the values of $\alpha * \beta$ on polyballs in $\mathbb{Z}_p^d$. To that end, consider the polyball $\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d$ and its characteristic function $g_{\boldsymbol{a},\boldsymbol{n}}$. Then

$$\begin{aligned} (\alpha * \beta)(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d) &= \int_{\mathbb{Z}_p^d} g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x})\, d(\alpha * \beta)(\boldsymbol{x}) \\ &= \int_{\mathbb{Z}_p^d} \left(\int_{\mathbb{Z}_p^d} g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x} + \boldsymbol{y})\, d\alpha(\boldsymbol{x}) \right) d\beta(\boldsymbol{y}). \end{aligned}$$

Now for a fixed $\boldsymbol{y} \in \mathbb{Z}_p^d$, we have

$$\begin{aligned} g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x} + \boldsymbol{y}) &= \begin{cases} 1 & : \boldsymbol{x} - \boldsymbol{y} \in \boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d \\ 0 & : \text{else} \end{cases} \\ &= \begin{cases} 1 & : \boldsymbol{x} \in -\boldsymbol{y} + (\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d) \\ 0 & : \text{else} \end{cases}, \end{aligned}$$

where

$$-\boldsymbol{y} + (\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d) = \{-\boldsymbol{y} + \boldsymbol{z} : \boldsymbol{z} \in \boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d\}$$

is the translate of $\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d$ by $-\boldsymbol{y}$. Translation by $-\boldsymbol{y}$ is an isometric isomorphism $\mathbb{Z}_p^d \to \mathbb{Z}_p^d$, so $-\boldsymbol{y} + (\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d) = (\boldsymbol{a} - \boldsymbol{y}) + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d$ is also a polyball in $L_{\boldsymbol{n}}$. We obtain

$$\int_{\mathbb{Z}_p^d} g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x} + \boldsymbol{y})\, d\alpha(\boldsymbol{x}) = \alpha(-\boldsymbol{y} + (\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d)).$$

As a function of $\boldsymbol{y}$, $\alpha(-\boldsymbol{y} + (\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d))$ is locally constant:

$$\alpha(-\boldsymbol{y} + (\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d)) = \sum_{\boldsymbol{b} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \alpha(-\boldsymbol{b} + (\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d)) g_{\boldsymbol{b},\boldsymbol{n}}(\boldsymbol{y}).$$

This last fact gives

$$\begin{aligned}
(\alpha * \beta)(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d) &= \int_{\mathbb{Z}_p^d} \left(\int_{\mathbb{Z}_p^d} g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x} + \boldsymbol{y})\, d\alpha(\boldsymbol{x}) \right) d\beta(\boldsymbol{y}) \\
&= \sum_{\boldsymbol{b} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \alpha(-\boldsymbol{b} + (\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d)) \beta(\boldsymbol{b} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d) \\
&= \sum_{\boldsymbol{b} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \alpha((\boldsymbol{a} - \boldsymbol{b}) + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d) \beta(\boldsymbol{b} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d).
\end{aligned}$$

Example 5. *Let* $\boldsymbol{s} \in \mathbb{Z}_p^d$ *and let* $\alpha \in \mathcal{M}_d$. *For any polyball* $\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d$,

$$\begin{aligned}
(\alpha * \delta_{\boldsymbol{s}})(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d) &= \sum_{\boldsymbol{b} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \alpha((\boldsymbol{a} - \boldsymbol{b}) + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d) \delta_{\boldsymbol{s}}(\boldsymbol{b} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d) \\
&= \alpha((\boldsymbol{a} - \boldsymbol{b}_{\boldsymbol{s}}) + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d),
\end{aligned}$$

where $\boldsymbol{b}_{\boldsymbol{s}} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}$ *is the unique element with* $\boldsymbol{s} \equiv \boldsymbol{b}_{\boldsymbol{s}} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d}$. *Of course,*

$$(\boldsymbol{a} - \boldsymbol{b}_{\boldsymbol{s}}) + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d = (\boldsymbol{a} - \boldsymbol{s}) + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d,$$

so

$$(\alpha * \delta_{\boldsymbol{s}})(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d) = \alpha((\boldsymbol{a} - \boldsymbol{s}) + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d).$$

Consequently,

$$\int_{\mathbb{Z}_p^d} g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x})\, d(\alpha * \delta_{\boldsymbol{s}})(\boldsymbol{x}) = \int_{\mathbb{Z}_p^d} g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{s}+\boldsymbol{x})\, d\alpha(\boldsymbol{x}).$$

Thus, for all $f \in C(\mathbb{Z}_p^d, R)$*,*

$$\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d(\alpha * \delta_{\boldsymbol{s}})(\boldsymbol{x}) = \int_{\mathbb{Z}_p^d} f(\boldsymbol{s}+\boldsymbol{x})\, d\alpha(\boldsymbol{x})$$

and for all $A \in CO_d$*,*

$$(\alpha * \delta_{\boldsymbol{s}})(A) = \alpha(-\boldsymbol{s} + A).$$

From this computation, we see that for $\boldsymbol{s}, \boldsymbol{t} \in \mathbb{Z}_p^d$*, and any* $f \in C(\mathbb{Z}_p^d, R)$*,*

$$\begin{aligned}\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d(\delta_{\boldsymbol{s}} * \delta_{\boldsymbol{t}})(\boldsymbol{x}) &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}+\boldsymbol{t})\, d\delta_{\boldsymbol{s}}(\boldsymbol{x}) \\ &= f(\boldsymbol{s}+\boldsymbol{t}) \\ &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\delta_{\boldsymbol{s}+\boldsymbol{t}}(\boldsymbol{x}),\end{aligned}$$

whence $\delta_{\boldsymbol{s}} * \delta_{\boldsymbol{t}} = \delta_{\boldsymbol{s}+\boldsymbol{t}}$*.* ◊

Example 6. *Fix* $\boldsymbol{n} \in \mathbb{N}^d$ *and consider the measure*

$$\alpha = \prod_{i=1}^{d} (\delta_{\boldsymbol{e}_i} - \delta_{\boldsymbol{0}})^{n_i},$$

where the product signifies convolution of measures and the exponent n_i *denotes the* n_i*-fold convolution of the measure* $(\delta_{\boldsymbol{e}_i} - \delta_{\boldsymbol{0}})$ *with itself. We will use the final result from Example 5. For each* $1 \le i \le d$*, we have*

$$\begin{aligned}(\delta_{\boldsymbol{e}_i} - \delta_{\boldsymbol{0}})^{n_i} &= \sum_{k_i=0}^{n_i} \binom{n_i}{k_i} (-1)^{n_i-k_i} \delta_{\boldsymbol{e}_i}^{k_i} * \delta_{\boldsymbol{0}}^{n_i-k_i} \\ &= \sum_{k_i=0}^{n_i} \binom{n_i}{k_i} (-1)^{n_i-k_i} \delta_{k_i \boldsymbol{e}_i}.\end{aligned}$$

Consequently,

$$\alpha = \sum_{\boldsymbol{k} \preccurlyeq_d \boldsymbol{n}} (-\mathbf{1})^{\boldsymbol{n}-\boldsymbol{k}} \binom{\boldsymbol{n}}{\boldsymbol{k}} \delta_{\boldsymbol{k}}.$$

Thus, for $f \in C(\mathbb{Z}_p^d, R)$, *we have*

$$\begin{aligned} \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) &= \sum_{\boldsymbol{k} \preccurlyeq_d \boldsymbol{n}} (-\mathbf{1})^{\boldsymbol{n}-\boldsymbol{k}} \binom{\boldsymbol{n}}{\boldsymbol{k}} \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\delta_{\boldsymbol{k}} \\ &= \sum_{\boldsymbol{k} \preccurlyeq_d \boldsymbol{n}} (-\mathbf{1})^{\boldsymbol{n}-\boldsymbol{k}} \binom{\boldsymbol{n}}{\boldsymbol{k}} f(\boldsymbol{k}) \\ &= m_{\boldsymbol{n}}(f). \end{aligned}$$

$\Diamond$

Example 7. *Let* $\alpha \in \mathcal{M}_d$ *and* $c \in R$. *Consider the measure* $c\delta_{\mathbf{0}}$. *Following along as in Example 5, we find*

$$(\alpha * c\delta_{\mathbf{0}}) = c\alpha(\mathbf{0} + A) = c\alpha(A)$$

for all $A \in CO_d$. *Consequently,* $c\alpha = \alpha * c\delta_{\mathbf{0}}$. $\Diamond$

Finally, we determine the bounded d-net of Mahler moments of the measure $\alpha * \beta$ in terms of the d-nets of Mahler moments of α and β by using the identities from Appendix C.

Let $\boldsymbol{n} \in \mathbb{N}^d$. The $\boldsymbol{n}$th Mahler moment of $\alpha * \beta$ is

$$\begin{aligned}
m_{\boldsymbol{n}}(\alpha * \beta) &= \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{n}} \, d(\alpha * \beta)(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} \left(\int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}+\boldsymbol{y}}{\boldsymbol{n}} \, d\alpha(\boldsymbol{x}) \right) d\beta(\boldsymbol{y}) \\
&= \int_{\mathbb{Z}_p^d} \left(\int_{\mathbb{Z}_p^d} \sum_{\boldsymbol{m} \preccurlyeq_d \boldsymbol{n}} \binom{\boldsymbol{x}}{\boldsymbol{m}} \binom{\boldsymbol{y}}{\boldsymbol{n}-\boldsymbol{m}} \, d\alpha(\boldsymbol{x}) \right) d\beta(\boldsymbol{y}) \\
&= \sum_{\boldsymbol{m} \preccurlyeq_d \boldsymbol{n}} \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{y}}{\boldsymbol{n}-\boldsymbol{m}} \left(\int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{m}} \, d\alpha(\boldsymbol{x}) \right) d\beta(\boldsymbol{y}) \\
&= \sum_{\boldsymbol{m} \preccurlyeq_d \boldsymbol{n}} \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{y}}{\boldsymbol{n}-\boldsymbol{m}} m_{\boldsymbol{m}}(\alpha) \, d\beta(\boldsymbol{y}) \\
&= \sum_{\boldsymbol{m} \preccurlyeq_d \boldsymbol{n}} m_{\boldsymbol{m}}(\alpha) \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{y}}{\boldsymbol{n}-\boldsymbol{m}} \, d\beta(\boldsymbol{y}) \\
&= \sum_{\boldsymbol{m} \preccurlyeq_d \boldsymbol{n}} m_{\boldsymbol{m}}(\alpha) m_{\boldsymbol{n}-\boldsymbol{m}}(\beta).
\end{aligned}$$

Thus, we have the following equality of d-nets of Mahler moments in $\ell^\infty_{(d)}(R)$:

$$(m_{\boldsymbol{n}}(\alpha * \beta))_{\boldsymbol{n}} = (m_{\boldsymbol{n}}(\alpha))_{\boldsymbol{n}} (m_{\boldsymbol{n}}(\beta))_{\boldsymbol{n}}.$$

Since $\ell^\infty_{(d)}(R)$ is an R-algebra under Cauchy multiplication, the next corollary reveals the full algebraic structure of $\mathcal{M}_d$.

Corollary 7. *The R-module isomorphism $\mathcal{M}_d \to \ell^\infty_{(d)}(R)$ given by $\alpha \mapsto (m_{\boldsymbol{n}}(\alpha))$ is such that*

$$(m_{\boldsymbol{n}}(\alpha * \beta))_{\boldsymbol{n}} = (m_{\boldsymbol{n}}(\alpha))_{\boldsymbol{n}} (m_{\boldsymbol{n}}(\beta))_{\boldsymbol{n}}.$$

Consequently, $\mathcal{M}_d$ is a ring under convolution of measures.

Corollary 8. *Let $R \subseteq S$, where S is a complete normed ring whose norm extends the norm on R. The inclusion $\mathcal{M}_d(R) \subseteq \mathcal{M}_d(S)$ is an R-algebra homomorphism.*

2.2 The Algebraic Viewpoint

In this section, we describe an interpretation of elements of $\mathcal{M}_d$ as elements of a profinite completed group ring. See Coates and Sujatha (2006) or the beginning of Chapter 12 of Washington (1997) for the $d = 1$ case.

Definition 15. *Let G be any profinite group. The profinite completed group ring of G over R is the ring*

$$R[[G]] := \lim_{\leftarrow} R[G/H]$$

where $R[G/H]$ is the group ring of G/H over R, and where the inverse limit is taken over the open normal subgroups H of G of finite index, with the maps on the group rings arising from the natural projections $G/H_1 \to G/H_2$ when $H_1 \subseteq H_2$.

When $G = \mathbb{Z}_p^d$, the open normal subgroups of finite index in G are of the form $H = \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d$, where $\boldsymbol{n} \in \mathbb{N}^d$. If we put $H_{\boldsymbol{n}} = \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d$, then $H_{\boldsymbol{b}} \subseteq H_{\boldsymbol{a}}$ precisely when $\boldsymbol{a} \preccurlyeq_d \boldsymbol{b}$. We can then view the inverse limit defining $R[[\mathbb{Z}_p^d]]$ as indexed by $\mathbb{N}^d$ under the product order, and it will be beneficial to adopt this viewpoint.

Let $\boldsymbol{n} \in \mathbb{N}^d$. The group ring $R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d]$ is the free R-algebra with basis consisting of elements of $\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d$. Thus, an element $\lambda_{\boldsymbol{n}} \in R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d]$ may be written in the form

$$\lambda_{\boldsymbol{n}} = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \lambda_{\boldsymbol{n}}(\boldsymbol{a})(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d),$$

where the $\lambda_{\boldsymbol{n}}(\boldsymbol{a}) \in R$ are uniquely determined (i.e., are independent of the choice of representative of $\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d$). We endow $R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d]$ with the maximum norm:

$$\|\lambda_{\boldsymbol{n}}\|_u = \max\{\|\lambda_{\boldsymbol{n}}(\boldsymbol{a})\|_R : \boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}\}.$$

Suppose that $\boldsymbol{n} \preccurlyeq_d \boldsymbol{m}$ and that $\lambda_{\boldsymbol{n}} \in R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d]$ is the image of $\lambda_{\boldsymbol{m}} \in R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge \boldsymbol{m}}\mathbb{Z}_p^d]$ under the map arising from the projection $\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge \boldsymbol{m}}\mathbb{Z}_p^d \to \mathbb{Z}_p^d/\boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d$.

Then for each $\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}$,

$$\lambda_{\boldsymbol{n}}(\boldsymbol{a}) = \sum_{\substack{\boldsymbol{b} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1} \\ \boldsymbol{b} \equiv \boldsymbol{a} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d}}} \lambda_{\boldsymbol{m}}(\boldsymbol{b}).$$

Since the norm on R is non-archimedean, this gives $\|\lambda_{\boldsymbol{n}}\|_u \leq \|\lambda_{\boldsymbol{m}}\|_u$, so the natural map $R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge \boldsymbol{m}}\mathbb{Z}_p^d] \to R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d]$ is continuous with respect to the maximum norm.

Definition 16. *Let $\lambda \in R[[\mathbb{Z}_p^d]]$ and let $\lambda_{\boldsymbol{n}}$ be the image of λ under the natural map $R[[\mathbb{Z}_p^d]] \to R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d]$. Define*

$$\|\lambda\|_u = \sup\{\|\lambda_{\boldsymbol{n}}\|_u : \boldsymbol{n} \in \mathbb{N}^d\} \in [0, \infty].$$

λ is bounded if $\|\lambda\|_u < \infty$.

Given elements $\lambda = (\lambda_{\boldsymbol{n}})$ and $\nu = (\nu_{\boldsymbol{n}})$ in $R[[\mathbb{Z}_p^d]]$, we have that $\lambda\nu = (\lambda_{\boldsymbol{n}}\nu_{\boldsymbol{n}})$ and $\lambda + \nu = (\lambda_{\boldsymbol{n}} + \nu_{\boldsymbol{n}})$, with the addition and multiplication of $\lambda_{\boldsymbol{n}}$ and $\nu_{\boldsymbol{n}}$ occurring in the group ring $R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d]$ for each $\boldsymbol{n} \in \mathbb{N}^d$. Since the norm on R is non-archimedean, we have

$$\|\lambda + \nu\|_u \leq \max\{\|\lambda\|_u, \|\nu\|_u\}$$
$$\|\lambda\nu\|_u \leq \|\lambda\|_u \|\nu\|_u.$$

Consequently, the bounded elements of $R[[\mathbb{Z}_p^d]]$ form a subring of $R[[\mathbb{Z}_p^d]]$.

Definition 17. *We put*

$$\Lambda_{(d)} = \Lambda_{(d)}(R) := \{\lambda \in R[[\mathbb{Z}_p^d]] : \|\lambda\|_u < \infty\}.$$

We begin by showing that elements of $\Lambda_{(d)}$ can be viewed as R-valued measures on $\mathbb{Z}_p^d$, and we obtain a ring isomorphism between $\mathcal{M}_d$ and $\Lambda_{(d)}$. To do so, we associate to each $\lambda \in \Lambda_{(d)}$ a continuous R-linear map $C(\mathbb{Z}_P^d, R) \to R$.

Fix an element $\lambda = (\lambda_{\boldsymbol{n}}) \in \Lambda_{(d)}$. We begin by defining an associated continuous R-linear map $LC(\mathbb{Z}_p^d, R) \to R$. Toward that end, suppose $f \in LC(\mathbb{Z}_p^d, R)$. Then there is $\boldsymbol{n} \in \mathbb{N}^d$ such that

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}).$$

Write

$$\lambda_{\boldsymbol{n}} = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \lambda_{\boldsymbol{n}}(\boldsymbol{a})(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d).$$

We define

$$I_\lambda(f) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \lambda_{\boldsymbol{n}}(\boldsymbol{a}) f(\boldsymbol{a}).$$

Note that because f is locally constant, the value $f(\boldsymbol{a})$ does not depend on the choice of coset representative; thus $I_\lambda(f)$ is independent of the choice of coset representatives for $\mathbb{Z}_p^d / \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d$.

As we have seen, if $\boldsymbol{n} \preccurlyeq_d \boldsymbol{m}$, then for each $\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}$,

$$\lambda_{\boldsymbol{n}}(\boldsymbol{a}) = \sum_{\substack{\boldsymbol{b} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1} \\ \boldsymbol{b} \equiv \boldsymbol{a} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d}}} \lambda_{\boldsymbol{m}}(\boldsymbol{b}).$$

We may also write

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{b} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1}} f(\boldsymbol{b}) g_{\boldsymbol{b},\boldsymbol{m}}(\boldsymbol{x}),$$

where for each $\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}$, $f(\boldsymbol{a}) = f(\boldsymbol{b})$ if $\boldsymbol{b} \equiv \boldsymbol{a} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d}$. We find

$$\sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \lambda_{\boldsymbol{n}}(\boldsymbol{a}) f(\boldsymbol{a}) = \sum_{\boldsymbol{b} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1}} \lambda_{\boldsymbol{m}}(\boldsymbol{b}) f(\boldsymbol{b}).$$

Thus, $I_\lambda : LC(\mathbb{Z}_p^d, R) \to R$ is well-defined.

Proposition 6. *Fix $\lambda \in \Lambda_d$. Then the map $I_\lambda : LC(\mathbb{Z}_p^d, R) \to R$ is a continuous R-linear map.*

Proof. Let $f_1, f_2 \in LC(\mathbb{Z}_p^d, R)$. There exists $\boldsymbol{n} \in \mathbb{N}^d$ such that

$$f_1(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\boldsymbol{n}} - \boldsymbol{1}} f_1(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x})$$

$$f_2(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\boldsymbol{n}} - \boldsymbol{1}} f_2(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}).$$

As before, $f_1 + f_2$ is locally constant with

$$(f_1 + f_2)(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\boldsymbol{n}} - \boldsymbol{1}} (f_1 + f_2)(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}),$$

and for $a \in R$, af_1 is locally constant with

$$(af_1)(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\boldsymbol{n}} - \boldsymbol{1}} af_1(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}) = a \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\boldsymbol{n}} - \boldsymbol{1}} f_1(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}).$$

Then clearly

$$I_\lambda(f_1 + f_2) = I_\lambda(f_1) + I_\lambda(f_2)$$

and

$$I_\lambda(af_1) = aI_\lambda(f_1).$$

This gives the R-linearity of I_λ on $LC(\mathbb{Z}_p^d, R)$.

For continuity, let $f_1, f_2 \in LC(\mathbb{Z}_p^d, R)$ and choose $\boldsymbol{n} \in \mathbb{N}^d$ as above. Then we may write

$$(f_1 - f_2)(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} (f_1 - f_2)(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}).$$

Since I_λ is R-linear, we have

$$\begin{aligned} \|I_\lambda(f_1) - I_\lambda(f_2)\| &= \|I_\lambda(f_1 - f_2)\|_R \\ &= \left\| \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \lambda_{\boldsymbol{n}}(\boldsymbol{a})(f_1 - f_2)(\boldsymbol{a}) \right\|_R \\ &\leq \max_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \{\|\lambda_{\boldsymbol{n}}(\boldsymbol{a})\|_R \|(f_1 - f_2)(\boldsymbol{a})\|_R\} \\ &\leq \|\lambda\|_u \|f_1 - f_2\|_\infty. \end{aligned}$$

Thus I_λ is continuous on $LC(\mathbb{Z}_p^d, R)$. □

Suppose $f \in C(\mathbb{Z}_p^d, R)$ with $f = \lim_{\boldsymbol{n}} f_{\boldsymbol{n}}$, where the locally constant function $f_{\boldsymbol{n}}$ factors through $L_{\boldsymbol{n}}$. For each $\boldsymbol{n}$ and $\boldsymbol{m}$ with $\boldsymbol{n} \preccurlyeq_d \boldsymbol{m}$, the previous result gives

$$\|I_\lambda(f_{\boldsymbol{m}}) - I_\lambda(f_{\boldsymbol{n}})\|_R \leq \|\lambda\|_u \|f_{\boldsymbol{m}} - f_{\boldsymbol{n}}\|_\infty.$$

Thus, the d-net of values $(I_\lambda(f_{\boldsymbol{n}}))_{\boldsymbol{n}}$ is a Cauchy net in R, so converges to an element of R.

In the event that $\lim f_{\boldsymbol{n}} = \lim g_{\boldsymbol{n}}$ for d-nets $(f_{\boldsymbol{n}})$ and $(g_{\boldsymbol{n}})$ of locally constant functions with the $\boldsymbol{n}$th term factoring through $L_{\boldsymbol{n}}$, then $(f_{\boldsymbol{n}} - g_{\boldsymbol{n}})$ is also a d-net of locally constant functions with

$$f_{\boldsymbol{n}} - g_{\boldsymbol{n}} \to 0 \in LC(\mathbb{Z}_p^d, R).$$

In this case, the continuity and linearity of I_λ on $LC(\mathbb{Z}_p^d, R)$ gives

$$I_\lambda(f_{\boldsymbol{n}}) - I_\lambda(g_{\boldsymbol{n}}) = I_\lambda(f_{\boldsymbol{n}} - g_{\boldsymbol{n}}) \to I_\lambda(0) = 0.$$

In light of the above considerations, for $f \in C(\mathbb{Z}_p^d, R)$ with $f = \lim f_{\boldsymbol{n}}$, where $f_{\boldsymbol{n}}$ is locally constant factoring through $L_{\boldsymbol{n}}$, we may put

$$I_\lambda(f) := \lim_{\boldsymbol{n}} I_\lambda(f_{\boldsymbol{n}}).$$

Then I_λ is well-defined on the set of continuous functions $\mathbb{Z}_p^d \to R$. We have extended the map I_λ from the dense R-subalgebra $LC(\mathbb{Z}_p^d, R)$ to all of $C(\mathbb{Z}_p^d, R)$ by continuity. We obtain

Proposition 7. *Fix $\lambda \in \Lambda_{(d)}$. The map $I_\lambda : C(\mathbb{Z}_p^d, R) \to R$ is a continuous R-linear map satisfying $\|I_\lambda\|_u = \|\lambda\|_u$. Consequently, $I_\lambda = I_\alpha$ for some $\alpha \in \mathcal{M}_d$.*

Proof. The only thing remaining to prove is $\|I_\lambda\|_u = \|\lambda\|_u$. First, for each $\boldsymbol{n} \in \mathbb{N}^d$ and each $\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}$,

$$\|\lambda_{\boldsymbol{n}}(\boldsymbol{a})\|_R = \|I(g_{\boldsymbol{a},\boldsymbol{n}})\|_R \leq \|I_\lambda\|_u \|g_{\boldsymbol{a},\boldsymbol{n}}\|_\infty = \|I_\lambda\|_u.$$

This gives $\|\lambda\|_u \leq \|I_\lambda\|_u$. If $f \in LC(\mathbb{Z}_p^d, R)$, we saw in the proof of Proposition 6 that

$$\|I_\lambda(f)\|_R \leq \|\lambda\|_u \|f\|_\infty.$$

For $f \in C(\mathbb{Z}_p^d, R)$, we may write f as a uniform limit of a d-net $(f_{\boldsymbol{n}})$ of locally constant functions for which $\|f_{\boldsymbol{n}}\|_\infty \leq \|f\|_\infty$ for all $\boldsymbol{n} \in \mathbb{N}^d$. Then

$$\begin{aligned}\|I_\lambda(f)\|_R &= \|\lim_{\boldsymbol{n}} I_\lambda(f_{\boldsymbol{n}})\|_R \\ &= \lim_{\boldsymbol{n}} \|I_\lambda(f_{\boldsymbol{n}})\|_R \\ &\leq \lim_{\boldsymbol{n}} \|\lambda\|_u \|f_{\boldsymbol{n}}\|_\infty \\ &\leq \|\lambda\|_u \|f\|_\infty.\end{aligned}$$

This gives $\|I_\lambda\|_u \leq \|\lambda\|_u$. □

Proposition 8. *Let $I : C(\mathbb{Z}_p^d, R) \to R$ be a continuous R-linear map. There is $\lambda \in \Lambda_{(d)}$ such that $I = I_\lambda$.*

Proof. For each $\boldsymbol{n} \in \mathbb{N}^d$, let

$$\lambda_{\boldsymbol{n}} = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} I(g_{\boldsymbol{a},\boldsymbol{n}})(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d) \in R[\mathbb{Z}_p^d / \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d].$$

If $\boldsymbol{n} \preccurlyeq_d \boldsymbol{m}$, then for each $\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}$,

$$g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}) = \sum_{\substack{\boldsymbol{b} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1} \\ \boldsymbol{b} \equiv \boldsymbol{a} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d}}} g_{\boldsymbol{b},\boldsymbol{m}}(\boldsymbol{x}).$$

Then by R-linearity,

$$I(g_{\boldsymbol{a},\boldsymbol{n}}) = \sum_{\substack{\boldsymbol{b} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1} \\ \boldsymbol{b} \equiv \boldsymbol{a} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d}}} I(g_{\boldsymbol{b},\boldsymbol{m}}).$$

Therefore, $\lambda_{\boldsymbol{m}} \mapsto \lambda_{\boldsymbol{n}}$ under the natural map $R[\mathbb{Z}_p^d / \boldsymbol{p}^{\wedge \boldsymbol{m}} \mathbb{Z}_p^d] \to R[\mathbb{Z}_p^d / \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d]$ and $\lambda = (\lambda_{\boldsymbol{n}}) \in R[[\mathbb{Z}_p^d]]$ is well-defined. Since $I : C(\mathbb{Z}_p^d, R) \to R$ is continuous, we have

$$\|I(g_{\boldsymbol{a},\boldsymbol{n}})\| \leq \|I\|_u \|g_{\boldsymbol{a},\boldsymbol{n}}\|_\infty = \|I\|_u$$

for each $\boldsymbol{n} \in \mathbb{N}^d$ and each $\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \mathbf{1}$. Consequently, $\|\lambda\|_u \leq \|I\|_u$, and $\lambda \in \Lambda_{(d)}$. It is now clear that I and I_λ agree on $LC(\mathbb{Z}_p^d, R)$, and thus agree on all of $C(\mathbb{Z}_p^d, R)$ by writing any element of $C(\mathbb{Z}_p^d, R)$ as a uniform limit of a net of locally constant functions and employing the continuity of I and I_λ on $LC(\mathbb{Z}_p^d, R)$. ☐

Corollary 9. *There is a bijection between $\mathcal{M}_d$ and $\Lambda_{(d)}$. Under this bijection, a measure $\alpha \in \mathcal{M}_d$ is associated to the element $\lambda = (\lambda_{\boldsymbol{n}}) \in \Lambda_{(d)}$ given by*

$$\lambda_{\boldsymbol{n}} = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}}} \alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d)(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d) \in R[\mathbb{Z}_p^d / \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d].$$

Proposition 9. *The bijection $\mathcal{M}_d \to \Lambda_{(d)}$ is an norm-preserving R-algebra isomorphism.*

Proof. If $\alpha \leftrightarrow \lambda$ under the bijection in Corollary 9, we have $I_\alpha = I_\lambda$. But now Proposition 7 gives

$$\|\alpha\|_u = \|I_\alpha\|_u = \|I_\lambda\|_u = \|\lambda\|_u.$$

Let $c \in R$, $\lambda, \nu \in \Lambda_{(d)}$, and let $\alpha, \beta \in \mathcal{M}_d$ be the measures associated to λ and ν, respectively by Corollary 9. We need only note that for each $\boldsymbol{n} \in \mathbb{N}^d$ and each $\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \mathbf{1}$,

$$\begin{aligned}
(c\lambda)_{\boldsymbol{n}}(\boldsymbol{a}) &= c\lambda_{\boldsymbol{n}}(\boldsymbol{a}) \\
(\lambda + \nu)_{\boldsymbol{n}}(\boldsymbol{a}) &= \lambda_{\boldsymbol{n}}(\boldsymbol{a}) + \nu_{\boldsymbol{n}}(\boldsymbol{a}) \\
(\lambda\nu)_{\boldsymbol{n}}(\boldsymbol{a}) &= \sum_{\boldsymbol{b} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \mathbf{1}} \lambda_{\boldsymbol{n}}(-\boldsymbol{b} + \boldsymbol{a}) \nu_{\boldsymbol{n}}(\boldsymbol{b}).
\end{aligned}$$

From the above equations, we find that for any polyball A with characteristic function f,

$$I_{c\lambda}(f) = cI_\lambda(f) = cI_\alpha(f) = I_{c\alpha}(f) = c\alpha(A)$$

$$I_{\lambda+\nu}(f) = I_\lambda(f) + I_\nu(f) = I_\alpha(f) + I_\beta(f) = I_{\alpha+\beta}(f) = (\alpha + \beta)(A)$$

$$I_{\lambda\nu}(f) = I_{\alpha*\beta}(f) = (\alpha * \beta)(A).$$

□

Recall that the group ring of $\mathbb{Z}_p^d$ over R, $R[\mathbb{Z}_p^d]$, consists of all elements of the form

$$\lambda = \sum_{\boldsymbol{s}\in\mathbb{Z}_p^d} a_{\boldsymbol{s}}\boldsymbol{s},$$

where $a_{\boldsymbol{s}} \in R$ for all $\boldsymbol{s} \in \mathbb{Z}_p^d$, and $a_{\boldsymbol{s}} = 0$ for all but finitely many $\boldsymbol{s} \in \mathbb{Z}_p^d$. For each $\boldsymbol{n} \in \mathbb{N}^d$, there is a natural surjective ring homomorphism $R[\mathbb{Z}_p^d] \to R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge\boldsymbol{n}}\mathbb{Z}_p^d]$ given by

$$\lambda = \sum_{\boldsymbol{s}\in\mathbb{Z}_p^d} a_{\boldsymbol{s}}\boldsymbol{s} \mapsto \lambda_{\boldsymbol{n}} = \sum_{\boldsymbol{s}\in\mathbb{Z}_p^d} a_{\boldsymbol{s}}(\boldsymbol{s} + \boldsymbol{p}^{\wedge\boldsymbol{n}}\mathbb{Z}_p^d).$$

Note that for $\lambda \in R[\mathbb{Z}_p^d]$ and any $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}^d$ with $\boldsymbol{n} \preccurlyeq_d \boldsymbol{m}$, $\lambda_{\boldsymbol{n}}$ is the image of $\lambda_{\boldsymbol{m}}$ under the natural map $R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge\boldsymbol{m}}\mathbb{Z}_p^d] \to R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge\boldsymbol{n}}\mathbb{Z}_p^d]$. Thus, we obtain a ring homomorphism $R[\mathbb{Z}_p^d] \to R[[\mathbb{Z}_p^d]]$. Suppose

$$\lambda = \sum_{\boldsymbol{s}\in\mathbb{Z}_p^d} a_{\boldsymbol{s}}\boldsymbol{s} \in R[\mathbb{Z}_p^d]$$

is in the kernel of this map. Let

$$S = \{\boldsymbol{s} \in \mathbb{Z}_p^d : a_{\boldsymbol{s}} \neq 0\},$$

and for each $1 \leq i \leq d$ put

$$N_i = \max\{\operatorname{ord}(s_i - t_i) : \boldsymbol{s}, \boldsymbol{t} \in S \text{ and } s_i \neq t_i\},$$

with the convention that $\max(\emptyset) = -1$. Set $\boldsymbol{N} = (N_1 + 1, \dots, N_d + 1) \in \mathbb{N}^d$. Then in $R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge\boldsymbol{N}}\mathbb{Z}_p^d]$,

$$\begin{aligned} 0 = \lambda_{\boldsymbol{N}} &= \sum_{\boldsymbol{s}\in\mathbb{Z}_p^d} a_{\boldsymbol{s}}(\boldsymbol{s} + \boldsymbol{p}^{\wedge\boldsymbol{N}}\mathbb{Z}_p^d) \\ &= \sum_{\boldsymbol{s}\in S} a_{\boldsymbol{s}}(\boldsymbol{s} + \boldsymbol{p}^{\wedge\boldsymbol{N}}\mathbb{Z}_p^d). \end{aligned}$$

But by the definition of the N_i, the $\boldsymbol{s} + \boldsymbol{p}^{\wedge \boldsymbol{N}}\mathbb{Z}_p^d$ for $\boldsymbol{s} \in S$ are distinct elements of $\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge \boldsymbol{N}}\mathbb{Z}_p^d$. Consequently, $a_{\boldsymbol{s}} = 0$ for all $\boldsymbol{s} \in S$, and $\lambda = 0$. We therefore have an injection $R[\mathbb{Z}_p^d] \to R[[\mathbb{Z}_p^d]]$, and we will view $R[\mathbb{Z}_p^d]$ as a subring of $R[[\mathbb{Z}_p^d]]$ via this map.

Example 8. *Let*

$$\lambda = \sum_{\boldsymbol{s} \in \mathbb{Z}_p^d} a_{\boldsymbol{s}} \boldsymbol{s} \in R[\mathbb{Z}_p^d],$$

let $\alpha \in \mathcal{M}_d$ be the measure associated to λ, and let $I_\lambda = I_\alpha$ be the associated continuous R-linear map $C(\mathbb{Z}_p^d, R) \to R$. Suppose $f \in LC(\mathbb{Z}_p^d, R)$, and write

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}).$$

Then

$$I_\lambda(f) = \sum_{\boldsymbol{a} \preccurlyeq \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f(\boldsymbol{a}) \lambda_{\boldsymbol{n}}(\boldsymbol{a}).$$

Fix $\boldsymbol{a} \preccurlyeq \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}$; then

$$\lambda_{\boldsymbol{n}}(\boldsymbol{a}) = \sum_{\substack{\boldsymbol{s} \in \mathbb{Z}_p^d \\ \boldsymbol{s} \equiv \boldsymbol{a} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{n}}}}} a_{\boldsymbol{s}}.$$

Since f is locally constant,

$$\begin{aligned} f(\boldsymbol{a}) \lambda_{\boldsymbol{n}}(\boldsymbol{a}) &= f(\boldsymbol{a}) \sum_{\substack{\boldsymbol{s} \in \mathbb{Z}_p^d \\ \boldsymbol{s} \equiv \boldsymbol{a} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d}}} a_{\boldsymbol{s}} \\ &= \sum_{\substack{\boldsymbol{s} \in \mathbb{Z}_p^d \\ \boldsymbol{s} \equiv \boldsymbol{a} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d}}} f(\boldsymbol{a}) a_{\boldsymbol{s}} \\ &= \sum_{\substack{\boldsymbol{s} \in \mathbb{Z}_p^d \\ \boldsymbol{s} \equiv \boldsymbol{a} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d}}} f(\boldsymbol{s}) a_{\boldsymbol{s}}. \end{aligned}$$

It follows that

$$I_\lambda(f) = \sum_{\boldsymbol{s} \in \mathbb{Z}_p^d} a_{\boldsymbol{s}} f(\boldsymbol{s}).$$

By R-linearity and continuity, we have that

$$I_\lambda(f) = \sum_{\boldsymbol{s}\in\mathbb{Z}_p^d} a_{\boldsymbol{s}} f(\boldsymbol{s})$$

for all $f \in C(\mathbb{Z}_p^d, R)$, *so that*

$$\alpha = \sum_{\boldsymbol{s}\in\mathbb{Z}_p^d} a_{\boldsymbol{s}} \delta_{\boldsymbol{s}},$$

is a finite R-linear combination of Dirac measures. $\Diamond$

The following observation regarding the ring $R[[\mathbb{Z}_p^d]]$ will be useful in what follows (see Appendix B). Let $R[\boldsymbol{T}-\boldsymbol{1}]$ denote the polynomial ring $R[T_1-1,\ldots,T_d-1]$. For $\boldsymbol{n}\in\mathbb{N}^d$, let $I_{\boldsymbol{n}}$ denote the ideal of $R[\boldsymbol{T}-\boldsymbol{1}]$ generated by the components of the d-tuple $\boldsymbol{T}^{\wedge\boldsymbol{p}^{\wedge\boldsymbol{n}}}-\boldsymbol{1}$. The map $(\boldsymbol{a}+\boldsymbol{p}^{\wedge\boldsymbol{n}}\mathbb{Z}_p^d)\mapsto\boldsymbol{T}^{\boldsymbol{a}}$ extends by R-linearity to an R-algebra isomorphism

$$R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge\boldsymbol{n}}\mathbb{Z}_p^d]\to R[\boldsymbol{T}-\boldsymbol{1}]/I_{\boldsymbol{n}};$$

in particular, observe that this map is well-defined. When $\boldsymbol{n}\preccurlyeq_d\boldsymbol{m}$, $I_{\boldsymbol{m}}\subseteq I_{\boldsymbol{n}}$, and the natural maps

$$R[\boldsymbol{T}-\boldsymbol{1}]/I_{\boldsymbol{m}}\to R[\boldsymbol{T}-\boldsymbol{1}]/I_{\boldsymbol{n}}$$

correspond to the natural maps

$$R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge\boldsymbol{m}}\mathbb{Z}_p^d]\to R[\mathbb{Z}_p^d/\boldsymbol{p}^{\wedge\boldsymbol{n}}\mathbb{Z}_p^d].$$

This gives

$$R[[\mathbb{Z}_p^d]]\cong\lim_{\leftarrow} R[\boldsymbol{T}-\boldsymbol{1}]/I_{\boldsymbol{n}}.$$

2.3 The Power Series Viewpoint

In this section, we explore yet another way to view elements of $\mathcal{M}_d$. We refer especially to the work of Sinnott for an illustration of the utility of this last perspective (see Sinnott (1984), Sinnott (1987b)), and Sinnott (1987a)). Let $R[[\boldsymbol{T}-\mathbf{1}]]$ denote the ring of formal power series $R[[T_1-1,\ldots,T_d-1]]$.

Recall that the map $\alpha \mapsto (m_{\boldsymbol{n}}(\alpha))_{\boldsymbol{n}}$ is a ring isomorphism $\mathcal{M}_d \to \ell_\infty^{(d)}(R)$ (Corollary 7). At the same time, the map

$$(a_{\boldsymbol{n}})_{\boldsymbol{n}} \mapsto \sum_{\boldsymbol{n}\in\mathbb{N}^d} a_{\boldsymbol{n}}(\boldsymbol{T}-\mathbf{1})^{\boldsymbol{n}}$$

is an injective ring homomorphism $\ell_\infty^{(d)}(R) \to R[[\boldsymbol{T}-\mathbf{1}]]$ with image equal to the subring of $R[[\boldsymbol{T}-\mathbf{1}]]$ of power series with bounded coefficients, which we denote by $\Lambda_d = \Lambda_d(R)$. For a measure $\alpha \in \mathcal{M}_d$, we put

$$\widehat{\alpha}(\boldsymbol{T}) = \sum_{\boldsymbol{n}\in\mathbb{N}^d} m_{\boldsymbol{n}}(\alpha)(\boldsymbol{T}-\mathbf{1})^{\boldsymbol{n}} \in \Lambda_d.$$

Then the map $\alpha \mapsto \widehat{\alpha}$ is a ring isomorphism $\mathcal{M}_d \to \Lambda_d$, called the *Iwasawa isomorphism.* Note that we write $\widehat{\alpha}(\boldsymbol{T})$ rather than the more cumbersome $\widehat{\alpha}(\boldsymbol{T}-\mathbf{1})$.

Note that Proposition 4 and Corollary 5 give that if $\alpha \in \mathcal{M}_d$ with

$$\widehat{\alpha}(\boldsymbol{T}) = \sum_{\boldsymbol{n}\in\mathbb{N}^d} m_{\boldsymbol{n}}(\alpha)(\boldsymbol{T}-\mathbf{1})^{\boldsymbol{n}} \in \Lambda_d,$$

then for $f \in C(\mathbb{Z}_p^d, R)$,

$$\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{N}^d} m_{\boldsymbol{n}}(\alpha) a_{\boldsymbol{n}}(f),$$

where $a_{\boldsymbol{n}}(f)$ is the $\boldsymbol{n}$th Mahler coefficient of f.

Definition 18. *Let $\alpha \in \mathcal{M}_d$. The power series associated to α is the image $\widehat{\alpha}(\boldsymbol{T})$ of α under the Iwasawa isomorphism $\mathcal{M}_d \to \Lambda_d$:*

$$\widehat{\alpha}(\boldsymbol{T}) = \sum_{\boldsymbol{n}\in\mathbb{N}^d} \left(\int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{n}} d\alpha(\boldsymbol{x}) \right) (\boldsymbol{T}-\mathbf{1})^{\boldsymbol{n}} = \int_{\mathbb{Z}_p^d} \boldsymbol{T}^{\boldsymbol{x}}\, d\alpha(\boldsymbol{x}).$$

Note that since $\widehat{\alpha}$ has bounded coefficients in R and since $\|\cdot\|_R$ is non-archimedean, $\widehat{\alpha}(\boldsymbol{z})$ converges in R for all $\boldsymbol{z} \in R^d$ satisfying $\|z_i - 1\|_R < 1$ for all $1 \leq i \leq d$, where for

$$\widehat{\alpha}(\boldsymbol{T}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} a_{\boldsymbol{n}} (\boldsymbol{T} - \mathbf{1})^{\boldsymbol{n}},$$

we put

$$\widehat{\alpha}(\boldsymbol{z}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} a_{\boldsymbol{n}} (\boldsymbol{z} - \mathbf{1})^{\boldsymbol{n}}.$$

Example 9. *Let $\alpha \in \mathcal{M}_d$. Recall that*

$$\binom{\boldsymbol{x}}{\mathbf{0}} = 1$$

for all $\boldsymbol{x} \in \mathbb{Z}_p^d$, and the constant function 1 is the characteristic function of $\mathbb{Z}_p^d$. Consequently,

$$\begin{aligned} m_{\mathbf{0}}(\alpha) &= \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\mathbf{0}} \, d\alpha(\boldsymbol{x}) \\ &= \int_{\mathbb{Z}_p^d} d\alpha(\boldsymbol{x}) \\ &= \alpha(\mathbb{Z}_p^d). \end{aligned}$$

This gives that

$$\widehat{\alpha}(\mathbf{1}) = m_{\mathbf{0}}(\alpha) = \alpha(\mathbb{Z}_p^d).$$

$\diamondsuit$

Example 10. *Let $\alpha \in \mathcal{M}_d$ have associated power series*

$$\widehat{\alpha}(\boldsymbol{T}) = (\boldsymbol{T} - \mathbf{1})^{\boldsymbol{n}} \in \Lambda_d$$

for some fixed $\boldsymbol{n} \in \mathbb{N}^d$. Then for $f \in C(\mathbb{Z}_p^d, R)$, we have

$$\int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) \, d\alpha(\boldsymbol{x}) = a_{\boldsymbol{n}}(f).$$

That is, the continuous R-linear map $I_\alpha : C(\mathbb{Z}_p^d, R) \to R$ is given by $I_\alpha : f \mapsto a_{\boldsymbol{n}}(f)$.

More generally, suppose $\widehat{\alpha}(\boldsymbol{T})$ is a polynomial in $(\boldsymbol{T}-\boldsymbol{1})$ over R:

$$\widehat{\alpha}(\boldsymbol{T}) = \sum_{\boldsymbol{k} \preccurlyeq_d \boldsymbol{n}} c_{\boldsymbol{k}} (\boldsymbol{T}-\boldsymbol{1})^{\boldsymbol{k}} \in R[\boldsymbol{T}-\boldsymbol{1}]$$

for some $\boldsymbol{n} \in \mathbb{N}^d$. Then for $f \in C(\mathbb{Z}_p^d, R)$,

$$\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = \sum_{\boldsymbol{k} \preccurlyeq_d \boldsymbol{n}} c_{\boldsymbol{k}} a_{\boldsymbol{k}}(f).$$

$\diamondsuit$

Example 11. *Fix $\boldsymbol{s} \in \mathbb{Z}_p^d$ and let $\delta_{\boldsymbol{s}}$ denote Dirac measure of mass one centered at $\boldsymbol{s}$ (Example 3). From Example 4, we have*

$$\widehat{\delta}_{\boldsymbol{s}}(\boldsymbol{T}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} \binom{\boldsymbol{s}}{\boldsymbol{n}} (\boldsymbol{T}-\boldsymbol{1})^{\boldsymbol{n}} = \boldsymbol{T}^{\boldsymbol{s}}.$$

$\diamondsuit$

We can now give a formula for the measure of polyballs in $\mathbb{Z}_p^d$ using associated power series. First, the characteristic function of the ball $p^n\mathbb{Z}_p$ in $\mathbb{Z}_p$ is given by

$$g_{0,n}(x) = \frac{1}{p^n} \sum_{\zeta^{p^n}=1} \zeta^x,$$

where the sum ranges over all p^nth roots of unity in $\mathbb{C}_p$. The characteristic function of the ball $a + p^n\mathbb{Z}_p$ is thus

$$g_{a,n}(x) = \frac{1}{p^n} \sum_{\zeta^{p^n}=1} \zeta^x \zeta^{-a}.$$

Consequently, the characteristic function of the polyball $\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d$ in $\mathbb{Z}_p^d$ is

$$g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}) = \prod_{i=1}^d g_{a_i,n_i}(x_i) = \prod_{i=1}^d \left(\frac{1}{p^{n_i}} \sum_{\zeta_i^{p^{n_i}}=1} \zeta_i^{x_i} \zeta^{-a_i} \right) = \frac{1}{p^{|\boldsymbol{n}|}} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}}=\boldsymbol{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \boldsymbol{\zeta}^{\boldsymbol{x}},$$

where the rightmost sum extends over all d-tuples $\boldsymbol{\zeta} \in \mathbb{C}_p^d$ satisfying $\boldsymbol{\zeta}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}} = \mathbf{1}$.

Let $S = R\,\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{C}_p$ (see Grothendieck (1954), Serre (1962), or Schneider (2002)). For any $\boldsymbol{\zeta} \in \mathbb{C}_p^d$ as above, $|\zeta_i - 1|_p < 1$ (in $\mathbb{C}_p$) for all $1 \le i \le d$. We have in S,

$$\begin{aligned}
\alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d) &= \int_{\mathbb{Z}_p^d} g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \frac{1}{p^{|\boldsymbol{n}|}} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}} = \mathbf{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \int_{\mathbb{Z}_p^d} \boldsymbol{\zeta}^{\boldsymbol{x}}\, d\alpha(\boldsymbol{x}) \\
&= \frac{1}{p^{|\boldsymbol{n}|}} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}} = \mathbf{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \widehat{\alpha}(\boldsymbol{\zeta}).
\end{aligned}$$

Note that each of the series in the last line above converges in S. We record this result below.

Theorem 3. *Let $\alpha \in \mathcal{M}_d$ with associated power series $\widehat{\alpha} \in \Lambda_d$. Then for all $\boldsymbol{n} \in \mathbb{N}^d$ and $\boldsymbol{a} \in \mathbb{Z}_p^d$,*

$$\alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d) = \frac{1}{p^{|\boldsymbol{n}|}} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}} = \mathbf{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \widehat{\alpha}(\boldsymbol{\zeta}),$$

where the sum extends over all d-tuples $\boldsymbol{\zeta} \in \mathbb{C}_p^d$ satisfying $\boldsymbol{\zeta}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}} = \mathbf{1}$.

2.4 Operations on Measures and the Γ-Transform

In this section, we consider methods of constructing new measures on $\mathbb{Z}_p^d$ from a given measure. We also interpret these operations as maps on power series.

Definition 19. *Let $\alpha \in \mathcal{M}_d$. For $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$, define a new measure $\alpha \circ \boldsymbol{a} \in \mathcal{M}_d$ by*

$$(\alpha \circ \boldsymbol{a})(A) = \alpha(\boldsymbol{a}A).$$

When integrating, we write $d\alpha(\boldsymbol{a}\boldsymbol{x})$ rather than $d(\alpha \circ \boldsymbol{a})(\boldsymbol{x})$. For $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$, we put $\boldsymbol{a}^{-1} = (a_1^{-1}, \ldots, a_d^{-1})$, as in Appendix A.

Clearly, $\|\alpha \circ \boldsymbol{a}\|_u \leq \|\alpha\|_u$. Since $\boldsymbol{a}\mathbb{Z}_p^d = \mathbb{Z}_p^d$, we have $\|\alpha \circ \boldsymbol{a}\|_u = \|\alpha\|_u$. For $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$, $\boldsymbol{a}(A \cup B) = \boldsymbol{a}A \cup \boldsymbol{a}B$ for all $A, B \in CO_d$ and $\boldsymbol{a}A \cap \boldsymbol{a}B = \emptyset$ whenever $A \cap B = \emptyset$ (multiplication by $\boldsymbol{a}$ is an isometric isomorphism $\mathbb{Z}_p^d \to \mathbb{Z}_p^d$). Thus, $\alpha \circ \boldsymbol{a} \in \mathcal{M}_d$, as claimed.

Example 12. *Since* $\mathbf{1}A = A$ *for all* $A \in CO_d$, *we have* $\alpha \circ \mathbf{1} = \alpha$ *for all* $\alpha \in \mathcal{M}_d$. $\Diamond$

Example 13. *Let* $\boldsymbol{s} \in \mathbb{Z}_p^d$ *and let* $\delta_{\boldsymbol{s}}$ *denote Dirac measure of mass one centered at* $\boldsymbol{s}$. *Fix* $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$. *For any* $A \in CO_d$,

$$\begin{aligned}(\delta_{\boldsymbol{s}} \circ \boldsymbol{a})(A) &= \delta_{\boldsymbol{s}}(\boldsymbol{a}A) \\ &= \begin{cases} 1 & : \boldsymbol{s} \in \boldsymbol{a}A \\ 0 & : else \end{cases} \\ &= \begin{cases} 1 & : \boldsymbol{a}^{-1}\boldsymbol{s} \in A \\ 0 & : else \end{cases} \\ &= \delta_{\boldsymbol{a}^{-1}\boldsymbol{s}}(A).\end{aligned}$$

Thus, $\delta_{\boldsymbol{s}} \circ \boldsymbol{a} = \delta_{\boldsymbol{a}^{-1}\boldsymbol{s}}$. *Consequently,*

$$(\widehat{\delta_{\boldsymbol{s}} \circ \boldsymbol{a}})(\boldsymbol{T}) = \widehat{\delta_{\boldsymbol{a}^{-1}\boldsymbol{s}}}(\boldsymbol{T}) = \boldsymbol{T}^{\boldsymbol{a}^{-1}\boldsymbol{s}} = (\boldsymbol{T}^{\boldsymbol{a}^{-1}})^{\boldsymbol{s}} = \widehat{\delta}_{\boldsymbol{s}}(\boldsymbol{T}^{\boldsymbol{a}^{-1}}).$$

$\Diamond$

Lemma 10. *Let* $\alpha \in \mathcal{M}_d$ *and* $\boldsymbol{a}, \boldsymbol{b} \in (\mathbb{Z}_p^\times)^d$. *Then*

$$(\alpha \circ \boldsymbol{a}) \circ \boldsymbol{b} = \alpha \circ (\boldsymbol{ab}).$$

Proof. This follows from the fact that $\boldsymbol{a}(\boldsymbol{b}A) = (\boldsymbol{ab})A$ for all $A \in CO_d$. $\square$

Lemma 11. *If* $\alpha \in \mathcal{M}_d$ *and* $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$, *then*

$$\int_{\mathbb{Z}_p^d} f(\boldsymbol{ax}) d\alpha(\boldsymbol{ax}) = \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) d\alpha(\boldsymbol{x})$$

whenever $f \in C(\mathbb{Z}_p^d, R)$.

Proof. By linearity and continuity, it suffices to consider the case where f is the characteristic function of some polyball A. Now $\boldsymbol{ax} \in A$ if and only if $\boldsymbol{x} \in \boldsymbol{a}^{-1}A$, i.e., $g(\boldsymbol{x}) = f(\boldsymbol{ax})$ is the characteristic function of $\boldsymbol{a}^{-1}A$. Since $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$, multiplication by $\boldsymbol{a}^{-1}$ is an isometry; thus, $\boldsymbol{a}^{-1}A$ is also a polyball in the same level as A. But now

$$\begin{aligned}\int_{\mathbb{Z}_p^d} f(\boldsymbol{ax})d\alpha(\boldsymbol{ax}) &= \int_{\mathbb{Z}_p^d} g(\boldsymbol{x})d\alpha(\boldsymbol{ax}) \\ &= (\alpha \circ \boldsymbol{a})(\boldsymbol{a}^{-1}A) \\ &= (\alpha \circ \mathbf{1})(A) \\ &= \alpha(A) \\ &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})d\alpha(\boldsymbol{x}).\end{aligned}$$

□

Corollary 10. *If* $\alpha \in \mathcal{M}_d$ *and* $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$, *then* $\widehat{\alpha \circ \boldsymbol{a}}(\boldsymbol{T}) = \widehat{\alpha}(\boldsymbol{T}^{\wedge \boldsymbol{a}^{-1}})$

Proof. Note first that $\widehat{\alpha}(\boldsymbol{T}^{\wedge \boldsymbol{a}^{-1}})$ denotes the composition of $\widehat{\alpha}(\boldsymbol{T})$ with the power series $\boldsymbol{T}^{\wedge \boldsymbol{a}^{-1}} - \mathbf{1}$, which is defined since $\boldsymbol{T}^{\wedge \boldsymbol{a}^{-1}} - \mathbf{1}$ is a d-tuple of one-variable power series without constant term.

The $\boldsymbol{m}$th coefficient of $\widehat{\alpha \circ \boldsymbol{a}}(\boldsymbol{T})$ is

$$\begin{aligned}\int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{m}} d\alpha(\boldsymbol{ax}) &= \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{a}\boldsymbol{a}^{-1}\boldsymbol{x}}{\boldsymbol{m}} d\alpha(\boldsymbol{ax}) \\ &= \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{a}^{-1}\boldsymbol{x}}{\boldsymbol{m}} d\alpha(\boldsymbol{x})\end{aligned}$$

and this is the $\boldsymbol{m}$th coefficient of $\widehat{\alpha}(\boldsymbol{T}^{\wedge \boldsymbol{a}^{-1}})$. □

Lemma 12. *Let* $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$. *For any* $\alpha, \beta \in \mathcal{M}_d$ *and* $c \in R$, *we have* $(c\alpha) \circ \boldsymbol{a} = c(\alpha \circ \boldsymbol{a})$, $(\alpha + \beta) \circ \boldsymbol{a} = (\alpha \circ \boldsymbol{a}) + (\beta \circ \boldsymbol{a})$, *and* $(\alpha * \beta) \circ \boldsymbol{a} = (\alpha \circ \boldsymbol{a}) * (\beta \circ \boldsymbol{a})$. *Moreover,* *for fixed* $\boldsymbol{a} \in \mathbb{Z}_p^\times$*, the map* $\alpha \mapsto \alpha \circ \boldsymbol{a}$ *is a norm-preserving* R*-algebra isomorphism* $\mathcal{M}_d \to \mathcal{M}_d$*.*

Proof. We have that

$$\begin{aligned}
(\widehat{(\alpha + \beta) \circ \boldsymbol{a}})(\boldsymbol{T}) &= (\widehat{\alpha + \beta})(\boldsymbol{T}^{\boldsymbol{a}^{-1}}) \\
&= \widehat{\alpha}(\boldsymbol{T}^{\boldsymbol{a}^{-1}}) + \widehat{\beta}(\boldsymbol{T}^{\boldsymbol{a}^{-1}}) \\
&= (\widehat{\alpha \circ \boldsymbol{a}})(\boldsymbol{T}) + (\widehat{\beta \circ \boldsymbol{a}})(\boldsymbol{T}) \\
&= (\alpha \circ \widehat{\boldsymbol{a}) + (}\beta \circ \boldsymbol{a})(\boldsymbol{T})
\end{aligned}$$

and

$$\begin{aligned}
(\widehat{(\alpha * \beta) \circ \boldsymbol{a}})(\boldsymbol{T}) &= (\widehat{\alpha * \beta})(\boldsymbol{T}^{\boldsymbol{a}^{-1}}) \\
&= \widehat{\alpha}(\boldsymbol{T}^{\boldsymbol{a}^{-1}})\widehat{\beta}(\boldsymbol{T}^{\boldsymbol{a}^{-1}}) \\
&= (\widehat{\alpha \circ \boldsymbol{a}})(\boldsymbol{T})(\widehat{\beta \circ \boldsymbol{a}})(\boldsymbol{T}) \\
&= (\alpha \circ \widehat{\boldsymbol{a}) * (}\beta \circ \boldsymbol{a})(\boldsymbol{T}).
\end{aligned}$$

From this last part, we also obtain (recall Example 7)

$$(c\alpha) \circ \boldsymbol{a} = c(\alpha \circ \boldsymbol{a}).$$

The above gives that the map $\alpha \mapsto \alpha \circ \boldsymbol{a}$ is an R-algebra homomorphism. For $\alpha \in \mathcal{M}_d$, we have

$$\alpha = \alpha \circ \mathbf{1} = \alpha \circ (\boldsymbol{a}^{-1}\boldsymbol{a}) = (\alpha \circ \boldsymbol{a}^{-1}) \circ \boldsymbol{a},$$

so that $\alpha \mapsto \alpha \circ \boldsymbol{a}$ is surjective. On the other hand, if $\alpha \circ \boldsymbol{a} = \alpha_0$, then for any $A \in CO_d$,

$$\alpha(A) = \alpha(\boldsymbol{a}\boldsymbol{a}^{-1}A) = (\alpha \circ \boldsymbol{a})(\boldsymbol{a}^{-1}A) = 0,$$

so that $\alpha = \alpha_0$. Consequently, the map $\alpha \mapsto \alpha \circ \boldsymbol{a}$ is injective. We have already noted that $\|\alpha\|_u = \|\alpha \circ \boldsymbol{a}\|_u$. □

Definition 20. *Let* $\alpha \in \mathcal{M}_d$ *and* $g \in C(\mathbb{Z}_p^d, R)$*. Define the measure* α_g *by*

$$\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha_g(\boldsymbol{x}) = \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) g(\boldsymbol{x})\, d\alpha(\boldsymbol{x})$$

for $f \in C(\mathbb{Z}_p^d, R)$*. If* g *is constant, say with* $g(\boldsymbol{x}) = c$ *for all* $\boldsymbol{x} \in \mathbb{Z}_p^d$*, then we will write* $c\alpha$ *for* α_g*.*

The map $I_{\alpha_g} : C(\mathbb{Z}_p^d, R) \to R$ is certainly R-linear, and we note that

$$\begin{aligned} \left\| I_{\alpha_g}(f) \right\|_R &= \| I_\alpha(fg) \|_R \\ &\leq \|\alpha\|_u \|fg\|_\infty \\ &\leq \|\alpha\|_u \|f\|_\infty \|g\|_\infty. \end{aligned}$$

Thus, $I_{\alpha_g} : C(\mathbb{Z}_p^d, R) \to R$ is R-linear and continuous, so $\alpha_g \in \mathcal{M}_d$. In terms of power series, note that for each $\boldsymbol{m} \in \mathbb{N}^d$, we have

$$\int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{m}}\, d\alpha_g(\boldsymbol{x}) = \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{m}} g(\boldsymbol{x})\, d\alpha(\boldsymbol{x}),$$

from which we obtain

$$\widehat{\alpha_g}(\boldsymbol{T}) = \int_{\mathbb{Z}_p^d} \boldsymbol{T}^{\boldsymbol{x}}\, d\alpha_g(\boldsymbol{x}) = \int_{\mathbb{Z}_p^d} \boldsymbol{T}^{\boldsymbol{x}} g(\boldsymbol{x})\, d\alpha(\boldsymbol{x})$$

as an identity of power series.

Note further that if g is constant, with $g \equiv c$ for $c \in R$, then $\alpha \mapsto \alpha_g$ agrees with the scalar multiplication by R on $\mathcal{M}_d$.

Example 14. *Let* $\boldsymbol{s} \in \mathbb{Z}_p^d$ *and* $g \in C(\mathbb{Z}_p^d, R)$*. Then* $(\delta_{\boldsymbol{s}})_g = g(\boldsymbol{s})\delta_{\boldsymbol{s}}$*. Indeed, for*

$f \in C(\mathbb{Z}_p^d, R)$,

$$\begin{aligned}
\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d(\delta_{\boldsymbol{s}})_g(\boldsymbol{x}) &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) g(\boldsymbol{x})\, d\delta_{\boldsymbol{s}}(\boldsymbol{x}) \\
&= f(\boldsymbol{s}) g(\boldsymbol{s}) \\
&= g(\boldsymbol{s}) \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\delta_{\boldsymbol{s}}(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d(g(\boldsymbol{s})\delta_{\boldsymbol{s}})(\boldsymbol{x}).
\end{aligned}$$

◊

Lemma 13. *Let* $g \in C(\mathbb{Z}_p^d, R)$ *and* $c \in R$. *For any* $\alpha, \beta \in \mathcal{M}_d$, *we have* $(\alpha + \beta)_g = \alpha_g + \beta_g$ *and* $(c\alpha)_g = c(\alpha_g) = \alpha_{cg}$.

Proof. Let $f \in C(\mathbb{Z}_p^d, R)$. Then

$$\begin{aligned}
\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d((\alpha+\beta)_g)(\boldsymbol{x}) &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) g(\boldsymbol{x})\, d(\alpha+\beta)(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) g(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) + \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) g(\boldsymbol{x})\, d\beta(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha_g(\boldsymbol{x}) + \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\beta_g(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d(\alpha_g + \beta_g)(\boldsymbol{x}).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d((c\alpha)_g)(\boldsymbol{x}) &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) g(\boldsymbol{x})\, d(c\alpha)(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) c g(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha_{cg}(\boldsymbol{x})
\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\,d\alpha_{cg}(\boldsymbol{x}) &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})cg(\boldsymbol{x})\,d\alpha(\boldsymbol{x}) \\ &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})c\,d\alpha_g(\boldsymbol{x}) \\ &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\,d(c(\alpha_g))(\boldsymbol{x}).\end{aligned}$$

□

Two special instances of the above construction will be used later. A simple application is when the function g is actually locally constant. As in the previous section, let $S = R\,\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{C}_p$.

Lemma 14. *Let $g \in LC(\mathbb{Z}_p^d, R)$, say with*

$$g(\boldsymbol{x}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge\boldsymbol{n}}-\mathbf{1}} g(\boldsymbol{a})g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}).$$

For each d-tuple $\boldsymbol{\zeta} \in \mathbb{C}_p^d$ satisfying $\boldsymbol{\zeta}^{\wedge\boldsymbol{p}^{\wedge\boldsymbol{n}}} = \mathbf{1}$, put

$$\widehat{g}(\boldsymbol{\zeta}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge\boldsymbol{n}}-\mathbf{1}} g(\boldsymbol{a})\boldsymbol{\zeta}^{-\boldsymbol{a}} \in S.$$

Then, in S,

$$\widehat{\alpha_g}(\boldsymbol{T}) = \frac{1}{p^{|\boldsymbol{n}|}} \sum_{\boldsymbol{\zeta}^{\wedge\boldsymbol{p}^{\wedge\boldsymbol{n}}}=\mathbf{1}} \widehat{g}(\boldsymbol{\zeta})\widehat{\alpha}(\boldsymbol{\zeta}\boldsymbol{T}),$$

where the sum extends over all d-tuples $\boldsymbol{\zeta} \in \mathbb{C}_p^d$ satisfying $\boldsymbol{\zeta}^{\wedge\boldsymbol{p}^{\wedge\boldsymbol{n}}} = \mathbf{1}$.

Proof. Observe first that for $\boldsymbol{\zeta}$ as in the statement of the lemma,

$$\boldsymbol{\zeta}\boldsymbol{T} - \mathbf{1} = (\boldsymbol{\zeta} - \mathbf{1}) + \boldsymbol{\zeta}(\boldsymbol{T} - \mathbf{1}),$$

which is a d-tuple of one-variable power series over S which are topologically nilpotent in $S[[T]]$. Thus, $\widehat{\alpha}(\boldsymbol{\zeta T})$ is a well-defined element of $S[[\boldsymbol{T}]]$. We have

$$\begin{aligned}\widehat{\alpha_g}(\boldsymbol{T}) &= \int_{\mathbb{Z}_p^d} \boldsymbol{T}^{\boldsymbol{x}}\, d\alpha_g(\boldsymbol{x}) \\ &= \int_{\mathbb{Z}_p^d} \boldsymbol{T}^{\boldsymbol{x}} g(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\ &= \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} g(\boldsymbol{a}) \int_{\mathbb{Z}_p^d} \boldsymbol{T}^{\boldsymbol{x}} g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x})\, d\alpha(\boldsymbol{x})\end{aligned}$$

Let $\boldsymbol{a} \in \mathbb{N}^d$ with $\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}$. We have seen that

$$g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}) = \frac{1}{p^{|\boldsymbol{n}|}} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}} = \boldsymbol{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \boldsymbol{\zeta}^{\boldsymbol{x}}.$$

Thus,

$$\begin{aligned}\int_{\mathbb{Z}_p^d} \boldsymbol{T}^{\boldsymbol{x}} g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) &= \frac{1}{p^{|\boldsymbol{n}|}} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}} = \boldsymbol{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \int_{\mathbb{Z}_p^d} \boldsymbol{T}^{\boldsymbol{x}} \boldsymbol{\zeta}^{\boldsymbol{x}}\, d\alpha(\boldsymbol{x}) \\ &= \frac{1}{p^{|\boldsymbol{n}|}} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}} = \boldsymbol{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \widehat{\alpha}(\boldsymbol{\zeta T}).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\widehat{\alpha_g}(\boldsymbol{T}) &= \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} g(\boldsymbol{a}) \int_{\mathbb{Z}_p^d} \boldsymbol{T}^{\boldsymbol{x}} g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\ &= \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} g(\boldsymbol{a}) \frac{1}{p^{|\boldsymbol{n}|}} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}} = \boldsymbol{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \widehat{\alpha}(\boldsymbol{\zeta T}) \\ &= \frac{1}{p^{|\boldsymbol{n}|}} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}} = \boldsymbol{1}} \widehat{g}(\boldsymbol{\zeta}) \widehat{\alpha}(\boldsymbol{\zeta T}).\end{aligned}$$

□

Corollary 11. *For each* $\boldsymbol{n} \in \mathbb{N}^d$*, in* S*,*

$$\widehat{\alpha}(\boldsymbol{T}) = \frac{1}{p^{|\boldsymbol{n}|}} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}} = \boldsymbol{1}} \left(\sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \right) \widehat{\alpha}(\boldsymbol{\zeta T}),$$

where the sum extends over all d-tuples $\boldsymbol{\zeta} \in \mathbb{C}_p^d$ *satisfying* $\boldsymbol{\zeta}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}} = \boldsymbol{1}$.

Proof. Let $g \equiv 1$. Then $\alpha_g = \alpha$, so $\widehat{\alpha_g} = \widehat{\alpha}$. Since g is constant, g is locally constant, factoring through any $L_{\boldsymbol{n}}$. Certainly, for each $\boldsymbol{\zeta} \in \mathbb{C}_p^d$ satisfying $\boldsymbol{\zeta}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}} = \mathbf{1}$, we have

$$\widehat{g}(\boldsymbol{\zeta}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \mathbf{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}}.$$

Accordingly,

$$\widehat{\alpha}(\boldsymbol{T}) = \widehat{\alpha_g}(\boldsymbol{T}) = \frac{1}{p^{|\boldsymbol{n}|}} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}} = \mathbf{1}} \left(\sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \mathbf{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \right) \widehat{\alpha}(\boldsymbol{\zeta T}).$$

□

Lemma 15. *Let $\alpha \in \mathcal{M}_d$ and let $\delta_{\boldsymbol{s}}$ denote Dirac measure of mass one centered at $\boldsymbol{s} \in \mathbb{Z}_p^d$. Then for $g \in C(\mathbb{Z}_p^d, R)$,*

$$(\alpha * \delta_{\boldsymbol{s}})_g = \alpha_{g_{\boldsymbol{s}}} * \delta_{\boldsymbol{s}},$$

where for $\boldsymbol{x} \in \mathbb{Z}_p^d$, $g_{\boldsymbol{s}}(\boldsymbol{x}) = g(\boldsymbol{s} + \boldsymbol{x})$.

Proof. Note first that since translation by $\boldsymbol{s}$ is an isometry $\mathbb{Z}_p^d \to \mathbb{Z}_p^d$, $g_{\boldsymbol{s}} \in C(\mathbb{Z}_p^d, R)$. Let $f \in C(\mathbb{Z}_p^d, R)$. We have by Example 5

$$\begin{aligned}
\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d((\alpha * \delta_{\boldsymbol{s}})_g)(\boldsymbol{x}) &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) g(\boldsymbol{x})\, d(\alpha * \delta_{\boldsymbol{s}})(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{s} + \boldsymbol{x}) g(\boldsymbol{s} + \boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{s} + \boldsymbol{x}) g_{\boldsymbol{s}}(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{s} + \boldsymbol{x})\, d\alpha_{g_{\boldsymbol{s}}}(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d((\alpha_{g_{\boldsymbol{s}}} * \delta_{\boldsymbol{s}})(\boldsymbol{x}).
\end{aligned}$$

□

Let D_i be the derivation $T_i \dfrac{\partial}{\partial T_i}$ on $R[[\boldsymbol{T} - \mathbf{1}]]$, and for $\boldsymbol{n} \in \mathbb{N}^d$, let $\boldsymbol{D}^{\boldsymbol{n}}$ denote the derivation $D_1^{n_1} \cdots D_d^{n_d}$ on $R[[\boldsymbol{T} - \mathbf{1}]]$ (see Appendix B).

Proposition 10. *Let $\alpha \in \mathcal{M}_d$ and $\boldsymbol{n} \in \mathbb{N}^d$. Let $g \in C(\mathbb{Z}_p^d, R)$ be the function given by $g(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{n}}$. Then*

$$\widehat{\alpha_g}(\boldsymbol{T}) = \boldsymbol{D}^{\boldsymbol{n}} \widehat{\alpha}(\boldsymbol{T}).$$

Proof. The Proposition amounts to showing that

$$\boldsymbol{D}^{\boldsymbol{n}} \boldsymbol{T}^{\boldsymbol{x}} = \boldsymbol{x}^{\boldsymbol{n}} \boldsymbol{T}^{\boldsymbol{x}}$$

for all $\boldsymbol{n} \in \mathbb{N}^d$ and all $\boldsymbol{x} \in \mathbb{Z}_p^d$. Fix $\boldsymbol{x} \in \mathbb{Z}_p^d$ and $1 \leq i \leq d$. Then

$$\boldsymbol{T}^{\boldsymbol{x}} = \sum_{\boldsymbol{m} \in \mathbb{N}^d} \binom{\boldsymbol{x}}{\boldsymbol{m}} (\boldsymbol{T} - \boldsymbol{1})^{\boldsymbol{m}}.$$

From this, we find that the $\boldsymbol{m}$th coefficient of $D_i \boldsymbol{T}^{\boldsymbol{x}}$ is

$$m_i \binom{\boldsymbol{x}}{\boldsymbol{m}} + (m_i + 1) \binom{\boldsymbol{x}}{\boldsymbol{m} + \boldsymbol{e}_i}.$$

Using the identities from Appendix C, we see that the $\boldsymbol{m}$th coefficient of $D_i \boldsymbol{T}^{\boldsymbol{x}}$ is

$$\begin{aligned} m_i \binom{\boldsymbol{x}}{\boldsymbol{m}} + (m_i + 1) \binom{\boldsymbol{x}}{\boldsymbol{m} + \boldsymbol{e}_i} &= \left(m_i \binom{x_i}{m_i} + (m_i + 1) \binom{x_i}{m_i + 1} \right) \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \binom{x_j}{m_j} \\ &= x_i \binom{\boldsymbol{x}}{\boldsymbol{m}}. \end{aligned}$$

Thus

$$\begin{aligned} D_i \boldsymbol{T}^{\boldsymbol{x}} &= \sum_{\boldsymbol{m} \in \mathbb{N}^d} x_i \binom{\boldsymbol{x}}{\boldsymbol{m}} (\boldsymbol{T} - \boldsymbol{1})^{\boldsymbol{m}} \\ &= x_i \sum_{\boldsymbol{m} \in \mathbb{N}^d} \binom{\boldsymbol{x}}{\boldsymbol{m}} (\boldsymbol{T} - \boldsymbol{1})^{\boldsymbol{m}} \\ &= x_i \boldsymbol{T}^{\boldsymbol{x}}. \end{aligned}$$

Now suppose that there is some $\boldsymbol{n} \in \mathbb{N}^d$ such that

$$\boldsymbol{D}^{\boldsymbol{n}} \boldsymbol{T}^{\boldsymbol{x}} = \boldsymbol{x}^{\boldsymbol{n}} \boldsymbol{T}^{\boldsymbol{x}}.$$

Then for each $1 \le i \le d$, we have

$$\begin{aligned} \boldsymbol{D}^{\boldsymbol{n}+\boldsymbol{e}_i}\boldsymbol{T}^{\boldsymbol{x}} &= D_i(\boldsymbol{D}^{\boldsymbol{n}}\boldsymbol{T}^{\boldsymbol{x}}) \\ &= D_i(\boldsymbol{x}^{\boldsymbol{n}}\boldsymbol{T}^{\boldsymbol{x}}) \\ &= \boldsymbol{x}^{\boldsymbol{n}} D_i \boldsymbol{T}^{\boldsymbol{x}} \\ &= \boldsymbol{x}^{\boldsymbol{n}} x_i \boldsymbol{T}^{\boldsymbol{x}} \\ &= \boldsymbol{x}^{\boldsymbol{n}+\boldsymbol{e}_i}\boldsymbol{T}^{\boldsymbol{x}} \end{aligned}$$

By induction, we have

$$\boldsymbol{D}^{\boldsymbol{n}}\boldsymbol{T}^{\boldsymbol{x}} = \boldsymbol{x}^{\boldsymbol{n}}\boldsymbol{T}^{\boldsymbol{x}}$$

for all $\boldsymbol{n} \in \mathbb{N}^d$ and all $\boldsymbol{x} \in \mathbb{Z}_p^d$. □

Corollary 12. *The* $\boldsymbol{n}$*th moment of the measure* α *is*

$$M_{\boldsymbol{n}}(\alpha) = \boldsymbol{D}^{\boldsymbol{n}}\widehat{\alpha}(\boldsymbol{T})|_{\boldsymbol{T}=\boldsymbol{1}}.$$

Proof. Let $\boldsymbol{n} \in \mathbb{N}^d$. From Proposition 10, with $g(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{n}}$, we see, as in Example 9,

$$\begin{aligned} \boldsymbol{D}^{\boldsymbol{n}}\widehat{\alpha}(\boldsymbol{T})|_{\boldsymbol{T}=\boldsymbol{1}} &= \widehat{\alpha_g}(\boldsymbol{1}) \\ &= \int_{\mathbb{Z}_p^d} d\alpha_g(\boldsymbol{x}) \\ &= \int_{\mathbb{Z}_p^d} \boldsymbol{x}^{\boldsymbol{n}}\, d\alpha(\boldsymbol{x}) \\ &= M_{\boldsymbol{n}}(\alpha). \end{aligned}$$

□

Definition 21. *For* $A \in CO_d$*, define a new measure* $\alpha|_A \in \mathcal{M}_d$ *by* $\alpha|_A(O) = \alpha(O \cap A)$ *for* $O \in CO_d$*. When* $A = (\mathbb{Z}_p^\times)^d$*, we put* $\alpha|_A = \alpha^*$*.*

For $A \in CO_d$, $\alpha \in \mathcal{M}_d$, and $f \in C(\mathbb{Z}_p^d, R)$, we introduce the notation

$$\int_A f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha|_A(\boldsymbol{x}).$$

The following shows that $\alpha|_A \in \mathcal{M}_d$, as claimed.

Lemma 16. *Let $\alpha \in \mathcal{M}_d$ and $A \in CO_d$. Let $g_A(\boldsymbol{x})$ be the characteristic function of A. Then for all $f \in C(\mathbb{Z}_p^d, R)$ we have*

$$\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha_{g_A}(\boldsymbol{x}) = \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) g_A(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = \int_A f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}).$$

Proof. By linearity and continuity, it will suffice to verify the Lemma for characteristic functions of polyballs in $\mathbb{Z}_p^d$. Let $\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d$ be a polyball in $\mathbb{Z}_p^d$ and let f be its characteristic function. We have

$$\begin{aligned}\int_A f(\boldsymbol{x})\, d\alpha(x) &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha|_A(\boldsymbol{x}) \\ &= \alpha|_A(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d) \\ &= \alpha(A \cap \boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d).\end{aligned}$$

But $f(\boldsymbol{x}) g_A(\boldsymbol{x})$ is the characteristic function of the set $A \cap \boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d$, so we likewise have

$$\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha_{g_A}(\boldsymbol{x}) = \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) g_A(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = \alpha(A \cap \boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d).$$

□

Note that for $A, B \in CO_d$, and $\alpha \in \mathcal{M}_d$,

$$(\alpha|_A)|_B = \alpha|_{A \cap B} = (\alpha|_B)|_A.$$

Definition 22. *Let $\alpha \in \mathcal{M}_d$. We say that α is supported on $A \in CO_d$, if $\alpha|_A = \alpha$, i.e., if for all $f \in C(\mathbb{Z}_p^d, R)$ we have*

$$\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = \int_A f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}).$$

Note that if α is supported on both $A, B \in CO_d$, then α is supported on $A \cap B \in CO_d$. This observation motivates the following definition.

Definition 23. *Let $\alpha \in \mathcal{M}_d$. Let $\{A_i : i \in I\} \subseteq CO_d$ be the collection of all compact open sets on which α is supported. The support of α is the closed set*

$$\operatorname{supp}(\alpha) = \bigcap_{i \in I} A_i.$$

Note that $\operatorname{supp}(\alpha)$ need not be an element of CO_d. By definition, however, $\operatorname{supp}(\alpha)$ is the largest closed set contained in every compact open subset of $\mathbb{Z}_p^d$ on which α is supported.

Example 15. *The zero measure, α_0 (Example 2) has $\operatorname{supp}(\alpha_0) = \emptyset$. Indeed, α_0 is supported on all $A \in CO_d$, and in particular on $\emptyset$.* $\Diamond$

Example 16. *Let $\boldsymbol{s} \in \mathbb{Z}_p^d$ and let $\delta_{\boldsymbol{s}}$ denote Dirac measure of mass 1 centered at $\boldsymbol{s}$. For each $\boldsymbol{n} \in \mathbb{N}^d$, $\delta_{\boldsymbol{s}}$ is supported on $\boldsymbol{s} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d$, the translate of $\boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d$ by $\boldsymbol{s}$. Thus,*

$$\operatorname{supp}(\delta_{\boldsymbol{s}}) \subseteq \bigcap_{\boldsymbol{n} \in \mathbb{N}^d} (\boldsymbol{s} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d) = \{\boldsymbol{s}\}.$$

Now let $A \in CO_d$. Then for any $B \in CO_d$, we have

$$\delta_{\boldsymbol{s}}|_A(B) = \delta_{\boldsymbol{s}}(A \cap B) = \begin{cases} 1 & : \boldsymbol{s} \in A \cap B \\ 0 & : \textit{else} \end{cases},$$

so $\delta_{\boldsymbol{s}}$ is supported on $A \in CO_d$ if and only if $\boldsymbol{s} \in A$. Consequently, $\boldsymbol{s} \in \operatorname{supp}(\delta_{\boldsymbol{s}})$, giving $\operatorname{supp}(\delta_{\boldsymbol{s}}) = \{\boldsymbol{s}\}$. $\Diamond$

Lemma 17. *Let $\alpha \in \mathcal{M}_d$. Fix $\boldsymbol{s} \in \mathbb{Z}_p^d$ and let $\delta_{\boldsymbol{s}}$ denote Dirac measure of mass 1 centered at $\boldsymbol{s}$. Then $\operatorname{supp}(\alpha * \delta_{\boldsymbol{s}}) = \boldsymbol{s} + \operatorname{supp}(\alpha)$.*

Proof. If $\boldsymbol{s} = \boldsymbol{0}$, then $\alpha * \delta_{\boldsymbol{s}} = \alpha$ and the result holds trivially. Suppose now that $\boldsymbol{s} \neq \boldsymbol{0}$. Let $A \in CO_d$ be any set on which α is supported. From Lemma 15 and Lemma 16, we have

$$(\alpha * \delta_{\boldsymbol{s}})|_{\boldsymbol{s}+A} = \alpha|_{-\boldsymbol{s}+\boldsymbol{s}+A} * \delta_{\boldsymbol{s}} = \alpha|_A * \delta_{\boldsymbol{s}} = \alpha * \delta_{\boldsymbol{s}}.$$

Since translation by $\boldsymbol{s}$ is an isometry $\mathbb{Z}_p^d \to \mathbb{Z}_p^d$, this gives

$$\operatorname{supp}(\alpha * \delta_{\boldsymbol{s}}) \subseteq \boldsymbol{s} + \operatorname{supp}(\alpha).$$

On the other hand, if $\alpha * \delta_{\boldsymbol{s}}$ is supported on $A \in CO_d$, then

$$\alpha * \delta_{\boldsymbol{s}} = (\alpha * \delta_{\boldsymbol{s}})|_A = \alpha|_{-\boldsymbol{s}+A} * \delta_{\boldsymbol{s}}.$$

Since $\delta_{\boldsymbol{s}} \neq \alpha_0$, this gives $\alpha = \alpha|_{-\boldsymbol{s}+A}$. Consequently,

$$\operatorname{supp}(\alpha) \subseteq -\boldsymbol{s} + \operatorname{supp}(\alpha * \delta_{\boldsymbol{s}}).$$

Finally, $\operatorname{supp}(\alpha * \delta_{\boldsymbol{s}}) = \boldsymbol{s} + \operatorname{supp}(\alpha)$, as claimed. □

Corollary 13. *Let $A \in CO_d$ and $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$. For all $f \in C(\mathbb{Z}_p^d, R)$ we have*

$$\int_{\boldsymbol{a}A} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = \int_A f(\boldsymbol{a}\boldsymbol{x})\, d\alpha(\boldsymbol{a}\boldsymbol{x}).$$

Proof. Let g_A and $g_{\boldsymbol{a}A}$ denote the characteristic functions of A and $\boldsymbol{a}A$, respectively. Then for all $\boldsymbol{x} \in \mathbb{Z}_p^d$, $g_{\boldsymbol{a}A}(\boldsymbol{a}\boldsymbol{x}) = g_A(\boldsymbol{x})$. Lemma 11 yields

$$\begin{aligned}
\int_{\boldsymbol{a}A} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) g_{\boldsymbol{a}A}(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{a}\boldsymbol{x}) g_{\boldsymbol{a}A}(\boldsymbol{a}\boldsymbol{x})\, d\alpha(\boldsymbol{a}\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{a}\boldsymbol{x}) g_A(\boldsymbol{x})\, d\alpha(\boldsymbol{a}\boldsymbol{x}) \\
&= \int_A f(\boldsymbol{a}\boldsymbol{x})\, d\alpha(\boldsymbol{a}\boldsymbol{x}),
\end{aligned}$$

as claimed. □

Corollary 14. *Let* $\alpha \in \mathcal{M}_d$ *and* $A, B \in CO_d$ *with* $A \cap B = \emptyset$. *Then for any* $f \in C(\mathbb{Z}_p^d, R)$,

$$\int_{A \cup B} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = \int_A f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) + \int_B f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}).$$

Proof. Let g_A denote the characteristic function of A, g_B denote the characteristic function of B and $g_{A\cup B}$ denote the characteristic function of $A \cup B$. In this case,

$$g_{A\cup B}(\boldsymbol{x}) = g_A(\boldsymbol{x}) + g_B(\boldsymbol{x})$$

for all $\boldsymbol{x} \in \mathbb{Z}_p^d$. Consequently, for $f \in C(\mathbb{Z}_p^d, R)$,

$$\begin{aligned}
\int_{A\cup B} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) g_{A\cup B}(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})(g_A(\boldsymbol{x}) + g_B(\boldsymbol{x}))\, d\alpha(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) g_A(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) + \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) g_B(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \int_A f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) + \int_B f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}).
\end{aligned}$$

□

Example 17. *Let* $\boldsymbol{s} \in \mathbb{Z}_p^d$ *and let* $\delta_{\boldsymbol{s}}$ *denote Dirac measure of mass 1 centered at* $\boldsymbol{s}$. *In Example 14, we saw* $(\delta_{\boldsymbol{s}})_g = g(\boldsymbol{s})\delta_{\boldsymbol{s}}$ *for any* $g \in C(\mathbb{Z}_p^d, R)$. *By taking* g *to be the characteristic function of some* $A \in CO_d$, *we readily obtain*

$$\delta_{\boldsymbol{s}}|_A = \begin{cases} \delta_{\boldsymbol{s}} & : \boldsymbol{s} \in A \\ \alpha_0 & : \textit{else} \end{cases}.$$

This agrees with the discussion in Example 16. ◊

Proposition 11. *Let* $\alpha \in \mathcal{M}_d$. *The power series associated to the measure* $\alpha^* = \alpha|_{(\mathbb{Z}_p^\times)^d}$ *is given by*

$$\widehat{\alpha^*}(\boldsymbol{T}) = \widehat{\alpha}(\boldsymbol{T}) - \frac{1}{p^d} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}} = \boldsymbol{1}} \left(\sum_{\boldsymbol{a} \prec \boldsymbol{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \right) \widehat{\alpha}(\boldsymbol{\zeta} \boldsymbol{T}),$$

where the outermost sum is over all d-tuples $\boldsymbol{\zeta} \in \mathbb{C}_p^d$ *satisfying* $\boldsymbol{\zeta}^{\wedge \boldsymbol{p}} = \mathbf{1}$.

Proof. Let $A = (\mathbb{Z}_p^\times)^d$ and let g_A denote the characteristic function of A. Recall that $\mathbb{Z}_p^\times = \mathbb{Z}_p \setminus p\mathbb{Z}_p$. Thus, g_A is locally constant, factoring through $L_{\mathbf{1}}$. Lemma 14 gives

$$\widehat{\alpha^*}(\boldsymbol{T}) = \frac{1}{p^d} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}} = \mathbf{1}} \widehat{g_A}(\boldsymbol{\zeta}) \widehat{\alpha}(\boldsymbol{\zeta T}).$$

For each such d-tuple $\boldsymbol{\zeta} \in \mathbb{C}_p^d$, recall

$$\widehat{g_A}(\boldsymbol{\zeta}) = \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p} - \mathbf{1}} g_A(\boldsymbol{a}) \boldsymbol{\zeta}^{-\boldsymbol{a}}.$$

But for $\boldsymbol{a} \in \mathbb{N}^d$ with $\boldsymbol{a} \preccurlyeq_d \boldsymbol{p} - \mathbf{1}$, $g_A(\boldsymbol{a}) \neq 0$ if and only if $a_i \neq 0$ for all $1 \leq i \leq d$. Since g_A is a characteristic function, we obtain

$$\begin{aligned}
\widehat{g_A}(\boldsymbol{\zeta}) &= \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p} - \mathbf{1}} g_A(\boldsymbol{a}) \boldsymbol{\zeta}^{-\boldsymbol{a}} \\
&= \sum_{\mathbf{1} \preccurlyeq_d \boldsymbol{a} \preccurlyeq_d \boldsymbol{p} - \mathbf{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \\
&= \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p} - \mathbf{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} - \sum_{\boldsymbol{a} \prec \mathbf{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}}
\end{aligned}$$

This gives

$$\widehat{\alpha^*}(\boldsymbol{T}) = \frac{1}{p^d} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}} = \mathbf{1}} \left(\sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p} - \mathbf{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \right) \widehat{\alpha}(\boldsymbol{\zeta T}) - \frac{1}{p^d} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}} = \mathbf{1}} \left(\sum_{\boldsymbol{a} \prec_d \mathbf{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \right) \widehat{\alpha}(\boldsymbol{\zeta T})$$

But Corollary 11 gives

$$\widehat{\alpha}(\boldsymbol{T}) = \frac{1}{p^d} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}} = \mathbf{1}} \left(\sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p} - \mathbf{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \right) \widehat{\alpha}(\boldsymbol{\zeta T}),$$

so that

$$\widehat{\alpha^*}(\boldsymbol{T}) = \widehat{\alpha}(\boldsymbol{T}) - \frac{1}{p^d} \sum_{\boldsymbol{\zeta}^{\wedge \boldsymbol{p}} = 1} \left(\sum_{\boldsymbol{a} \prec_d \mathbf{1}} \boldsymbol{\zeta}^{-\boldsymbol{a}} \right) \widehat{\alpha}(\boldsymbol{\zeta T}),$$

as claimed. $\square$

Lemma 18. *If $\alpha \in \mathcal{M}_d$, $A \in CO_d$, and $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$, then $(\alpha \circ \boldsymbol{a})|_A = \alpha|_{\boldsymbol{a}A} \circ \boldsymbol{a}$.*

Proof. Let g_A denote the characteristic function of A. Let $f \in C(\mathbb{Z}_p^d, R)$ be arbitrary. Note that $g_A(\boldsymbol{a}^{-1}\boldsymbol{x})$ gives the characteristic function of $\boldsymbol{a}A$, which we denote by $g_{\boldsymbol{a}A}$. We have

$$\begin{aligned}
\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d(\alpha \circ \boldsymbol{a})|_A(\boldsymbol{x}) &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) g_A(\boldsymbol{x})\, d\alpha(\boldsymbol{a}\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{a}^{-1}\boldsymbol{x}) g_A(\boldsymbol{a}^{-1}\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{a}^{-1}\boldsymbol{x}) g_{\boldsymbol{a}A}(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{a}^{-1}\boldsymbol{x})\, d\alpha|_{\boldsymbol{a}A}(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha|_{\boldsymbol{a}A}(\boldsymbol{a}\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d(\alpha|_{\boldsymbol{a}A} \circ \boldsymbol{a})(\boldsymbol{x}).
\end{aligned}$$

This gives the claimed equality of measures. □

Corollary 15. *For any $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$, $(\alpha \circ \boldsymbol{a})^* = \alpha^* \circ \boldsymbol{a}$.*

Proof. Let $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$. Then $\boldsymbol{a}(\mathbb{Z}_p^\times)^d = (\mathbb{Z}_p^\times)^d$, and we may apply Lemma 18. □

Corollary 16. *Let $\alpha \in \mathcal{M}_d$ and $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$. Then*

$$\mathrm{supp}(\alpha \circ \boldsymbol{a}) = \boldsymbol{a}^{-1}\mathrm{supp}(\alpha).$$

Proof. Suppose α is supported on $A \in CO_d$, then

$$(\alpha \circ \boldsymbol{a})|_{\boldsymbol{a}^{-1}A} = \alpha|_{\boldsymbol{a}\boldsymbol{a}^{-1}A} \circ \boldsymbol{a} = \alpha|_A \circ \boldsymbol{a} = \alpha \circ \boldsymbol{a}.$$

Consequently,

$$\mathrm{supp}(\alpha \circ \boldsymbol{a}) \subseteq \boldsymbol{a}^{-1}\mathrm{supp}(\alpha).$$

Now suppose $\alpha \circ \boldsymbol{a}$ is supported on $A \in CO_d$, then

$$\alpha \circ \boldsymbol{a} = (\alpha \circ \boldsymbol{a})|_A = \alpha_{\boldsymbol{a}A} \circ \boldsymbol{a}.$$

By the injectivity of the map $\alpha \mapsto \alpha \circ \boldsymbol{a}$ (Lemma 12), we have $\alpha|_{\boldsymbol{a}A} = \alpha$. This gives

$$\mathrm{supp}(\alpha) \subseteq \boldsymbol{a}\mathrm{supp}(\alpha \circ \boldsymbol{a}).$$

□

Corollary 17. *Let* $\alpha \in \mathcal{M}_d$ *and suppose* $\mathrm{supp}(\alpha) \subseteq (\mathbb{Z}_p^\times)^d$. *Then* $\mathrm{supp}(\alpha \circ \boldsymbol{a}) \subseteq (\mathbb{Z}_p^\times)^d$ *for all* $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$.

Recall $\mathbb{Z}_p^\times \cong V \times U$, where V is the set of $(p-1)$th roots of unity in $\mathbb{Z}_p$ and $U = 1 + p\mathbb{Z}_p$, which is topologically cyclic (see Appendix A). Fix an element $\boldsymbol{u} \in U^d$ such that u_i is a topological generator of U for each $1 \leq i \leq d$. Then the map $\varphi_{\boldsymbol{u}} : \mathbb{Z}_p^d \to U^d$ given by $\varphi_{\boldsymbol{u}} : \boldsymbol{x} \mapsto \boldsymbol{u}^{\wedge \boldsymbol{x}}$ is a topological group isomorphism. Moreover, $\varphi_{\boldsymbol{u}} : L_{\boldsymbol{n}} \to L_{\boldsymbol{n}+\boldsymbol{1}}$ for all $\boldsymbol{n} \in \mathbb{N}^d$. If $A = A_1 \times \cdots \times A_d$ with each A_i compact open in $\mathbb{Z}_p$, then

$$\varphi_{\boldsymbol{u}}(A) = \varphi_{u_1}(A_1) \times \cdots \times \varphi_{u_d}(A_d),$$

where for $1 \leq i \leq d$, $\varphi_{u_i} : \mathbb{Z}_p \to U$ is as in Appendix A. The inverse of $\varphi_{\boldsymbol{u}}$ is the map $\ell_{\boldsymbol{u}} : U^d \to \mathbb{Z}_p$ given by

$$\ell_{\boldsymbol{u}}(\boldsymbol{y}) = \left(\frac{\log y_1}{\log u_1}, \ldots, \frac{\log y_d}{\log u_d} \right),$$

where log is the p-adic logarithm.

Definition 24. *For* $\alpha \in \mathcal{M}_d$, *define the measure* $\alpha \circ \varphi_{\boldsymbol{u}}$ *by* $(\alpha \circ \varphi_{\boldsymbol{u}})(A) = \alpha|_{U^d}(\varphi_{\boldsymbol{u}}(A))$ *for* $A \in CO_d$.

Clearly

$$\|\alpha \circ \varphi_{\boldsymbol{u}}\|_u \leq \|\alpha|_{U^d}\|_u \leq \|\alpha\|_u.$$

Since $\varphi_{\boldsymbol{u}}(O_1 \cup O_2) = \varphi_{\boldsymbol{u}}(O_1) \cup \varphi_{\boldsymbol{u}}(O_2)$ for all $O_1, O_2 \in CO_d$, and $\varphi_{\boldsymbol{u}}(O_1) \cap \varphi_{\boldsymbol{u}}(O_2) = \emptyset$ whenever $O_1 \cap O_2 = \emptyset$, we have that $\alpha \circ \varphi_{\boldsymbol{u}} \in \mathcal{M}_d$ as claimed.

Example 18. *Let $\delta_{\boldsymbol{s}}$ denote Dirac measure of mass 1 centered at $\boldsymbol{s}$. In Example 17, we saw*

$$\delta_{\boldsymbol{s}}|_{U^d} = \begin{cases} \delta_{\boldsymbol{s}} & : \boldsymbol{s} \in U^d \\ \alpha_0 & : else \end{cases} .$$

If $\boldsymbol{s} \in U^d$, then $\varphi_{\boldsymbol{u}}^{-1}(\boldsymbol{s}) \in \mathbb{Z}_p^d$, and we have for $O \in CO_d$,

$$\begin{aligned} (\delta_{\boldsymbol{s}} \circ \varphi_{\boldsymbol{u}})(O) = \delta_{\boldsymbol{s}}|_{U^d}(\varphi_{\boldsymbol{u}}(O)) &= \delta_{\boldsymbol{s}}(\varphi_{\boldsymbol{u}}(O)) \\ &= \begin{cases} 1 & : \boldsymbol{s} \in \varphi_{\boldsymbol{u}}(O) \\ 0 & : else \end{cases} \\ &= \begin{cases} 1 & : \varphi_{\boldsymbol{u}}^{-1}(\boldsymbol{s}) \in O \\ 0 & : else \end{cases} \\ &= \delta_{\varphi_{\boldsymbol{u}}^{-1}(\boldsymbol{s})}(O). \end{aligned}$$

If $\boldsymbol{s} \notin U^d$; then for $O \in CO_d$,

$$(\delta_{\boldsymbol{s}} \circ \varphi_{\boldsymbol{u}})(O) = \delta_{\boldsymbol{s}}|_{U^d}(\varphi_{\boldsymbol{u}}(O)) = \alpha_0(\varphi_{\boldsymbol{u}}(O)) = 0.$$

Therefore,

$$\delta_{\boldsymbol{s}} \circ \varphi_{\boldsymbol{u}} = \begin{cases} \delta_{\varphi_{\boldsymbol{u}}^{-1}(\boldsymbol{s})} & : \boldsymbol{s} \in U^d \\ \alpha_0 & : else \end{cases} .$$

◊

Lemma 19. *For any $\alpha, \beta \in \mathcal{M}_d$ and $c \in R$, $(\alpha + \beta) \circ \varphi_{\boldsymbol{u}} = (\alpha \circ \varphi_{\boldsymbol{u}}) + (\beta \circ \varphi_{\boldsymbol{u}})$ and $(c\alpha) \circ \varphi_{\boldsymbol{u}} = c(\alpha \circ \varphi_{\boldsymbol{u}})$.*

Proof. By Lemmas 13 and 16, $(\alpha+\beta)|_{U^d} = \alpha|_{U^d} + \beta|_{U^d}$ and $(c\alpha)|_{U^d} = c(\alpha|_{U^d})$. Thus, for $O \in CO_d$,

$$\begin{aligned}((\alpha+\beta)\circ\varphi_{\boldsymbol{u}})(O) &= (\alpha+\beta)|_{U^d}(\varphi_{\boldsymbol{u}}(O)) \\ &= \alpha|_{U^d}(\varphi_{\boldsymbol{u}}(O)) + \beta|_{U^d}(\varphi_{\boldsymbol{u}}(O)) \\ &= (\alpha\circ\varphi_{\boldsymbol{u}})(O) + (\beta\circ\varphi_{\boldsymbol{u}})(O)\end{aligned}$$

and

$$\begin{aligned}(c\alpha)\circ\varphi_{\boldsymbol{u}})(O) &= (c\alpha)|_{U^d}(\varphi_{\boldsymbol{u}}(O)) \\ &= c\alpha|_{U^d}(\varphi_{\boldsymbol{u}}(O)) \\ &= c(\alpha\circ\varphi_{\boldsymbol{u}})(O).\end{aligned}$$

□

Lemma 20. *For any $f \in C(\mathbb{Z}_p^d, R)$, we have*

$$\int_{\mathbb{Z}_p^d} f(\varphi_{\boldsymbol{u}}(\boldsymbol{x}))\, d(\alpha\circ\varphi_{\boldsymbol{u}})(\boldsymbol{x}) = \int_{U^d} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}).$$

Proof. By linearity and continuity, it suffices to consider the case where f is the characteristic function of a polyball A. Since $\varphi_{\boldsymbol{u}}$ is a topological group isomorphism, the function $f(\varphi_{\boldsymbol{u}}(\boldsymbol{x}))$ is the characteristic function of the compact open set $\varphi_{\boldsymbol{u}}^{-1}(U^d \cap A) \subseteq \mathbb{Z}_p^d$. Thus,

$$\begin{aligned}\int_{\mathbb{Z}_p^d} f(\varphi_{\boldsymbol{u}}(\boldsymbol{x}))\, d(\alpha\circ\varphi_{\boldsymbol{u}})(\boldsymbol{x}) &= (\alpha\circ\varphi_{\boldsymbol{u}})(\varphi_{\boldsymbol{u}}^{-1}(U^d\cap A)) \\ &= \alpha|_{U^d}(\varphi_{\boldsymbol{u}}(\varphi_{\boldsymbol{u}}^{-1}(U^d\cap A))) \\ &= \alpha|_{U^d}(U^d\cap A) \\ &= \alpha|_{U^d}(A) \\ &= \int_{U^d} f(\boldsymbol{x})\, d\alpha(\boldsymbol{x}).\end{aligned}$$

□

For $f \in C(\mathbb{Z}_p^d, R)$, the composition $f \circ \varphi_{\boldsymbol{u}}^{-1} : U^d \to R$ is continuous when U^d is given the subspace topology. Since U^d is both open and closed in $\mathbb{Z}_p^d$, we may extend this composition by 0 to obtain a continuous function $\mathbb{Z}_p^d \to R$, and we will not distinguish notationally between the function with domain U^d and the function with domain $\mathbb{Z}_p^d$. That is, for $\boldsymbol{x} \in \mathbb{Z}_p^d$, we put

$$f(\varphi_{\boldsymbol{u}}^{-1}(\boldsymbol{x})) = \begin{cases} f(\varphi_{\boldsymbol{u}}^{-1}(\boldsymbol{x})) & : \boldsymbol{x} \in U^d \\ 0 & : \boldsymbol{x} \notin U^d \end{cases}.$$

Corollary 18. *For any $f \in C(\mathbb{Z}_p^d, R)$, we have*

$$\int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d(\alpha \circ \varphi_{\boldsymbol{u}})(\boldsymbol{x}) = \int_{U^d} f(\varphi_{\boldsymbol{u}}^{-1}(\boldsymbol{x}))\, d\alpha(\boldsymbol{x}).$$

Proof. Since $\varphi_{\boldsymbol{u}}(\boldsymbol{x}) \in U^d$ for all $\boldsymbol{x} \in \mathbb{Z}_p^d$,

$$f(\varphi_{\boldsymbol{u}}^{-1}(\varphi_{\boldsymbol{u}}(\boldsymbol{x}))) = f(\boldsymbol{x})$$

for all $\boldsymbol{x} \in \mathbb{Z}_p^d$. The result now holds by Lemma 20. □

2.4.1 The Γ-Transform

Let ω and $\langle \cdot \rangle$ denote the projections onto the first and second components of the decomposition $\mathbb{Z}_p^\times \cong V \times U$, respectively. For $\boldsymbol{x} \in (\mathbb{Z}_p^\times)^d$, we let $\boldsymbol{\omega}(\boldsymbol{x}) = (\omega(x_1), \ldots, \omega(x_d)) \in V^d$ and $\langle \boldsymbol{x} \rangle = (\langle x_1 \rangle, \ldots \langle x_d \rangle) \in U^d$. Note that for any $\boldsymbol{x} \in U^d$ and any $\boldsymbol{\eta} \in V^d$, we have

$$\langle \boldsymbol{\eta x} \rangle = \langle \boldsymbol{x} \rangle = \boldsymbol{x} \qquad \text{and} \qquad \boldsymbol{\omega}(\boldsymbol{\eta x}) = \boldsymbol{\omega}(\boldsymbol{\eta}) = \boldsymbol{\eta}.$$

Since $(\mathbb{Z}_p^\times)^d$ is both open and closed in $\mathbb{Z}_p^d$, we may extend $\boldsymbol{\omega}$ and $\langle \cdot \rangle$ by zero to obtain continuous functions $\mathbb{Z}_p^d \to R$, and we will not distinguish notationally between the functions on $\mathbb{Z}_p^d$ and on $(\mathbb{Z}_p^\times)^d$. Using the above, we can write $(\mathbb{Z}_p^\times)^d$ as a disjoint union:

$$(\mathbb{Z}_p^\times)^d = \bigsqcup_{\boldsymbol{\eta} \in V^d} \boldsymbol{\eta} U^d.$$

For $\boldsymbol{i} \in \mathbb{N}^d$ and $\boldsymbol{x} \in (\mathbb{Z}_p^\times)^d$, we put

$$\boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{x}) = (\boldsymbol{\omega}(\boldsymbol{x}))^{\boldsymbol{i}} = \prod_{j=1}^{d} \omega^{i_j}(x_j).$$

We likewise extend $\boldsymbol{\omega}^{\boldsymbol{i}}$ by zero to a continuous function $\mathbb{Z}_p^d \to R$.

Example 19. *Fix $\boldsymbol{i} \preccurlyeq_d \boldsymbol{p} - \boldsymbol{2}$, and consider $\boldsymbol{\omega}^{\boldsymbol{i}} \in C(\mathbb{Z}_p^d, R)$. Let g denote the characteristic function of $(\mathbb{Z}_p^\times)^d$. Then $\boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{x}) = \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{x})g(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{Z}_p^d$. Consequently, for $f \in C(\mathbb{Z}_p^d, R)$, we have*

$$\begin{aligned}
\int_{(\mathbb{Z}_p^\times)^d} f(\boldsymbol{x})\, d\alpha_{\boldsymbol{\omega}^{\boldsymbol{i}}} &= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) g(\boldsymbol{x}) \omega^{\boldsymbol{i}}(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x}) \omega^{\boldsymbol{i}}(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} f(\boldsymbol{x})\, d\alpha_{\boldsymbol{\omega}^{\boldsymbol{i}}}.
\end{aligned}$$

Thus, $(\alpha_{\boldsymbol{\omega}^{\boldsymbol{i}}})^* = \alpha_{\boldsymbol{\omega}^{\boldsymbol{i}}}$. ♢

Definition 25. *Let $\boldsymbol{i} \in \mathbb{N}^d$ with $\boldsymbol{i} \preccurlyeq_d \boldsymbol{p} - \boldsymbol{2}$. The $\gamma^{(\boldsymbol{i})}$-transform of a measure $\alpha \in \mathcal{M}_d$ is the measure*

$$\gamma^{(\boldsymbol{i})}(\alpha) = \sum_{\boldsymbol{\eta} \in V^d} \boldsymbol{\eta}^{\boldsymbol{i}} \alpha \circ \boldsymbol{\eta} = \sum_{\boldsymbol{\eta} \in V^d} \boldsymbol{\eta}^{\boldsymbol{i}} (\alpha \circ \boldsymbol{\eta}).$$

When $\boldsymbol{i} = \boldsymbol{0}$, we instead write $\gamma(\alpha)$ for $\gamma^{(\boldsymbol{0})}(\alpha)$.

Lemma 21. *Let $\alpha, \beta \in \mathcal{M}_d$ and $c \in R$. Then for each $\boldsymbol{i} \in \mathbb{N}^d$ with $\boldsymbol{i} \preccurlyeq_d \boldsymbol{p} - \boldsymbol{2}$, $\gamma^{(\boldsymbol{i})}(\alpha + \beta) = \gamma^{(\boldsymbol{i})}(\alpha) + \gamma^{(\boldsymbol{i})}(\beta)$ and $\gamma^{(\boldsymbol{i})}(c\alpha) = c\gamma^{(\boldsymbol{i})}(\alpha)$.*

Proof. This follows immediately from Lemmas 12 and 13. □

Lemma 22. *Suppose* $\alpha \in \mathcal{M}_d$ *with* $\mathrm{supp}(\alpha) \subseteq (\mathbb{Z}_p^\times)^d$. *Then for any* $\boldsymbol{i} \preccurlyeq_d \boldsymbol{p-2}$, $\mathrm{supp}(\gamma^{(\boldsymbol{i})}(\alpha)) \subseteq (\mathbb{Z}_p^\times)^d$.

Proof. By Lemma 13 and Lemma 18, we find

$$\begin{aligned}
\gamma^{(\boldsymbol{i})}(\alpha)|_{(\mathbb{Z}_p^\times)^d} &= \left(\sum_{\boldsymbol{\eta}\in V^d} \boldsymbol{\eta}^{\boldsymbol{i}}(\alpha \circ \boldsymbol{\eta})\right)\Bigg|_{(\mathbb{Z}_p^\times)^d} \\
&= \sum_{\boldsymbol{\eta}\in V^d} \boldsymbol{\eta}^{\boldsymbol{i}}(\alpha \circ \boldsymbol{\eta})|_{(\mathbb{Z}_p^\times)^d} \\
&= \sum_{\boldsymbol{\eta}\in V^d} \boldsymbol{\eta}^{\boldsymbol{i}}(\alpha|_{\boldsymbol{\eta}(\mathbb{Z}_p^\times)^d} \circ \boldsymbol{\eta}) \\
&= \sum_{\boldsymbol{\eta}\in V^d} \boldsymbol{\eta}^{\boldsymbol{i}}(\alpha|_{(\mathbb{Z}_p^\times)^d} \circ \boldsymbol{\eta}) \\
&= \sum_{\boldsymbol{\eta}\in V^d} \boldsymbol{\eta}^{\boldsymbol{i}}(\alpha \circ \boldsymbol{\eta}) \\
&= \gamma^{(\boldsymbol{i})}(\alpha).
\end{aligned}$$

□

Definition 26. *Let* $\boldsymbol{i} \preccurlyeq_d \boldsymbol{p-2}$. *The* $\Gamma^{(\boldsymbol{i})}$*-transform of a measure* $\alpha \in \mathcal{M}_d$ *is the function* $\Gamma_\alpha^{(\boldsymbol{i})} : \mathbb{Z}_p^d \to R$ *given by*

$$\Gamma_\alpha^{(\boldsymbol{i})}(\boldsymbol{s}) = \lim_{\boldsymbol{k}} \int_{\mathbb{Z}_p^d} \boldsymbol{x}^{\boldsymbol{k}} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = \int_{(\mathbb{Z}_p^\times)^d} \langle \boldsymbol{x} \rangle^{\boldsymbol{s}} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{x})\, d\alpha(\boldsymbol{x})$$

where the d-net $\boldsymbol{k}$ *in* $\mathbb{N}^d$ *is such that for each* i, *the* k_i *form a sequence such that* $k_i \to s_i$ *p-adically,* $k_i \to \infty$ *in the Archimedean sense, and* $k_i \equiv 0 \pmod{p-1}$. *When* $\boldsymbol{i} = \boldsymbol{0}$, *we instead write* $\Gamma_\alpha(\boldsymbol{s})$ *for* $\Gamma_\alpha^{(\boldsymbol{0})}(\boldsymbol{s})$.

Notice that for each $\boldsymbol{i} \preccurlyeq_d \boldsymbol{p-2}$, $\Gamma_\alpha^{(\boldsymbol{i})} = \Gamma_{\alpha_{\boldsymbol{\omega}^{\boldsymbol{i}}}}$, where $\boldsymbol{\omega}^{\boldsymbol{i}} \in C(\mathbb{Z}_p^d, R)$. Notice further that $\Gamma_\alpha^{(\boldsymbol{i})} = \Gamma_{\alpha^*}^{(\boldsymbol{i})}$ for each $\boldsymbol{i} \preccurlyeq_d \boldsymbol{p-2}$.

Lemma 23. *Let* $\boldsymbol{i} \preccurlyeq_d \boldsymbol{p-2}$*. For any* $\boldsymbol{k} \in \mathbb{N}^d$ *with* $\boldsymbol{k} \equiv \boldsymbol{i} \pmod{\boldsymbol{p-2}}$*,*

$$\Gamma_\alpha^{(\boldsymbol{i})}(\boldsymbol{k}) = \boldsymbol{D}^{\boldsymbol{k}} \widehat{\alpha^*}(\boldsymbol{T})|_{\boldsymbol{T}=\boldsymbol{1}},$$

where $\boldsymbol{D}^{\boldsymbol{k}}$ *is as in Proposition 10.*

Proof. Let $\boldsymbol{k} \in \mathbb{N}^d$ with $\boldsymbol{k} \equiv \boldsymbol{i} \pmod{\boldsymbol{p-2}}$. Let $g \in C(\mathbb{Z}_p^d, R)$ denote the characteristic function of $(\mathbb{Z}_p^\times)^d$. Then for any $\boldsymbol{x} \in \mathbb{Z}_p^d$, we have

$$\boldsymbol{x}^{\boldsymbol{k}} g(\boldsymbol{x}) = \langle \boldsymbol{x} \rangle^{\boldsymbol{k}} \boldsymbol{\omega}^{\boldsymbol{k}}(\boldsymbol{x}) g(\boldsymbol{x}) = \langle \boldsymbol{x} \rangle^{\boldsymbol{k}} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{x}) g(\boldsymbol{x}).$$

Thus, from Proposition 10, we obtain

$$\begin{aligned}
\Gamma_\alpha^{(\boldsymbol{i})}(\boldsymbol{k}) &= \int_{(\mathbb{Z}_p^\times)^d} \langle \boldsymbol{x} \rangle^{\boldsymbol{k}} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} \langle \boldsymbol{x} \rangle^{\boldsymbol{k}} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{x}) g(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} \langle \boldsymbol{x} \rangle^{\boldsymbol{k}} \boldsymbol{\omega}^{\boldsymbol{k}}(\boldsymbol{x}) g(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} \boldsymbol{x}^{\boldsymbol{k}} g(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \int_{\mathbb{Z}_p^d} \boldsymbol{x}^{\boldsymbol{k}}\, d\alpha^*(\boldsymbol{x}) \\
&= m_{\boldsymbol{k}}(\alpha^*) \\
&= \boldsymbol{D}^{\boldsymbol{k}} \widehat{\alpha^*}(\boldsymbol{T})|_{\boldsymbol{T}=\boldsymbol{1}}.
\end{aligned}$$

□

Lemma 24. *Let* $\alpha, \beta \in \mathcal{M}_d$ *and* $c \in R$*. Then for each* $\boldsymbol{i} \preccurlyeq_d \boldsymbol{p-2}$*,* $\Gamma_{(\alpha+\beta)}^{(\boldsymbol{i})} = \Gamma_\alpha^{(\boldsymbol{i})} + \Gamma_\beta^{(\boldsymbol{i})}$ *and* $\Gamma_{c\alpha}^{(\boldsymbol{i})} = c\Gamma_\alpha^{(\boldsymbol{i})}$*.*

Proof. This follows immediately from Lemma 13. □

Theorem 4. *Let* $\alpha \in \mathcal{M}_d$*. For each* $\boldsymbol{i} \preccurlyeq_d \boldsymbol{p-2}$*, there exists a unique power series* $G_\alpha^{(\boldsymbol{i})}(\boldsymbol{T}) \in \Lambda_d$ *such that* $\Gamma_\alpha^{(\boldsymbol{i})}(\boldsymbol{s}) = G_\alpha^{(\boldsymbol{i})}(\boldsymbol{u}^{\wedge \boldsymbol{s}})$*.*

Proof. Recall that $\boldsymbol{u} \in U^d$ is such that u_i is a topological generator of U for each $1 \leq i \leq d$. We will use the partition of $(\mathbb{Z}_p^\times)^d$ given above:

$$(\mathbb{Z}_p^\times)^d = \bigsqcup_{\boldsymbol{\eta} \in V^d} \boldsymbol{\eta} U^d.$$

We have from Corollary 14

$$\begin{aligned}
\Gamma_\alpha^{(\boldsymbol{i})}(\boldsymbol{s}) &= \int_{(\mathbb{Z}_p^\times)^d} \langle \boldsymbol{x} \rangle^{\boldsymbol{s}} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) \\
&= \sum_{\boldsymbol{\eta} \in V^d} \int_{\boldsymbol{\eta} U^d} \langle \boldsymbol{x} \rangle^{\boldsymbol{s}} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{x})\, d\alpha(\boldsymbol{x}).
\end{aligned}$$

But for each $\boldsymbol{\eta} \in V^d$, Corollary 13 and Lemma 12 give

$$\int_{\boldsymbol{\eta} U^d} \langle \boldsymbol{x} \rangle^{\boldsymbol{s}} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = \int_{U^d} \langle \boldsymbol{\eta x} \rangle^{\boldsymbol{s}} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{\eta x})\, d\alpha(\boldsymbol{\eta x}).$$

Let $g \in C(\mathbb{Z}_p^d, R)$ denote the characteristic function of U^d. For $\boldsymbol{x} \in U^d$,

$$\langle \boldsymbol{\eta x} \rangle^{\boldsymbol{s}} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{\eta x}) g(\boldsymbol{x}) = \langle \boldsymbol{x} \rangle^{\boldsymbol{s}} \boldsymbol{\eta}^{\boldsymbol{i}} g(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{s}} \boldsymbol{\eta}^{\boldsymbol{i}} g(\boldsymbol{x}),$$

while for $\boldsymbol{x} \notin U^d$,

$$0 = \langle \boldsymbol{\eta x} \rangle^{\boldsymbol{s}} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{\eta x}) g(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{s}} \boldsymbol{\eta}^{\boldsymbol{i}} g(\boldsymbol{x}),$$

where we extend $\boldsymbol{x} \mapsto \boldsymbol{x}^{\boldsymbol{s}}$ by zero to a continuous function $\mathbb{Z}_p^d \to R$. Consequently,

$$\begin{aligned}
\int_{U^d} \langle \boldsymbol{\eta x} \rangle^{\boldsymbol{s}} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{\eta x})\, d\alpha(\boldsymbol{\eta x}) &= \int_{U^d} \langle \boldsymbol{x} \rangle^{\boldsymbol{s}} \boldsymbol{\eta}^{\boldsymbol{i}}\, d\alpha(\boldsymbol{\eta x}) \\
&= \int_{U^d} \boldsymbol{x}^{\boldsymbol{s}}\, d(\boldsymbol{\eta}^{\boldsymbol{i}}(\alpha \circ \boldsymbol{\eta}))(\boldsymbol{x}) \\
&= \int_{U^d} \boldsymbol{x}^{\boldsymbol{s}}\, d(\boldsymbol{\eta}^{\boldsymbol{i}} \alpha \circ \boldsymbol{\eta})(\boldsymbol{x}).
\end{aligned}$$

Thus,

$$\begin{aligned}\Gamma_\alpha^{(\boldsymbol{i})}(\boldsymbol{s}) &= \sum_{\boldsymbol{\eta}\in V^d}\int_{\boldsymbol{\eta}U^d}\langle\boldsymbol{x}\rangle^{\boldsymbol{s}}\boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{x})\,d\alpha(\boldsymbol{x})\\ &= \sum_{\boldsymbol{\eta}\in V^d}\int_{U^d}\boldsymbol{x}^{\boldsymbol{s}}\,d(\boldsymbol{\eta}^{\boldsymbol{i}}\alpha\circ\boldsymbol{\eta})(\boldsymbol{x})\\ &= \int_{U^d}\boldsymbol{x}^{\boldsymbol{s}}\,d\left(\sum_{\boldsymbol{\eta}\in V^d}\boldsymbol{\eta}^{\boldsymbol{i}}\alpha\circ\boldsymbol{\eta}\right)(\boldsymbol{x})\\ &= \int_{U^d}\boldsymbol{x}^{\boldsymbol{s}}\,d(\gamma^{(\boldsymbol{i})}(\alpha))(\boldsymbol{x}).\end{aligned}$$

But now Lemma 20 gives

$$\begin{aligned}\Gamma_\alpha^{(\boldsymbol{i})}(\boldsymbol{s}) &= \int_{U^d}\boldsymbol{x}^{\boldsymbol{s}}\,d(\gamma^{(\boldsymbol{i})}(\alpha))(\boldsymbol{x})\\ &= \int_{\mathbb{Z}_p^d}(\boldsymbol{u}^{\wedge\boldsymbol{x}})^{\boldsymbol{s}}\,d(\gamma^{(\boldsymbol{i})}(\alpha)\circ\varphi_{\boldsymbol{u}})(\boldsymbol{x})\\ &= \int_{\mathbb{Z}_p^d}(\boldsymbol{u}^{\wedge\boldsymbol{s}})^{\boldsymbol{x}}\,d(\gamma^{(\boldsymbol{i})}(\alpha)\circ\varphi_{\boldsymbol{u}})(\boldsymbol{x})\\ &= (\gamma^{(\boldsymbol{i})}\widehat{(\alpha)\circ}\varphi_{\boldsymbol{u}})(\boldsymbol{u}^{\wedge\boldsymbol{s}}).\end{aligned}$$

Taking $G_\alpha^{(\boldsymbol{i})}(\boldsymbol{T}) = (\gamma^{(\boldsymbol{i})}\widehat{(\alpha)\circ}\varphi_{\boldsymbol{u}})(\boldsymbol{T})$ gives the existence result. Any such power series is clearly unique. □

The power series $G_\alpha^{(\boldsymbol{i})}(\boldsymbol{T})$ is called the *Iwasawa series* associated to $\Gamma_\alpha^{(\boldsymbol{i})}(\boldsymbol{s})$.

Corollary 19. *Let $\alpha,\beta\in\mathcal{M}_d$ and $c\in R$. Then for each $\boldsymbol{i}\in\mathbb{N}^d$ with $\boldsymbol{i}\preccurlyeq_d \boldsymbol{p}-\boldsymbol{2}$, $G_{(\alpha+\beta)}^{(\boldsymbol{i})}(\boldsymbol{T}) = G_\alpha^{(\boldsymbol{i})}(\boldsymbol{T}) + G_\beta^{(\boldsymbol{i})}(\boldsymbol{T})$ and $G_{c\alpha}^{(\boldsymbol{i})}(\boldsymbol{T}) = cG_\alpha^{(\boldsymbol{i})}(\boldsymbol{T})$.*

Proof. Since the proof of Theorem 4 gives $G_\alpha^{(\boldsymbol{i})}(\boldsymbol{T}) = \gamma^{(\boldsymbol{i})}\widehat{(\alpha)\circ}\varphi_{\boldsymbol{u}}(\boldsymbol{T})$, the Corollary follows from Lemma 19. □

Example 20. *Let* $\boldsymbol{s} \in \mathbb{Z}_p^d$*, and let* $\delta_{\boldsymbol{s}}$ *denote Dirac measure of mass 1 centered at* $\boldsymbol{s}$ *(Example 3). Let* $\boldsymbol{i} \preccurlyeq_d \boldsymbol{p-2}$*. From Example 13, we have*

$$\begin{aligned}\gamma^{(\boldsymbol{i})}(\delta_{\boldsymbol{s}}) &= \sum_{\boldsymbol{\eta} \in V^d} \boldsymbol{\eta}^{\boldsymbol{i}} \delta_{\boldsymbol{s}} \circ \boldsymbol{\eta} \\ &= \sum_{\boldsymbol{\eta} \in V^d} \boldsymbol{\eta}^{\boldsymbol{i}} (\delta_{\boldsymbol{s}} \circ \boldsymbol{\eta}) \\ &= \sum_{\boldsymbol{\eta} \in V^d} \boldsymbol{\eta}^{\boldsymbol{i}} \delta_{\boldsymbol{\eta}^{-1}\boldsymbol{s}}.\end{aligned}$$

As in Example 17, for each $\boldsymbol{\eta} \in V^d$*,*

$$\delta_{\boldsymbol{\eta}^{-1}\boldsymbol{s}}|_{U^d} = \begin{cases} \delta_{\boldsymbol{\eta}^{-1}\boldsymbol{s}} & : \boldsymbol{\eta}^{-1}\boldsymbol{s} \in U^d \\ 0 & : \textit{else} \end{cases}.$$

Of course, if $\boldsymbol{s} \notin (\mathbb{Z}_p^\times)^d$*, then* $\boldsymbol{\eta}^{-1}\boldsymbol{s} \notin U^d$ *for all* $\boldsymbol{\eta} \in V^d$*. If* $\boldsymbol{s} \in (\mathbb{Z}_p^\times)^d$*, then* $\boldsymbol{\eta}^{-1}\boldsymbol{s} \in U^d$ *if and only if* $\boldsymbol{s} \in \boldsymbol{\eta} U^d$*, which occurs if and only if* $\boldsymbol{\omega}(\boldsymbol{s}) = \boldsymbol{\eta}$*. Therefore,*

$$\begin{aligned}\gamma^{(\boldsymbol{i})}(\delta_{\boldsymbol{s}})|_{U^d} &= \sum_{\boldsymbol{\eta} \in V^d} (\boldsymbol{\eta}^{\boldsymbol{i}} \delta_{\boldsymbol{\eta}^{-1}\boldsymbol{s}})|_{U^d} \\ &= \sum_{\boldsymbol{\eta} \in V^d} \boldsymbol{\eta}^{\boldsymbol{i}} \delta_{\boldsymbol{\eta}^{-1}\boldsymbol{s}}|_{U^d} \\ &= \begin{cases} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{s}) \delta_{(\boldsymbol{\omega}(\boldsymbol{s}))^{-1}\boldsymbol{s}} & : \boldsymbol{s} \in (\mathbb{Z}_p^\times)^d \\ \alpha_0 & : \textit{else} \end{cases}.\end{aligned}$$

Now, Lemma 19 and Example 18 give

$$\begin{aligned}\gamma^{(\boldsymbol{i})}(\delta_{\boldsymbol{s}}) \circ \varphi_{\boldsymbol{u}} &= \begin{cases} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{s}) \delta_{\ell_{\boldsymbol{u}}((\boldsymbol{\omega}(\boldsymbol{s}))^{-1}\boldsymbol{s})} & : \boldsymbol{s} \in (\mathbb{Z}_p^\times)^d \\ \alpha_0 & : \textit{else} \end{cases} \\ &= \begin{cases} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{s}) \delta_{\ell_{\boldsymbol{u}}(\boldsymbol{s})} & : \boldsymbol{s} \in (\mathbb{Z}_p^\times)^d \\ \alpha_0 & : \textit{else} \end{cases}.\end{aligned}$$

Finally, Example 4 gives

$$G^{(\boldsymbol{i})}_{\delta_{\boldsymbol{s}}}(\boldsymbol{T}) = \begin{cases} \boldsymbol{\omega}^{\boldsymbol{i}}(\boldsymbol{s})\boldsymbol{T}^{\ell_u(\boldsymbol{s})} & : \boldsymbol{s} \in (\mathbb{Z}_p^\times)^d \\ 0 & : \textit{else} \end{cases} .$$

◊

2.5 Iwasawa Invariants of Measures and Power Series

In this section, we assume R is a discrete valuation ring with valuation $\mathrm{ord}_R : R \to \mathbb{N} \cup \{\infty\}$ inducing the topology on R. Let $\pi \in R$ be a fixed uniformizer. In this case, $\Lambda_{(d)} \cong \Lambda_d = R[[\boldsymbol{T} - \boldsymbol{1}]]$. Thus, from the discussion at the end of Section 2.2, we have that

$$R[[\boldsymbol{T} - \boldsymbol{1}]] \cong \lim_{\leftarrow} R[\boldsymbol{T} - \boldsymbol{1}]/I_{\boldsymbol{n}},$$

where $I_{\boldsymbol{n}}$ is the ideal of $R[\boldsymbol{T} - \boldsymbol{1}]$ generated by the components of the d-tuple $\boldsymbol{T}^{\wedge \boldsymbol{p}^{\wedge \boldsymbol{n}}} - \boldsymbol{1}$.

Definition 27. *Let $F \in \Lambda_d$ be a nonzero power series, and let $a_{\boldsymbol{m}}$ denote the $\boldsymbol{m}$th coefficient of F. The Iwasawa μ-invariant of F is the non-negative integer*

$$\mu(F) = \min\{\mathrm{ord}_R(a_{\boldsymbol{m}}) : \boldsymbol{m} \in \mathbb{N}^d\}.$$

Let $\preccurlyeq$ be any linear extension of $\preccurlyeq_d$. The Iwasawa λ-invariant associated to $\preccurlyeq$ is the element of $\mathbb{N}^d$ given by

$$\lambda_{\preccurlyeq}(F) = \min_{\preccurlyeq}\{\boldsymbol{m} : \mathrm{ord}_R(a_{\boldsymbol{m}}) = \mu(F)\},$$

where the minimum is taken with respect to the well order $\preccurlyeq$. For a non-zero measure $\alpha \in \mathcal{M}_d$, we set

$$\mu(\alpha) = \mu(\widehat{\alpha}) \text{ and } \lambda_{\preccurlyeq}(\alpha) = \lambda_{\preccurlyeq}(\widehat{\alpha}).$$

For each $\boldsymbol{i} \in \mathbb{N}^d$ *with* $\boldsymbol{i} \preccurlyeq_d \boldsymbol{p}-\mathbf{2}$, *we set*

$$\mu(\Gamma_\alpha^{(\boldsymbol{i})}) = \mu(G_\alpha^{(\boldsymbol{i})}) \text{ and } \lambda_\preccurlyeq(\Gamma_\alpha^{(\boldsymbol{i})}) = \lambda_\preccurlyeq(G_\alpha^{(\boldsymbol{i})}).$$

Definition 28. *For a nonzero power series* $F \in \Lambda_d$, *let*

$$L(F) = \{\lambda_\preccurlyeq(F) : \preccurlyeq \text{ is a linear extension of } \preccurlyeq_d\} \subseteq \mathbb{N}^d.$$

For a non-zero measure $\alpha \in \mathcal{M}_d$, *let* $L(\alpha) = L(\widehat{\alpha})$, *and for each* $\boldsymbol{i} \in \mathbb{N}^d$ *with* $\boldsymbol{i} \preccurlyeq_d \boldsymbol{p}-\mathbf{2}$, *let* $L(\Gamma_\alpha^{(\boldsymbol{i})}) = L(G_\alpha^{(\boldsymbol{i})})$.

We pause to make one observation that will be used frequently in what follows. Let $F \in \Lambda_d$ be a nonzero power series. Then we may write $F(\boldsymbol{T}) = \pi^{\mu(F)}G(\boldsymbol{T})$ where $\mu(G) = 0$ and $\lambda_\preccurlyeq(F) = \lambda_\preccurlyeq(G)$ for any linear extension $\preccurlyeq$ of $\preccurlyeq_d$. Consequently, $L(F) = L(G)$.

When $d = 1$, there is a unique λ-invariant for each nonzero power series $F \in \Lambda_1$ since $(\mathbb{N}, \leq)$ is totally ordered. We denote this unique λ-invariant simply by $\lambda(F)$.

Lemma 25. *Suppose* $F \in \Lambda_d$ *is such that*

$$F(\boldsymbol{T}) = \prod_{i=1}^{d} F_i(T_i)$$

for $F_1, \ldots, F_d \in \Lambda_1$. *Then*

$$\mu(F) = \sum_{i=1}^{d} \mu(F_i),$$

and $L(F) = \{(\lambda(F_1), \ldots, \lambda(F_d))\}$, *where* $\lambda(F_i)$ *is the unique* λ*-invariant for* $F_i \in \Lambda_1$.

Proof. For each i, we may write

$$F_i = \pi^{\mu(F_i)}G_i,$$

where $G_i \in \Lambda_1$ is such that $\mu(G_i) = 0$ and $\lambda(F_i) = \lambda(G_i)$. Then

$$F(\boldsymbol{T}) = \pi^{\mu(F_1)+\cdots+\mu(F_d)} \prod_{i=1}^{d} G_i(T_i).$$

Put

$$G(\boldsymbol{T}) = \prod_{i=1}^{d} G_i(T_i),$$

and write

$$G(\boldsymbol{T}) = \sum_{\boldsymbol{m}\in\mathbb{N}^d} a_{\boldsymbol{m}}(\boldsymbol{T}-\mathbf{1})^{\boldsymbol{m}}$$

and for each i,

$$G_i(T_i) = \sum_{m\in\mathbb{N}} a_m^{(i)}(T_i - 1)^m.$$

Since

$$a_{\boldsymbol{m}} = \prod_{i=1}^{d} a_{m_i}^{(i)},$$

for all $\boldsymbol{m} \in \mathbb{N}^d$, we have that

$$\mathrm{ord}(a_{(\lambda(F_1),\ldots,\lambda(F_d))}) = \sum_{i=1}^{d} \mathrm{ord}(a_{\lambda(F_i)}^{(i)}) = 0.$$

In particular, this gives that $\mu(G) = 0$, so that

$$\mu(F) = \sum_{i=1}^{d} \mu(F_i),$$

as claimed. We also have $L(G) = L(F)$, so if $\boldsymbol{m} \in L(F)$, then

$$0 = \mathrm{ord}(a_{\boldsymbol{m}}) = \sum_{i=1}^{d} \mathrm{ord}(a_{m_i}).$$

Consequently, $\mathrm{ord}(a_{m_i}) = 0$ for all i, so that $\lambda(G_i) = \lambda(F_i) \leq m_i$ for all i. But this gives $(\lambda(F_1), \ldots, \lambda(F_d)) \preccurlyeq_d \boldsymbol{m}$. Since $(\lambda(F_1), \ldots, \lambda(F_d))$ is a candidate λ-invariant for G, we conclude $L(F) = \{(\lambda(F_1), \ldots, \lambda(F_d))\}$. □

Example 21. *Let $\boldsymbol{s} \in \mathbb{Z}_p^d$ and let $\delta_{\boldsymbol{s}}$ denote Dirac measure of mass 1 centered at $\boldsymbol{s}$ (Example 3). Recall from Example 11 that*

$$\widehat{\delta_{\boldsymbol{s}}}(\boldsymbol{T}) = \boldsymbol{T}^{\boldsymbol{s}} = \sum_{\boldsymbol{m}\in\mathbb{N}^d} \binom{\boldsymbol{s}}{\boldsymbol{m}} (\boldsymbol{T}-\mathbf{1})^{\boldsymbol{m}} \in \mathbb{Z}_p[[\boldsymbol{T}-\mathbf{1}]].$$

Since

$$\binom{\boldsymbol{s}}{\boldsymbol{0}} = 1,$$

we have $\mu(\delta_{\boldsymbol{s}}) = 0$ *and* $L(\delta_{\boldsymbol{s}}) = \{\boldsymbol{0}\}$.

This example also serves as an illustration of Lemma 25, since

$$\boldsymbol{T}^{\boldsymbol{s}} = \prod_{i=1}^{d} T_i^{s_i} = \prod_{i=1}^{d} \left(\sum_{m_i \in \mathbb{N}} \binom{s_i}{m_i} (T_i - 1)^{m_i} \right).$$

We have that

$$\binom{s_i}{0} = 1$$

for all $1 \leq i \leq d$*, so that* $\mu(T_i^{s_i}) = 0$ *and* $\lambda(T_i^{s_i}) = 0$ *for all* $1 \leq i \leq d$. ♢

Example 22. *Let* $\boldsymbol{s} \in \mathbb{Z}_p^d$ *and let* $\delta_{\boldsymbol{s}}$ *denote Dirac measure of mass 1 centered at* $\boldsymbol{s}$ *(Example 3). Let* $\alpha = \delta_{\boldsymbol{s}} - \delta_{\boldsymbol{0}}$*. Then*

$$\widehat{\alpha}(\boldsymbol{T}) = \boldsymbol{T}^{\boldsymbol{s}} - 1 = \sum_{\boldsymbol{m} \in \mathbb{N}^d \setminus \{\boldsymbol{0}\}} \binom{\boldsymbol{s}}{\boldsymbol{m}} (\boldsymbol{T} - \boldsymbol{1})^{\boldsymbol{m}} \in \mathbb{Z}_p[[\boldsymbol{T} - \boldsymbol{1}]].$$

In particular, $\boldsymbol{0} \notin L(\alpha)$*. We assume* $\boldsymbol{s} \neq \boldsymbol{0}$ *(else* $\widehat{\alpha} = 0$*). Fix an index* $1 \leq i \leq d$ *for which* $s_i \neq 0$*. Put* $n_i = \operatorname{ord}(s_i)$*, and let* a_i *be the unique integer with* $0 \leq a_i < p^{n_i+1}$ *and* $s_i \equiv a_i \pmod{p^{n_i+1}}$*. Then the only non-zero digit in the* p*-adic expansion of* a_i *is the* n_i*th digit, and we have by Lemma 51 in Appendix C,*

$$\begin{aligned} \binom{\boldsymbol{s}}{p^{n_i}\boldsymbol{e}_i} &= \binom{s_i}{p^{n_i}} \\ &\equiv \binom{a_i}{p^{n_i}} \pmod{p} \\ &\not\equiv 0 \pmod{p}. \end{aligned}$$

On the other hand, for each $0 < k < p^{n_i}$, since k has some non-zero p-adic digit before the n_ith,

$$\begin{aligned}\binom{\boldsymbol{s}}{k\boldsymbol{e}_i} &= \binom{s_i}{k} \\ &\equiv \binom{a_i}{k} \pmod{p} \\ &\equiv 0 \pmod{p}.\end{aligned}$$

We have shown that $\mu(\alpha) = 0$. Moreover, we have shown that the $p^{n_i}\boldsymbol{e}_i$ for each $1 \le i \le d$ with $s_i \ne 0$ are candidate λ-invariants for α. Since $\mathbf{0} \notin L(\alpha)$ and since the $p^{n_i}\boldsymbol{e}_i$ are pairwise incomparable with respect to $\preccurlyeq_d$, we must have

$$\{p^{n_i}\boldsymbol{e}_i : 1 \le i \le d \text{ and } s_i \ne 0\} \subseteq L(\alpha).$$

On the other hand, suppose $\boldsymbol{m} \in L(\alpha)$. Of course, if $1 \le i \le d$ is an index for which $s_i = 0$, then $m_i = 0$ since

$$\binom{s_i}{m_i} = \begin{cases} 0 & : m_i > 0 \\ 1 & : m_i = 0 \end{cases}.$$

Consequently,

$$\boldsymbol{m} = \sum_{\substack{1 \le i \le d \\ s_i \ne 0}} m_i \boldsymbol{e}_i.$$

Suppose $\boldsymbol{m}$ is not one of the $p^{n_i}\boldsymbol{e}_i$ for $1 \le i \le d$ with $s_i \ne 0$. Because $\boldsymbol{m}$ is incomparable to each of these elements with respect to $\preccurlyeq_d$, the above work gives $0 \le m_i < p^{n_i}$ for all $1 \le i \le d$ with $s_i \ne 0$. As before, for each $1 \le i \le d$ with $s_i \ne 0$, if $m_i \ne 0$, then

$$\binom{s_i}{m_i} \equiv \binom{a_i}{m_i} \equiv 0 \pmod{p}.$$

This forces $m_i = 0$ for all i; since $\boldsymbol{m} \ne \mathbf{0}$, we have reached a contradiction. We conclude

$$L(\alpha) = \{p^{n_i}\boldsymbol{e}_i : 1 \le i \le d \text{ and } s_i \ne 0\}.$$

$\Diamond$

Lemma 26. *Let $\alpha \in \mathcal{M}_d$ be nonzero and let $g \in C(\mathbb{Z}_p^d, R)$. Then $\mu(\alpha_g) \geq \mu(\alpha)$ (If $\alpha_g = \alpha_0$, we put $\mu(\alpha_g) = \infty$).*

Proof. We assume $\alpha_g \neq \alpha_0$, since there is nothing to show in this case. Let $\boldsymbol{n} \in \mathbb{N}^d$. The $\boldsymbol{n}$th coefficient of $\widehat{\alpha}_g$ is

$$\int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{n}} \, d\alpha_g(\boldsymbol{x}) = \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{n}} g(\boldsymbol{x}) \, d\alpha(\boldsymbol{x}).$$

For $\boldsymbol{m} \in \mathbb{N}^d$, let $b_{\boldsymbol{m}}$ denote the $\boldsymbol{m}$th coefficient of $\widehat{\alpha}(\boldsymbol{T})$, and let $a_{\boldsymbol{m}}$ denote the $\boldsymbol{m}$th Mahler coefficient of the continuous function $\mathbb{Z}_p^d \to R$ given by

$$\boldsymbol{x} \mapsto \binom{\boldsymbol{x}}{\boldsymbol{n}} g(\boldsymbol{x}).$$

We have

$$\int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{n}} \, d\alpha_g(\boldsymbol{x}) = \sum_{\boldsymbol{m} \in \mathbb{N}^d} a_{\boldsymbol{m}} b_{\boldsymbol{m}}.$$

It follows that $\mu(\alpha_g) \geq \mu(\alpha)$. $\square$

The following two lemmas are the multivariate analogues of Lemma 8 in Rosenberg (2004). These results are significant because they give information on the Iwasawa invariants of a measure in terms of the measure itself, rather than in terms of associated power series.

Lemma 27. *Let $\alpha \in \mathcal{M}_d$ be nonzero. Then*

$$\mu(\alpha) = \min_{\boldsymbol{n} \in \mathbb{N}^d} \{\mathrm{ord}_R(\alpha(A)) : A \in L_{\boldsymbol{n}}\}.$$

Proof. Write

$$\widehat{\alpha}(\boldsymbol{T}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} b_{\boldsymbol{n}} (\boldsymbol{T} - \boldsymbol{1})^{\boldsymbol{n}}.$$

Let $\boldsymbol{n} \in \mathbb{N}^d$, and consider any $A \in L_{\boldsymbol{n}}$. Let $g \in C(\mathbb{Z}_p^d, R)$ denote the characteristic function of A and $a_{\boldsymbol{n}}(g)$ the $\boldsymbol{n}$th Mahler coefficient of g. Then

$$\alpha(A) = \int_{\mathbb{Z}_p^d} g(\boldsymbol{x})\, d\alpha(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} a_{\boldsymbol{n}}(g) b_{\boldsymbol{n}}.$$

This gives that $\operatorname{ord}_R(\alpha(A)) \geq \mu(\alpha)$, from which we find

$$\mu(\alpha) \leq \min_{\boldsymbol{n} \in \mathbb{N}^d} \{\operatorname{ord}_R(\alpha(A)) : A \in L_{\boldsymbol{n}}\}.$$

To establish the reverse inequality, it suffices to consider the case $\mu(\alpha) = 0$. Fix $\boldsymbol{k} \in \mathbb{N}^d$ with $\operatorname{ord}_R(b_{\boldsymbol{k}}) = 0$, and choose $\boldsymbol{m} \in \mathbb{N}^d$ such that $\boldsymbol{k} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1}$. Recall that by definition

$$\begin{aligned} b_{\boldsymbol{k}} &= \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{k}} d\alpha(\boldsymbol{x}) \\ &= \lim_{\boldsymbol{n} \in \mathbb{N}^d} \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \binom{\boldsymbol{a}}{\boldsymbol{k}} \alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d). \end{aligned}$$

Consider any $\boldsymbol{n} \in \mathbb{N}^d$ with $\boldsymbol{m} \preccurlyeq_d \boldsymbol{n}$. Fix $\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1}$. If $\boldsymbol{b} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}$ and $\boldsymbol{b} \equiv \boldsymbol{a} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{m}} \mathbb{Z}_p^d}$, then Lemma 51 gives

$$\binom{\boldsymbol{b}}{\boldsymbol{k}} \equiv \binom{\boldsymbol{a}}{\boldsymbol{k}} \pmod{p}.$$

Since

$$\sum_{\substack{\boldsymbol{b} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1} \\ \boldsymbol{b} \equiv \boldsymbol{a} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{m}} \mathbb{Z}_p^d}}} \alpha(\boldsymbol{b} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d) = \alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{m}} \mathbb{Z}_p^d),$$

we have that

$$\sum_{\boldsymbol{b} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \binom{\boldsymbol{b}}{\boldsymbol{k}} \alpha(\boldsymbol{b} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d) \equiv \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1}} \binom{\boldsymbol{a}}{\boldsymbol{k}} \alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{m}} \mathbb{Z}_p^d) \pmod{\pi}.$$

Thus, the d-net

$$\left(\sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \binom{\boldsymbol{a}}{\boldsymbol{k}} \alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d) \pmod{\pi} \right)_{\boldsymbol{n}}$$

is constant for $\boldsymbol{m} \preccurlyeq_d \boldsymbol{n}$. Therefore,

$$b_{\boldsymbol{k}} \equiv \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1}} \binom{\boldsymbol{a}}{\boldsymbol{k}} \alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{m}} \mathbb{Z}_p^d) \pmod{\pi},$$

so it cannot be that $\mathrm{ord}_R(\alpha(A)) > 0$ for all $A \in L_{\boldsymbol{m}}$. This gives the claimed equality

$$\mu(\alpha) = \min_{\boldsymbol{n} \in \mathbb{N}^d} \{\mathrm{ord}_R(\alpha(A)) : A \in L_{\boldsymbol{n}}\}.$$

Moreover, by applying the above argument to the case $\boldsymbol{k} \in L(\alpha)$, we have shown that if $\boldsymbol{k} \in L(\alpha)$ and $\boldsymbol{m}$ is such that

$$\boldsymbol{k} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1},$$

then

$$\boldsymbol{m} \in \{\boldsymbol{n} : \mathrm{ord}_R(\alpha(A)) = \mu(\alpha) \text{ for some } A \in L_{\boldsymbol{n}}\}.$$

□

Corollary 20. *Let $\alpha \in \mathcal{M}_d$ be nonzero, let $A \in CO_d$, and let $A^c = \mathbb{Z}_p^d \setminus A$. Then*

$$\mu(\alpha) = \min\{\mu(\alpha|_A), \mu(\alpha|_{A^c})\}.$$

Consequently, if $\mu(\alpha|_{A^c}) > \mu(\alpha)$, then $\mu(\alpha) = \mu(\alpha|_A)$.

Proof. From Lemma 26, we have that $\mu(\alpha|_A) \geq \mu(\alpha)$ and $\mu(\alpha|_{A^c}) \geq \mu(\alpha)$. Thus,

$$\mu(\alpha) \leq \min\{\mu(\alpha|_A), \mu(\alpha|_{A^c})\}.$$

Now let B be any polyball. Corollary 14 gives

$$\alpha(B) = \alpha|_A(B) + \alpha|_{A^c}(B).$$

Employing Lemma 27 gives

$$\begin{aligned} \mathrm{ord}_R(\alpha(B)) &\geq \min\{\mathrm{ord}_R(\alpha|_A(B)), \mathrm{ord}_R(\alpha|_{A^c}(B))\} \\ &\geq \min\{\mu(\alpha|_A), \mu(\alpha|_{A^c})\}. \end{aligned}$$

Since the above holds for all polyballs B, a final application of Lemma 27 yields

$$\mu(\alpha) \geq \min\{\mu(\alpha|_A), \mu(\alpha|_{A^c})\}.$$

□

Observe that if

$$\boldsymbol{m} \in \{\boldsymbol{n} : \mathrm{ord}_R(\alpha(A)) = \mu(\alpha) \text{ for some } A \in L_{\boldsymbol{n}}\},$$

then for any $\boldsymbol{t} \in \mathbb{N}^d$ with $\boldsymbol{m} \preccurlyeq_d \boldsymbol{t}$, also

$$\boldsymbol{t} \in \{\boldsymbol{n} : \mathrm{ord}_R(\alpha(A)) = \mu(\alpha) \text{ for some } A \in L_{\boldsymbol{n}}\}.$$

Indeed, each $A \in L_{\boldsymbol{m}}$ can be written as a finite disjoint union of elements of $L_{\boldsymbol{t}}$.

Lemma 28. *Let $\alpha \in \mathcal{M}_d$ be nonzero. For each element $\boldsymbol{m}$ of the set*

$$\{\boldsymbol{n} : \mathrm{ord}_R(\alpha(A)) = \mu(\alpha) \textit{ for some } A \in L_{\boldsymbol{n}}\} \subseteq \mathbb{N}^d$$

which is minimal with respect to $\preccurlyeq_d$, there is some $\lambda \in L(\alpha)$ such that

$$\lfloor \boldsymbol{p}^{\wedge(\boldsymbol{m}-\boldsymbol{1})} \rfloor \preccurlyeq_d \lambda \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1}.$$

Proof. Write

$$\widehat{\alpha}(\boldsymbol{T}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} b_{\boldsymbol{n}} (\boldsymbol{T} - \boldsymbol{1})^{\boldsymbol{n}}.$$

Fix an element $\boldsymbol{m}$ of the set

$$\{\boldsymbol{n} : \mathrm{ord}_R(\alpha(A)) = \mu(\alpha) \text{ for some } A \in L_{\boldsymbol{n}}\} \subseteq \mathbb{N}^d$$

which is minimal with respect to $\preccurlyeq_d$. It suffices to consider the case $\mu(\alpha) = 0$. Let $\boldsymbol{n} \in \mathbb{N}^d$ and suppose that $\mathrm{ord}_R(b_{\boldsymbol{k}}) > 0$ for all $\boldsymbol{k} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}$. The proof of Lemma 27 showed that

$$b_{\boldsymbol{k}} \equiv \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \binom{\boldsymbol{a}}{\boldsymbol{k}} \alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d) \pmod{\pi},$$

yielding the linear system

$$0 \equiv \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \binom{\boldsymbol{a}}{\boldsymbol{k}} \alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d) \pmod{\pi}$$

in $R/\pi R$. Fix a linear extension $\preccurlyeq$ of $\preccurlyeq_d$. Then the integer matrix

$$\left[\binom{\boldsymbol{a}}{\boldsymbol{k}}\right]_{\boldsymbol{0} \preccurlyeq_d \boldsymbol{a}, \boldsymbol{k} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}}$$

(with the rows and columns indexed by those $\boldsymbol{j} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}$ and ordered by $\preccurlyeq$) is upper triangular with only 1 on the main diagonal, so is invertible. Indeed, if $\boldsymbol{a} \prec \boldsymbol{k}$, then $\boldsymbol{k} \not\preccurlyeq_d \boldsymbol{a}$, so that

$$\binom{\boldsymbol{a}}{\boldsymbol{k}} = 0,$$

while

$$\binom{\boldsymbol{a}}{\boldsymbol{a}} = 1$$

for all $\boldsymbol{a}$. In light of the linear system

$$0 \equiv \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \binom{\boldsymbol{a}}{\boldsymbol{k}} \alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d) \pmod{\pi}$$

in $R/\pi R$, we must have that $\operatorname{ord}_R(\alpha(A)) > 0$ for all $A \in L_{\boldsymbol{n}}$. By the definition of $\boldsymbol{m}$, we therefore must have that

$$L(\alpha) \cap \{\boldsymbol{k} \in \mathbb{N}^d : \boldsymbol{k} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1}\} \neq \emptyset.$$

On the other hand, let $\boldsymbol{n} \in \mathbb{N}^d$ and suppose that $\operatorname{ord}_R(\alpha(A)) > 0$ for all $A \in L_{\boldsymbol{n}}$. Let $\boldsymbol{k} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}$. The proof of Lemma 27 showed that

$$b_{\boldsymbol{k}} \equiv \sum_{\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} \binom{\boldsymbol{a}}{\boldsymbol{k}} \alpha(\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d) \pmod{\pi},$$

so that $\operatorname{ord}_\pi(b_{\boldsymbol{k}}) > 0$. By the definition of $\boldsymbol{m}$, we thus have that

$$L(\alpha) \cap \{\boldsymbol{k} \in \mathbb{N}^d : \boldsymbol{k} \preccurlyeq_d \boldsymbol{p}^{\wedge (\boldsymbol{m} - \boldsymbol{e}_i)} - \boldsymbol{1}\} = \emptyset$$

for all $1 \leq i \leq d$ with $m_i \neq 0$. Combining this with the above gives that there is some $\lambda \in L(\alpha)$ with

$$\lfloor \boldsymbol{p}^{\wedge(\boldsymbol{m}-\boldsymbol{1})} \rfloor \preccurlyeq_d \lambda \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1}.$$

$\square$

Example 23. *We return to the two power series considered in Example 21 and Example 22. Certainly,* $\delta_{\boldsymbol{s}}(\mathbb{Z}_p) = 1$*. Lemma 27 gives that* $\mu(\delta_{\boldsymbol{s}}) = 0$*, while Lemma 28 may be applied with* $\boldsymbol{m} = \boldsymbol{0}$ *to deduce that* $L(\delta_{\boldsymbol{s}}) = \boldsymbol{0}$*.*

In Example 22, we considered $\alpha = \delta_{\boldsymbol{s}} - \delta_{\boldsymbol{0}}$*, so* $\widehat{\alpha}(\boldsymbol{T}) = \boldsymbol{T}^{\boldsymbol{s}} - 1$ *for* $\boldsymbol{s} \neq \boldsymbol{0}$*. Let* $n_i = \operatorname{ord}(s_i)$ *if* $s_i \neq 0$ *and* $n_i = 0$ *if* $s_i = 0$*. Put* $\boldsymbol{n} = (n_1, \ldots, n_d)$*. Then*

$$\boldsymbol{s} \equiv \boldsymbol{0} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{n}} \mathbb{Z}_p^d},$$

but

$$\boldsymbol{s} \not\equiv \boldsymbol{0} \pmod{\boldsymbol{p}^{\wedge(\boldsymbol{n}+\boldsymbol{1})} \mathbb{Z}_p^d}.$$

Thus

$$\alpha(\boldsymbol{p}^{\wedge(\boldsymbol{n}+\boldsymbol{1})} \mathbb{Z}_p^d) = \delta_{\boldsymbol{s}}(\boldsymbol{p}^{\wedge(\boldsymbol{n}+\boldsymbol{1})} \mathbb{Z}_p^d) - \delta_{\boldsymbol{0}}(\boldsymbol{p}^{\wedge(\boldsymbol{n}+\boldsymbol{1})} \mathbb{Z}_p^d) = -1.$$

Lemma 27 gives that $\mu(\alpha) = 0$*. On the other hand, for each* $\boldsymbol{m} \preccurlyeq_d \boldsymbol{n}$*,*

$$\alpha(\boldsymbol{p}^{\wedge \boldsymbol{m}} \mathbb{Z}_p^d) = \delta_{\boldsymbol{s}}(\boldsymbol{p}^{\wedge \boldsymbol{m}} \mathbb{Z}_p^d) - \delta_{\boldsymbol{0}}(\boldsymbol{p}^{\wedge \boldsymbol{m}} \mathbb{Z}_p^d) = 0.$$

Now for each $1 \leq i \leq d$ *with* $s_i \neq 0$*,*

$$\boldsymbol{s} \not\equiv \boldsymbol{0} \pmod{\boldsymbol{p}^{\wedge((n_i+1)\boldsymbol{e}_i)}},$$

so that

$$\alpha(\boldsymbol{p}^{\wedge((n_i+1)\boldsymbol{e}_i)} \mathbb{Z}_p^d) = \delta_{\boldsymbol{s}}(\boldsymbol{p}^{\wedge((n_i+1)\boldsymbol{e}_i)} \mathbb{Z}_p^d) - \delta_{\boldsymbol{0}}(\boldsymbol{p}^{\wedge((n_i+1)\boldsymbol{e}_i)} \mathbb{Z}_p^d) = -1.$$

However, if $\boldsymbol{m} \prec_d (n_i+1)\boldsymbol{e}_i$*, then* $\boldsymbol{m} \preccurlyeq_d \boldsymbol{n}$*. Consequently, the* $(n_i+1)\boldsymbol{e}_i$ *for those* i *with* $s_i \neq 0$ *are minimal elements of the set*

$$\{\boldsymbol{n} : \operatorname{ord}_R(A) = \mu(\alpha) \textit{ for some } A \in L_{\boldsymbol{n}}\}.$$

Lemma 28 gives for each i *with* $s_i \neq 0$ *a* $\lambda_i \in L(\alpha)$ *satisfying*

$$p^{n_i}\boldsymbol{e}_i \preccurlyeq_d \lambda_i \preccurlyeq_d \boldsymbol{p}^{\wedge((n_i+1)\boldsymbol{e}_i)} - \mathbf{1} = (p^{n_i+1}-1)\boldsymbol{e}_i.$$

Consequently, $\lambda_i = m_i\boldsymbol{e}_i$ *for some* $p^{n_i} \leq m_i < p^{n_i+1}$*. Thus, the* λ_i *we have identified are distinct elements of* $\mathbb{N}^d$*. As we saw in Example 22,* $\lambda_i = p^{n_i}\boldsymbol{e}_i$*, and the collection of all* λ_i *for those* i *with* $s_i \neq 0$ *is precisely the set* $L(\alpha)$*.* ♢

Lemma 29. *Let* $\alpha \in \mathcal{M}_d$*. Suppose* $g, g_0 \in LC(\mathbb{Z}_p^d, R)$ *are such that for all* $\boldsymbol{n} \in \mathbb{N}^d$*,* $\operatorname{ord}_R(g(\boldsymbol{n})) > 0$ *if and only if* $\operatorname{ord}_R(g_0(\boldsymbol{n})) > 0$*. Then* $\mu(\alpha_g) = 0$ *if and only if* $\mu(\alpha_{g_0}) = 0$*.*

Proof. Fix $\boldsymbol{m} \in \mathbb{N}$ for which g and g_0 both factor through $L_{\boldsymbol{m}}$. For any $\boldsymbol{n} \in \mathbb{N}^d$ with $\boldsymbol{m} \preccurlyeq_d \boldsymbol{n}$ and $A = \boldsymbol{a} + \boldsymbol{p}^{\wedge\boldsymbol{n}}\mathbb{Z}_p^d \in L_{\boldsymbol{n}}$, we have

$$\alpha_g(A) = g(\boldsymbol{a})\alpha(A)$$
$$\alpha_{g_0}(A) = g_0(\boldsymbol{a})\alpha(A).$$

If $\mu(\alpha_g) = 0$, then we may choose $\boldsymbol{m} \preccurlyeq_d \boldsymbol{n}$ and $A = \boldsymbol{a} + \boldsymbol{p}^{\wedge\boldsymbol{n}}\mathbb{Z}_p^d \in L_{\boldsymbol{n}}$ such that $0 = \operatorname{ord}_R(\alpha_g(A))$. Using the first line above, we must have that $\operatorname{ord}_R(g(\boldsymbol{a})) = 0 = \operatorname{ord}_R(\alpha(A))$. But then $\operatorname{ord}_R(g_0(\boldsymbol{a})) = 0$ as well, and we find $0 = \operatorname{ord}_R(\alpha_{g_0}(A))$. Thus, $\mu(\alpha_{g_0}) = 0$. The converse follows in the same manner. □

In Λ_d, for $\boldsymbol{n} \in \mathbb{N}^d$ and $m \in \mathbb{N}$, we define the ideal

$$J_{\boldsymbol{n}}(\pi^m) := \langle \pi^m, (T_1-1)^{n_1}, \ldots, (T_d-1)^{n_d} \rangle.$$

Observe that if $\boldsymbol{m} \in L(\alpha)$, then for all $\boldsymbol{m}+\mathbf{1} \preccurlyeq_d \boldsymbol{n}$,

$$\widehat{\alpha}(\boldsymbol{T}) \not\equiv 0 \pmod{J_{\boldsymbol{n}}(\pi^{\mu(F)+1})}.$$

Conversely, if

$$\widehat{\alpha}(\boldsymbol{T}) \not\equiv 0 \pmod{J_{\boldsymbol{n}}(\pi^{m})}$$

for some $\boldsymbol{n} \in \mathbb{N}^d$ and $m \in \mathbb{N}$, then $\mu(\alpha) < m$ and there is $\boldsymbol{m} \in L(\alpha)$ with $\boldsymbol{m} \prec_d \boldsymbol{n}$.

Lemma 30. *Let $\alpha \in \mathcal{M}_d$ be nonzero, and let $b_{\boldsymbol{m}}$ denote the $\boldsymbol{m}$th coefficient of $\widehat{\alpha}(\boldsymbol{T})$. Let $\preccurlyeq$ be a linear extension of $\preccurlyeq_d$. Suppose $\mu(\alpha) = 0$ and set $\lambda = \lambda_{\preccurlyeq}(\alpha) \in \mathbb{N}^d$. For any $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$,*

$$\widehat{(\alpha \circ \boldsymbol{a})}(\boldsymbol{T}) \equiv b_\lambda (\boldsymbol{a}^{-1})^\lambda (\boldsymbol{T}-\mathbf{1})^\lambda \pmod{J_{\lambda+\mathbf{1}}(\pi)}.$$

In particular, $\mu(\alpha \circ \boldsymbol{a}) = 0$ and $\lambda \in L(\alpha \circ \boldsymbol{a})$.

Proof. We have

$$\begin{aligned}\widehat{(\alpha \circ \boldsymbol{a})}(\boldsymbol{T}) &= \widehat{\alpha}(\boldsymbol{T}^{\wedge \boldsymbol{a}^{-1}}) \\ &= \sum_{\boldsymbol{m} \in \mathbb{N}^d} b_{\boldsymbol{m}} (\boldsymbol{T}^{\wedge \boldsymbol{a}^{-1}} - \mathbf{1})^{\boldsymbol{m}} \\ &\equiv \sum_{\boldsymbol{m} \not\prec_d \lambda} b_{\boldsymbol{m}} (\boldsymbol{T}^{\wedge \boldsymbol{a}^{-1}} - \mathbf{1})^{\boldsymbol{m}} \pmod{\pi}.\end{aligned}$$

But for any $\boldsymbol{m} \in \mathbb{N}^d$,

$$(\boldsymbol{T}^{\wedge \boldsymbol{a}^{-1}} - \mathbf{1})^{\boldsymbol{m}} = \prod_{i=1}^{d} \left(\sum_{k_i > 0} \binom{a_i^{-1}}{k_i} (T_i - 1)^{k_i} \right)^{m_i}.$$

If $\boldsymbol{m} \not\preccurlyeq_d \lambda$, then $m_i \geq \lambda_i + 1$ for some i. But if $m_i \geq \lambda_i + 1$, then

$$\left(\sum_{k_i > 0} \binom{a_i^{-1}}{k_i} (T_i - 1)^{k_i} \right)^{m_i} \equiv 0 \pmod{(T_i - 1)^{\lambda_i + 1}}.$$

Thus,

$$\begin{aligned}\widehat{(\alpha \circ \boldsymbol{a})}(\boldsymbol{T}) &\equiv \sum_{\boldsymbol{m} \not<_d \lambda} b_{\boldsymbol{m}} (\boldsymbol{T}^{\wedge \boldsymbol{a}^{-1}} - \mathbf{1})^{\boldsymbol{m}} \pmod{\pi} \\ &\equiv b_\lambda (\boldsymbol{T}^{\wedge \boldsymbol{a}^{-1}} - \mathbf{1})^{\lambda} \pmod{J_{\lambda+\mathbf{1}}(\pi)} \\ &\equiv b_\lambda \prod_{i=1}^{d} \left(\binom{a_i^{-1}}{1} (T_i - 1) \right)^{\lambda_i} \pmod{J_{\lambda+\mathbf{1}}(\pi)} \\ &\equiv b_\lambda (\boldsymbol{a}^{-1})^{\lambda} (\boldsymbol{T} - \mathbf{1})^{\lambda} \pmod{J_{\lambda+\mathbf{1}}(\pi)}.\end{aligned}$$

This gives $\mu(\alpha \circ \boldsymbol{a}) = 0$ and $\lambda \in L(\alpha \circ \boldsymbol{a})$. □

Lemma 31. *Let $\alpha \in \mathcal{M}_d$, with $\alpha|_{U^d}$ nonzero. Then $\mu(\alpha|_{U^d}) = \mu(\alpha \circ \varphi_{\boldsymbol{u}})$.*

Proof. Write

$$\widehat{\alpha|_{U^d}}(\boldsymbol{T}) = \sum_{\boldsymbol{m} \in \mathbb{N}^d} a_{\boldsymbol{m}} (\boldsymbol{T} - \mathbf{1})^{\boldsymbol{m}} \quad \text{and} \quad \widehat{\alpha \circ \varphi_{\boldsymbol{u}}}(\boldsymbol{T}) = \sum_{\boldsymbol{m} \in \mathbb{N}^d} b_{\boldsymbol{m}} (\boldsymbol{T} - \mathbf{1})^{\boldsymbol{m}}.$$

Lemma 20 gives

$$a_{\boldsymbol{m}} = \int_{U^d} \binom{\boldsymbol{x}}{\boldsymbol{m}} d\alpha(\boldsymbol{x}) = \int_{\mathbb{Z}_p^d} \binom{\varphi_{\boldsymbol{u}}(\boldsymbol{x})}{\boldsymbol{m}} d(\alpha \circ \varphi_{\boldsymbol{u}})(\boldsymbol{x})$$

for all $\boldsymbol{m} \in \mathbb{N}^d$. Let $c_{\boldsymbol{n}}$ denote the $\boldsymbol{n}$th Mahler coefficient of the continuous function $f : \mathbb{Z}_p^d \to R$ given by $f(\boldsymbol{x}) = \binom{\varphi_{\boldsymbol{u}}(\boldsymbol{x})}{\boldsymbol{m}}$. Then

$$a_{\boldsymbol{m}} = \int_{\mathbb{Z}_p^d} \binom{\varphi_{\boldsymbol{u}}(\boldsymbol{x})}{\boldsymbol{m}} d(\alpha \circ \varphi_{\boldsymbol{u}})(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} c_{\boldsymbol{n}} b_{\boldsymbol{n}}$$

for all $\boldsymbol{m} \in \mathbb{N}^d$. This gives $\mu(\alpha|_{U^d}) \geq \mu(\alpha \circ \varphi_{\boldsymbol{u}})$. Similarly, Corollary 18 gives

$$b_{\boldsymbol{m}} = \int_{\mathbb{Z}_p^d} \binom{\boldsymbol{x}}{\boldsymbol{m}} d(\alpha \circ \varphi_{\boldsymbol{u}})(\boldsymbol{x}) = \int_{U^d} \binom{\varphi_{\boldsymbol{u}}^{-1}(\boldsymbol{x})}{\boldsymbol{m}} d\alpha(\boldsymbol{x})$$

for all $\boldsymbol{m} \in \mathbb{N}^d$. Let $d_{\boldsymbol{n}}$ denote the $\boldsymbol{n}$th Mahler coefficient of the continuous function $f : \mathbb{Z}_p^d \to R$ given by $f(\boldsymbol{x}) = \binom{\varphi_{\boldsymbol{u}}^{-1}(\boldsymbol{x})}{\boldsymbol{m}}$ (recall the discussion following Lemma 20). Then

$$b_{\boldsymbol{m}} = \int_{\mathbb{Z}_p^d} \binom{\varphi_{\boldsymbol{u}}^{-1}(\boldsymbol{x})}{\boldsymbol{m}} d\alpha|_{U^d}(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} d_{\boldsymbol{n}} a_{\boldsymbol{n}}$$

for all $\boldsymbol{m} \in \mathbb{N}^d$. This gives $\mu(\alpha|_{U^d}) \leq \mu(\alpha \circ \varphi_{\boldsymbol{u}})$. □

The following alternate argument for Lemma 31 mimics the proof in Rosenberg (2004).

Alternate Proof. Let $\boldsymbol{n} \in \mathbb{N}$ and take any $A \in L_{\boldsymbol{n}}$. Then since $\varphi_{\boldsymbol{u}}$ maps $L_{\boldsymbol{n}}$ to $L_{\boldsymbol{n}+\boldsymbol{1}}$, we have

$$\begin{aligned} \operatorname{ord}_R((\alpha \circ \varphi_{\boldsymbol{u}})(A)) &= \operatorname{ord}_R(\alpha|_{U^d}(\varphi_{\boldsymbol{u}}(A))) \\ &\geq \min\{\operatorname{ord}_R(\alpha|_{U^d}(B)) : B \in L_{\boldsymbol{n}+\boldsymbol{1}}\} \\ &\geq \mu(\alpha|_{U^d}). \end{aligned}$$

Applying Lemma 27 gives

$$\mu(\alpha \circ \varphi_{\boldsymbol{u}}) = \min_{\boldsymbol{n}}\{\operatorname{ord}_R((\alpha \circ \varphi_{\boldsymbol{u}})(A)) : A \in L_{\boldsymbol{n}}\} \geq \mu(\alpha|_{U^d}).$$

By Lemma 27, there is $\boldsymbol{n} \in \mathbb{N}^d$ and $A \in L_{\boldsymbol{n}}$ with

$$\mu(\alpha|_{U^d}) = \operatorname{ord}_R(\alpha|_{U^d}(A)).$$

For any $\boldsymbol{n} \preccurlyeq_d \boldsymbol{m}$, we can write A as a disjoint union of polyballs in $L_{\boldsymbol{m}}$, say

$$A = \bigsqcup_{k=1}^{t} B_k.$$

Since

$$0 = \operatorname{ord}_R(\alpha|_{U^d}(A))) \geq \min_{1 \leq k \leq t}\{\operatorname{ord}_R(\alpha|_{U^d}(B_k))\},$$

we may assume that $\boldsymbol{1} \preccurlyeq_d \boldsymbol{n}$ by replacing A with a suitable polyball contained in A if needed. Since $\operatorname{ord}_R(\alpha|_{U^d}(A)) = \operatorname{ord}_R(\alpha(A \cap U^d))$ is finite, necessarily $A \cap U^d \neq \emptyset$. But since A is a polyball of level greater than that of U, it must be that $A \subseteq U^d$ (see

Appendix A). Consequently, $A = \varphi_{\boldsymbol{u}}(B)$ for some $B \in L_{\boldsymbol{n-1}}$. But now

$$\begin{aligned}\mu(\alpha|_{U^d}) &= \mathrm{ord}_R(\alpha|_{U^d})(A) \\ &= \mathrm{ord}_R(\alpha|_{U^d}(\varphi_{\boldsymbol{u}}(B)) \\ &= \mathrm{ord}_R((\alpha \circ \varphi_{\boldsymbol{u}})(B)).\end{aligned}$$

Lemma 27 now gives $\mu(\alpha|_{U^d}) \geq \mu(\alpha \circ \varphi_{\boldsymbol{u}})$. □

The following result is a slight strengthening of Theorem 2 from Childress and Zinzer (2015). This proposition is a natural multivariate analogue of the univariate results found in Kida (1986), Childress (1989), and Satoh (1992).

Proposition 12. *For any nonzero $\alpha \in \mathcal{M}_d$, componentwise multiplication by p gives a bijection $L(\alpha|_{U^d}) \to L(\alpha \circ \varphi_{\boldsymbol{u}})$.*

Proof. By Lemma 31, it suffices to consider the case $\mu(\alpha|_{U^d}) = 0 = \mu(\alpha \circ \varphi_{\boldsymbol{u}})$. In Childress and Zinzer (2015), an application of Lemma 51 gives that $L(\alpha|_{U^d}) \subseteq p\mathbb{N}^d$; the proof then employs Lemma 7 and a result of Satoh (1992) to relate λ-invariants of $\alpha|_{U^d}$ and $\alpha \circ \varphi_{\boldsymbol{u}}$. Together with an application of Corollary 1, the proof in Childress and Zinzer (2015) gives that for each $\lambda \in L(\alpha \circ \varphi_{\boldsymbol{u}})$, there is $\boldsymbol{a} \in L(\alpha|_{U^d})$ with

$$\boldsymbol{a} \preccurlyeq_d p\lambda.$$

Since $\boldsymbol{a} \in p\mathbb{N}^d$, we have $p^{-1}\boldsymbol{a} \preccurlyeq_d \lambda$. The proof in Childress and Zinzer (2015) further gives that for a fixed linear extension $\preccurlyeq$,

$$\lambda_{\preccurlyeq}(\alpha \circ \varphi_{\boldsymbol{u}}) \preccurlyeq p^{-1}\boldsymbol{a}$$

for all $\boldsymbol{a} \in L(\alpha|_{U^d})$. When coupled with the fact that $p^{-1}\boldsymbol{a} \preccurlyeq_d \lambda$ for some $\boldsymbol{a} \in L(\alpha|_{U^d})$ discussed above, this gives that $p\lambda \in L(\alpha|_{U^d})$ for every $\lambda \in L(\alpha \circ \varphi_{\boldsymbol{u}})$. Therefore, multiplication by p is an injection $L(\alpha \circ \varphi_{\boldsymbol{u}}) \to L(\alpha|_{U^d})$.

Now let $\boldsymbol{a} \in L(\alpha|_{U^d})$. As above,

$$\lambda_{\preccurlyeq}(\alpha \circ \varphi_{\boldsymbol{u}}) \preccurlyeq p^{-1}\boldsymbol{a}$$

for every linear extension $\preccurlyeq$. By Lemma 4, it follows that $p^{-1}\boldsymbol{a}$ must be comparable with respect to $\preccurlyeq_d$ to some $\boldsymbol{b} \in L(\alpha \circ \varphi_{\boldsymbol{u}})$. But now $\boldsymbol{a}$ is comparable to $p\boldsymbol{b}$ with respect to $\preccurlyeq_d$, and $p\boldsymbol{b} \in L(\alpha|_{U^d})$, giving that $\boldsymbol{a} = p\boldsymbol{b}$ by the definition of $L(\alpha|_{U^d})$. This shows that multiplication by p is a surjection $L(\alpha \circ \varphi_{\boldsymbol{u}}) \to L(\alpha|_{U^d})$.

□

Corollary 21. *Let $\alpha \in \mathcal{M}_d$. If $\preccurlyeq$ is a linear extension for which multiplication by p preserves minimal elements when viewed as a map $L(\alpha \circ \varphi_{\boldsymbol{u}}) \to L(\alpha|_{U^d})$, then $\lambda_{\preccurlyeq}(\alpha|_{U^d}) = p\lambda_{\preccurlyeq}(\alpha \circ \varphi_{\boldsymbol{u}})$.*

Recall that if $\preccurlyeq$ is a monomial order on $\mathbb{N}^d$, then multiplication by any integer $n \geq 1$ is an order-preserving injection $(\mathbb{N}^d, \preccurlyeq) \to (\mathbb{N}^d, \preccurlyeq)$ (see Lemma 5). Consequently, any monomial order satisfies the hypotheses of Corollary 21. See Childress and Zinzer (2015) for some consequences of this last fact for λ-invariants associated to monomial orders.

Example 24. *It is worth noting that $\lambda_{\preccurlyeq}(\alpha|_{U^d}) = p\lambda_{\preccurlyeq}(\alpha \circ \varphi_{\boldsymbol{u}})$ need not hold for every $\alpha \in \mathcal{M}_d$ and every linear extension $\preccurlyeq$. For instance, take $d = 2$ and fix $\boldsymbol{u} \in U^2$ with u_i a topological generator for U for $i = 1, 2$. Let $\alpha = \delta_{\boldsymbol{u}} - \delta_{\boldsymbol{1}}$. Then $\alpha = \alpha|_{U^2}$ by Example 14, and $\alpha \circ \varphi_{\boldsymbol{u}} = \delta_{\boldsymbol{1}} - \delta_{\boldsymbol{0}}$ by Example 18. Example 22 gives that $L(\alpha \circ \varphi_{\boldsymbol{u}}) = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$, so Proposition 12 gives $L(\alpha|_{U^2}) = \{p\boldsymbol{e}_1, p\boldsymbol{e}_2\}$. By applying the initial construction in Lemma 4 twice and then Theorem 1, there is a linear extension $\preccurlyeq$ for which $\boldsymbol{e}_1 \preccurlyeq \boldsymbol{e}_2$, but $p\boldsymbol{e}_2 \preccurlyeq p\boldsymbol{e}_1$.* ♢

In these final two sections, we specialize to the cases $d = 1$ and $d = 2$, as these will be the setting for the applications in Chapter 3. We explore the construction of product measures to produce elements of $\mathcal{M}_2$ from elements of $\mathcal{M}_1$.

As we have noted before, a polyball A in $\mathbb{Z}_p^2$ is of the form $A = A_1 \times A_2$, with each A_i a ball in $\mathbb{Z}_p$.

Definition 29. *Let $\alpha, \beta \in \mathcal{M}_1$. Suppose $A \in CO_2$ and $A = A_1 \times A_2$ with $A_1, A_2 \in CO_1$. We define*

$$(\alpha \otimes \beta)(A) = \alpha(A_1)\beta(A_2).$$

At the moment $\alpha \otimes \beta$ is defined only on a subset of CO_2. We will show that $\alpha \otimes \beta \in \mathcal{M}_2$ by showing that we can extend $\alpha \otimes \beta$ to a function on all of CO_2 by additivity. That is to say, for $A \in CO_2$, we can write

$$A = \bigsqcup_{i=1}^{n} B_i,$$

with each B_i of the form $B_i = B_1^{(i)} \times B_2^{(i)}$ with $B_1^{(i)}, B_2^{(i)} \in CO_1$ (e.g., we may take the B_i to be polyballs in $\mathbb{Z}_p^2$). We wish to show that

$$(\alpha \otimes \beta)(A) = \sum_{i=1}^{n} (\alpha \otimes \beta)(B_i)$$

gives a well defined function $CO_2 \to R$. This will follow from the following two lemmas, the proofs of which were inspired by the introductory material in Petalas and Katsaras (2010).

Lemma 32. *Let $\alpha, \beta \in \mathcal{M}_1$. Let $A = A_1 \times A_2$ be compact open in $\mathbb{Z}_p^2$ with each $A_i \in CO_1$. Suppose*

$$A = \bigsqcup_{k=1}^{n} B_k$$

where each $B_k = B_1^{(k)} \times B_2^{(k)}$ *with* $B_1^{(k)}, B_2^{(k)} \in CO_1$. *Then*

$$(\alpha \otimes \beta)(A) = \sum_{k=1}^{n} (\alpha \otimes \beta)(B_k).$$

Proof. We induct on n. Certainly, the result holds for $n = 1$. Suppose now that there is $n > 1$ for which the result holds all $i < n$. Let $A = A_1 \times A_2$ be compact open in $\mathbb{Z}_p^2$ with $A_1, A_2 \in CO_1$, and write

$$A = \bigsqcup_{k=1}^{n} B_k$$

where each $B_k = B_1^{(k)} \times B_2^{(k)}$, with $B_1^{(k)}, B_2^{(k)} \in CO_1$. By the induction hypothesis, we may suppose $B_k \neq \emptyset$ for all $1 \leq k \leq n$. We have

$$\begin{aligned}
\bigsqcup_{k=1}^{n-1} B_k &= A \setminus B_n \\
&= A \cap B_n^c \\
&= (A_1 \times A_2) \cap \Big((B_1^{(n)})^c \times \mathbb{Z}_p) \cup (\mathbb{Z}_p \times (B_2^{(n)})^c \Big) \\
&= \Big((A_1 \cap (B_1^{(n)})^c) \times A_2) \cup (A_1 \times (A_2 \cap (B_2^{(n)})^c) \Big).
\end{aligned}$$

Note that $A_1 \cap B_1^{(k)} = B_1^{(k)}$ and $A_2 \cap B_2^{(k)} = B_2^{(k)}$ for all $1 \leq k \leq n$. The calculation above gives

$$\begin{aligned}
(A_1 \cap (B_1^{(n)})^c) \times A_2 &= \Big((A_1 \cap (B_1^{(n)})^c) \times A_2 \Big) \cap \left(\bigsqcup_{k=1}^{n-1} B_k \right) \\
&= \bigsqcup_{k=1}^{n-1} (A_1 \cap (B_1^{(n)})^c \cap B_1^{(k)}) \times (A_2 \cap B_2^{(k)}) \\
&= \bigsqcup_{k=1}^{n-1} ((B_1^{(n)})^c \cap B_1^{(k)}) \times B_2^{(k)}
\end{aligned}$$

and

$$\begin{aligned}
B_1^{(n)} \times (A_2 \cap (B_2^{(n)})^c) &= (A_1 \cap B_1^{(n)}) \times (A_2 \cap (B_2^{(n)})^c) \\
&= \left((A_1 \cap B_1^{(n)}) \times (A_2 \cap (B_2^{(n)})^c)\right) \cap \left(\bigsqcup_{k=1}^{n-1} B_k\right) \\
&= \bigsqcup_{k=1}^{n-1} (A_1 \cap B_1^{(n)} \cap B_1^{(k)}) \times (A_2 \cap (B_2^{(n)})^c \cap B_2^{(k)}) \\
&= \bigsqcup_{k=1}^{n-1} (B_1^{(n)} \cap B_1^{(k)}) \times ((B_2^{(n)})^c \cap B_2^{(k)}).
\end{aligned}$$

The sets $(B_1^{(n)})^c \cap B_1^{(k)}$, $B_1^{(n)} \cap B_1^{(k)}$, and $(B_2^{(n)})^c \cap B_2^{(k)}$ are compact open in $\mathbb{Z}_p$ for all $1 \leq k \leq n-1$. The induction hypothesis gives

$$(\alpha \otimes \beta)\left((A_1 \cap (B_1^{(n)})^c) \times A_2\right) = \sum_{k=1}^{n-1} (\alpha \otimes \beta)\left(((B_1^{(n)})^c \cap B_1^{(k)}) \times B_2^{(k)}\right)$$

$$(\alpha \otimes \beta)\left(B_1^{(n)} \times (A_2 \cap (B_2^{(n)})^c)\right) = \sum_{k=1}^{n-1} (\alpha \otimes \beta)\left((B_1^{(n)} \cap B_1^{(k)}) \times ((B_2^{(n)})^c \cap B_2^{(k)})\right).$$

Let $k < n$. We have

$$\begin{aligned}
(\alpha \otimes \beta)(B_k) &= \alpha(B_1^{(k)})\beta(B_2^{(k)}) \\
&= \alpha\left((B_1^{(k)} \cap B_1^{(n)}) \sqcup (B_1^{(k)} \cap (B_1^{(n)})^c)\right)\beta(B_2^{(k)}) \\
&= \alpha(B_1^{(k)} \cap B_1^{(n)})\beta(B_2^{(k)}) + \alpha(B_1^{(k)} \cap (B_1^{(n)})^c)\beta(B_2^{(k)}) \\
&= \alpha(B_1^{(k)} \cap B_1^{(n)})\beta\left((B_2^{(k)} \cap B_2^{(n)}) \sqcup (B_2^{(k)} \cap (B_2^{(n)})^c)\right) \\
&\qquad + \alpha(B_1^{(k)} \cap (B_1^{(n)})^c)\beta(B_2^{(k)}) \\
&= \alpha(B_1^{(k)} \cap B_1^{(n)})\beta(B_2^{(k)} \cap B_2^{(n)}) \\
&\qquad + \alpha(B_1^{(k)} \cap B_1^{(n)})\beta(B_2^{(k)} \cap (B_2^{(n)})^c) + \alpha(B_1^{(k)} \cap (B_1^{(n)})^c)\beta(B_2^{(k)}).
\end{aligned}$$

However, if both $B_1^{(k)} \cap B_1^{(n)} \neq \emptyset$ and $B_2^{(k)} \cap B_2^{(n)} \neq \emptyset$, then $B_k \cap B_n \neq \emptyset$, which is a contradiction. Thus,

$$\alpha(B_1^{(k)} \cap B_1^{(n)})\beta(B_2^{(k)} \cap B_2^{(n)}) = 0.$$

This gives

$$\begin{aligned}(\alpha\otimes\beta)(B_k) &= \alpha(B_1^{(k)}\cap B_1^{(n)})\beta(B_2^{(k)}\cap (B_2^{(n)})^c) + \alpha(B_1^{(k)}\cap (B_1^{(n)})^c)\beta(B_2^{(k)}) \\ &= (\alpha\otimes\beta)\left((B_1^{(k)}\cap (B_1^{(n)})^c)\times (B_2^{(k)}\cap (B_2^{(n)})^c)\right) \\ &\quad + (\alpha\otimes\beta)\left((B_1^{(k)}\cap (B_1^{(n)})^c)\times B_2^{(k)}\right).\end{aligned}$$

Finally,

$$\begin{aligned}(\alpha\otimes\beta)(A) &= \alpha(A_1)\beta(A_2) \\ &= \alpha\left((A_1\cap (B_1^{(n)})^c)\sqcup (A_1\cap B_1^{(n)})\right)\beta(A_2) \\ &= \alpha(A_1\cap (B_1^{(n)})^c)\beta(A_2) + \alpha(B_1^{(n)})\beta(A_2) \\ &= \alpha(A_1\cap (B_1^{(n)})^c)\beta(A_2) + \alpha(B_1^{(n)})\beta\left((A_2\cap (B_2^{(n)})^c)\sqcup (A_2\cap B_2^{(n)})\right) \\ &= \alpha(A_1\cap (B_1^{(n)})^c)\beta(A_2) + \alpha(B_1^{(n)})\beta(A_2\cap (B_2^{(n)})^c) + \alpha(B_1^{(n)})\beta(B_2^{(n)}) \\ &= (\alpha\otimes\beta)\left((A_1\cap (B_1^{(n)})^c)\times A_2\right) \\ &\quad + (\alpha\otimes\beta)\left(B_1^{(n)}\times (A_2\cap (B_2^{(n)})^c)\right) + (\alpha\otimes\beta)(B_n) \\ &= \left(\sum_{k=1}^{n-1}(\alpha\otimes\beta)\left(((B_1^{(n)})^c\cap B_1^{(k)})\times B_2^{(k)}\right)\right) \\ &\quad + \left(\sum_{k=1}^{n-1}(\alpha\otimes\beta)\left((B_1^{(n)}\cap B_1^{(k)})\times ((B_2^{(n)})^c\cap B_2^{(k)})\right)\right) + (\alpha\otimes\beta)(B_n) \\ &= \left(\sum_{k=1}^{n-1}(\alpha\otimes\beta)(B_k)\right) + (\alpha\otimes\beta)(B_n) \\ &= \sum_{k=1}^{n}(\alpha\otimes\beta)(B_k),\end{aligned}$$

as needed. □

Lemma 33. *Let $\alpha,\beta\in\mathcal{M}_1$. Suppose*

$$\bigsqcup_{k=1}^{n} A_k = \bigsqcup_{j=1}^{m} B_j,$$

where the $A_k, B_j \in CO_2$ *are such that* $A_k = A_1^{(k)} \times A_2^{(k)}$ *and* $B_j = B_1^{(j)} \times B_2^{(j)}$ *where* $A_1^{(k)}, A_2^{(k)}, B_1^{(j)}, B_2^{(j)} \in CO_1$ *for all* $1 \leq k \leq n$ *and* $1 \leq j \leq m$*. Then*

$$\sum_{k=1}^{n}(\alpha \otimes \beta)(A_k) = \sum_{j=1}^{m}(\alpha \otimes \beta)(B_j).$$

Proof. For each $1 \leq k \leq n$,

$$\begin{aligned} A_k &= A_k \cap \left(\bigsqcup_{i=1}^{n} A_i \right) \\ &= A_k \cap \left(\bigsqcup_{j=1}^{m} B_j \right) \\ &= \bigsqcup_{j=1}^{m} (A_k \cap B_j) \end{aligned}$$

and for each $1 \leq j \leq m$,

$$\begin{aligned} B_j &= B_j \cap \left(\bigsqcup_{i=1}^{m} B_j \right) \\ &= B_j \cap \left(\bigsqcup_{k=1}^{m} A_k \right) \\ &= \bigsqcup_{k=1}^{n} (B_j \cap A_k). \end{aligned}$$

For each $1 \leq k \leq n$ and $1 \leq j \leq m$,

$$A_k \cap B_j = (A_1^{(k)} \cap B_1^{(j)}) \times (A_2^{(k)} \cap B_2^{(j)}),$$

where $A_1^{(k)} \cap B_1^{(j)}$ and $A_2^{(k)} \cap B_2^{(j)}$ are compact open in $\mathbb{Z}_p$. By Lemma 32,

$$\begin{aligned} (\alpha \otimes \beta)(A_k) &= \sum_{j=1}^{m} (\alpha \otimes \beta)(A_k \cap B_j) \\ (\alpha \otimes \beta)(B_j) &= \sum_{k=1}^{n} (\alpha \otimes \beta)(A_k \cap B_j). \end{aligned}$$

But now

$$\begin{aligned}\sum_{k=1}^{n}(\alpha\otimes\beta)(A_k) &= \sum_{k=1}^{n}\sum_{j=1}^{m}(\alpha\otimes\beta)(A_k\cap B_j)\\ &= \sum_{j=1}^{m}\sum_{k=1}^{n}(\alpha\otimes\beta)(A_k\cap B_j)\\ &= \sum_{j=1}^{m}(\alpha\otimes\beta)(B_j).\end{aligned}$$

□

Proposition 13. *Let $\alpha, \beta \in \mathcal{M}_1$. Suppose $A \in CO_2$, and write*

$$A = \bigsqcup_{k=1}^{n} B_k,$$

where each $B_k \in CO_2$ is of the form $B_k = B_1^{(k)} \times B_2^{(k)}$ with $B_1^{(k)}, B_2^{(k)} \in CO_1$. Then

$$(\alpha\otimes\beta)(A) = \sum_{k=1}^{n}(\alpha\otimes\beta)(B_k)$$

gives a well-defined element $\alpha\otimes\beta$ in $\mathcal{M}_2$.

Proof. Lemma 33 gives that $\alpha\otimes\beta$ is a well-defined function $CO_2 \to R$ which is additive on disjoint unions. Further, for any $A \in CO_2$, it is clear that

$$\|(\alpha\otimes\beta)(A)\|_R \leq \|\alpha\|_u\|\beta\|_u,$$

the two norms on the right denoting the supremum norm on $\mathcal{M}_1$. □

Example 25. *Fix $s, t \in \mathbb{Z}_p$. Consider the measure $\delta_s \otimes \delta_t \in \mathcal{M}_2$. Let A be a polyball in $\mathbb{Z}_p^2$ and write $A = A_1 \times A_2$ with A_1, A_2 balls in $\mathbb{Z}_p$, then*

$$\begin{aligned}(\delta_s\otimes\delta_t)(A) &= \delta_s(A_1)\delta_t(A_2)\\ &= \begin{cases} 1 & : (s,t) \in A \\ 0 & : (s,t) \notin A \end{cases}\\ &= \delta_{(s,t)}(A).\end{aligned}$$

By Lemma 7, $\delta_s \otimes \delta_t = \delta_{(s,t)}$. $\Diamond$

We endow $\mathcal{M}_1 \times \mathcal{M}_1$ with the maximum norm:

$$\|(\alpha, \beta)\| = \max\{\|\alpha\|_u, \|\beta\|_u\},$$

with $\|\cdot\|_u$ in the line above representing the supremum norm on $\mathcal{M}_1$.

Lemma 34. *Let* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{M}_1$*, and* $c \in R$*. Then*

$$(\alpha_1 + \alpha_2) \otimes \beta_1 = (\alpha_1 \otimes \beta_1) + (\alpha_2 \otimes \beta_1)$$

$$\alpha_1 \otimes (\beta_1 + \beta_2) = (\alpha_1 \otimes \beta_1) + (\alpha_1 \otimes \beta_2)$$

$$(c\alpha_1) \otimes \beta_1 = c(\alpha \otimes \beta) = \alpha_1 \otimes (c\beta_1).$$

Furthermore, the map $(\alpha, \beta) \mapsto \alpha \otimes \beta$ *is continuous* $\mathcal{M}_1 \times \mathcal{M}_1 \to \mathcal{M}_2$*.*

Proof. Let A be a polyball in $\mathbb{Z}_p^2$ and write $A = A_1 \times A_2$ with A_1, A_2 balls in $\mathbb{Z}_p$. Then

$$\begin{aligned}
((\alpha_1 + \alpha_2) \otimes \beta_1)(A) &= (\alpha_1 + \alpha_2)(A_1)\beta_1(A_2) \\
&= (\alpha_1(A_1) + \alpha_2(A_1))\,\beta_1(A_2) \\
&= \alpha_1(A_1)\beta_1(A_2) + \alpha_2(A_1)\beta_1(A_2) \\
&= (\alpha_1 \otimes \beta_1)(A) + (\alpha_2 \otimes \beta_1)(A)
\end{aligned}$$

and

$$\begin{aligned}
(\alpha_1 \otimes (\beta_1 + \beta_2))(A) &= \alpha_1(A_1)(\beta_1 + \beta_2)(A_2) \\
&= \alpha_1(A_1)\,(\beta_1(A_2) + \beta_2(A_2)) \\
&= \alpha_1(A_1)\beta_1(A_2) + \alpha_1(A_1)\beta_2(A_2) \\
&= (\alpha_1 \otimes \beta_1)(A) + (\alpha_1 \otimes \beta_2)(A)
\end{aligned}$$

and

$$
\begin{aligned}
((c\alpha_1) \otimes \beta_1)(A) &= (c\alpha_1)(A_1)\beta_1(A_2) \\
&= (c\alpha_1(A_1))\,\beta_1(A_2) \\
&= c\,(\alpha_1(A_1)\beta_1(A_2)) \\
&= (c(\alpha_1 \otimes \beta_1))(A) \\
&= c\,(\alpha_1(A_1)\beta_1(A_2)) \\
&= \alpha_1(A_1)\,(c\beta_1(A_2)) \\
&= \alpha_1(A_1)(c\beta_1)(A_2) \\
&= (\alpha_1 \otimes (c\beta_1))\,(A).
\end{aligned}
$$

By Lemma 7, we have

$$
\begin{aligned}
(\alpha_1 + \alpha_2) \otimes \beta_1 &= (\alpha_1 \otimes \beta_1) + (\alpha_2 \otimes \beta_1) \\
\alpha_1 \otimes (\beta_1 + \beta_2) &= (\alpha_1 \otimes \beta_1) + (\alpha_1 \otimes \beta_2) \\
(c\alpha_1) \otimes \beta_1 = c(\alpha \otimes \beta) &= \alpha_1 \otimes (c\beta_1),
\end{aligned}
$$

as claimed.

For continuity, we saw in the proof of Proposition 13 that

$$\|(\alpha \otimes \beta)\|_u \leq \|\alpha\|_u \|\beta\|_u$$

for all $\alpha, \beta \in \mathcal{M}_1$.

Let $(\alpha, \beta) \in \mathcal{M}_1 \times \mathcal{M}_1$ and $\varepsilon > 0$. Fix

$$0 < \delta < \min\left\{1, \frac{\varepsilon}{1 + \|(\alpha, \beta)\|}\right\}.$$

Let $(\alpha_1, \beta_1) \in \mathcal{M}_1 \times \mathcal{M}_1$ with

$$\|(\alpha, \beta) - (\alpha_1, \beta_1)\| = \max\{\|\alpha - \alpha_1\|_u, \|\beta - \beta_1\|_u\} < \delta.$$

Then

$$\begin{aligned}
\|(\alpha_1 \otimes \beta_1) - (\alpha \otimes \beta)\|_u &= \|(\alpha \otimes (\beta_1 - \beta)) + ((\alpha_1 - \alpha) \otimes (\beta_1 - \beta)) + ((\alpha_1 - \alpha) \otimes \beta)\|_u \\
&\leq \max\{\|\alpha \otimes (\beta_1 - \beta)\|_u, \|(\alpha_1 - \alpha) \otimes (\beta_1 - \beta)\|_u, \|(\alpha_1 - \alpha) \otimes \beta\|_u\} \\
&= \max\{\|\alpha\|_u\|\beta_1 - \beta\|_u, \|\alpha_1 - \alpha\|_u\|\beta_1 - \beta\|_u, \|\alpha_1 - \alpha\|_u\|\beta\|_u\} \\
&< \max\{\|\alpha\|_u\delta, \delta^2, \|\beta\|_u\delta\} \\
&< \varepsilon.
\end{aligned}$$

This gives the continuity of the map $(\alpha, \beta) \mapsto \alpha \otimes \beta$. □

Lemma 35. *Let $\alpha, \beta \in \mathcal{M}_1$. Suppose $f \in C(\mathbb{Z}_p^2, R)$ is such that there are $g, h \in C(\mathbb{Z}_p, R)$ with*

$$f(\boldsymbol{x}) = g(x_1)h(x_2)$$

for all $\boldsymbol{x} \in \mathbb{Z}_p^2$. Then

$$\int_{\mathbb{Z}_p^2} f(\boldsymbol{x})\, d(\alpha \otimes \beta)(\boldsymbol{x}) = \left(\int_{\mathbb{Z}_p} g(x)\, d\alpha(x)\right)\left(\int_{\mathbb{Z}_p} h(x)\, d\beta(x)\right).$$

Proof. For $\boldsymbol{n} \in \mathbb{N}^2$ and $n \in \mathbb{N}$, we put

$$\begin{aligned}
f_{\boldsymbol{n}}(\boldsymbol{x}) &= \sum_{\boldsymbol{a} \preccurlyeq_2 \boldsymbol{p}^{\wedge \boldsymbol{n}} - \boldsymbol{1}} f(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}) \in LC(\mathbb{Z}_p^2, R) \\
g_n(x) &= \sum_{b=0}^{p^n - 1} g(b) g_{b,n}(x) \in LC(\mathbb{Z}_p, R) \\
h_n(x) &= \sum_{c=0}^{p^n - 1} h(c) g_{c,n}(x) \in LC(\mathbb{Z}_p, R).
\end{aligned}$$

The 2-net $(f_{\boldsymbol{n}})$ converges uniformly to f, and the sequences (g_n) and (h_n) converge uniformly to g and h, respectively. Note that

$$\begin{aligned}\int_{\mathbb{Z}_p^2} f_{\boldsymbol{n}}(\boldsymbol{x})\, d(\alpha\otimes\beta)(\boldsymbol{x}) &= \sum_{\boldsymbol{a}\preccurlyeq_2 \boldsymbol{p}^{\wedge\boldsymbol{n}}-\boldsymbol{1}} f(\boldsymbol{a})(\alpha\otimes\beta)(\boldsymbol{a}+\boldsymbol{p}^{\wedge\boldsymbol{n}}\mathbb{Z}_p^d) \\ &= \sum_{\boldsymbol{a}\preccurlyeq_2 \boldsymbol{p}^{\wedge\boldsymbol{n}}-\boldsymbol{1}} g(a_1)h(a_2)\alpha(a_1+p^{n_1}\mathbb{Z}_p)\beta(a_2+p^{n_2}\mathbb{Z}_p) \\ &= \left(\sum_{b=0}^{p^{n_1}-1} g(b)\alpha(b+p^{n_1}\mathbb{Z}_p)\right)\left(\sum_{c=0}^{p^{n_2}-1} h(c)\beta(c+p^{n_2}\mathbb{Z}_p)\right) \\ &= \left(\int_{\mathbb{Z}_p} g_{n_1}(x)\, d\alpha(x)\right)\left(\int_{\mathbb{Z}_p} h_{n_2}(x)\, d\beta(x)\right).\end{aligned}$$

Taking the limit on $\boldsymbol{n}\in\mathbb{N}^2$ gives

$$\int_{\mathbb{Z}_p^2} f(\boldsymbol{x})\, d(\alpha\otimes\beta)(\boldsymbol{x}) = \left(\int_{\mathbb{Z}_p} g(x)\, d\alpha(x)\right)\left(\int_{\mathbb{Z}_p} h(x)\, d\beta(x)\right),$$

as claimed. □

Corollary 22. *Let $\alpha,\beta\in\mathcal{M}_1$. Then*

$$\widehat{(\alpha\otimes\beta)}(\boldsymbol{T}) = \widehat{\alpha}(T_1)\widehat{\beta}(T_2).$$

Proof. Let $\boldsymbol{n}\in\mathbb{N}^2$. For $\boldsymbol{x}\in\mathbb{Z}_p^2$,

$$\binom{\boldsymbol{x}}{\boldsymbol{n}} = \binom{x_1}{n_1}\binom{x_2}{n_2}.$$

Using Lemma 35, we find that the $\boldsymbol{n}$th coefficient of $\widehat{(\alpha\otimes\beta)}$ is

$$\int_{\mathbb{Z}_p^2} \binom{\boldsymbol{x}}{\boldsymbol{n}}\, d(\alpha\otimes\beta)(\boldsymbol{x}) = \left(\int_{\mathbb{Z}_p} \binom{x}{n_1}\, d\alpha(x)\right)\left(\int_{\mathbb{Z}_p} \binom{x}{n_2}\, d\beta(x)\right),$$

which is the $\boldsymbol{n}$th coefficient of $\widehat{\alpha}(T_1)\widehat{\beta}(T_2)$. □

Corollary 23. *Let $\alpha,\alpha_1,\beta,\beta_1\in\mathcal{M}_1$. Then*

$$(\alpha\otimes\beta)*(\alpha_1\otimes\beta_1) = (\alpha*\alpha_1)\otimes(\beta*\beta_1).$$

Consequently, the map $\alpha\mapsto\alpha\otimes\delta_0$ is a norm-preserving embedding of R-algebras $\mathcal{M}_1\to\mathcal{M}_2$.

Proof. We have

$$\begin{aligned}\left((\alpha\otimes\beta)\widehat{)*(}\alpha_1\otimes\beta_1)\right)(\boldsymbol{T}) &= (\widehat{\alpha\otimes\beta})(\boldsymbol{T})(\widehat{\alpha_1\otimes\beta_1})(\boldsymbol{T})\\ &= \widehat{\alpha}(T_1)\widehat{\beta}(T_2)\widehat{\alpha}_1(T_1)\widehat{\beta}_1(T_2)\\ &= \widehat{\alpha}(T_1)\widehat{\alpha}_1(T_1)\widehat{\beta}(T_2)\widehat{\beta}_1(T_2)\\ &= (\widehat{\alpha*\alpha_1})(T_1)(\widehat{\beta*\beta_1})(T_2)\\ &= \left((\alpha*\alpha_1)\widehat{)\otimes(}\beta*\beta_1)\right)(\boldsymbol{T}).\end{aligned}$$

Now, we saw in Example 5 that

$$\delta_0*\delta_0=\delta_{0+0}=\delta_0.$$

The above work, together with Lemma 34 gives that $\alpha\mapsto\alpha\otimes\delta_0$ is an R-algebra homomorphism $\mathcal{M}_1\to\mathcal{M}_2$. If $\alpha\otimes\delta_0=\alpha_0$, then for any $A\in CO_1$,

$$0=(\alpha\otimes\delta_0)(A\times\mathbb{Z}_p)=\alpha(A)\delta_0(\mathbb{Z}_p)=\alpha(A).$$

Thus, the map $\alpha\mapsto\alpha\otimes\delta_0$ is injective. Similarly, we saw in the proof of Proposition 13 that

$$\|\alpha\otimes\delta_0\|_u\le\|\alpha\|_u\|\delta_0\|_u=\|\alpha\|_u.$$

Since $\alpha(A)=(\alpha\otimes\delta_0)(A\times\mathbb{Z}_p)$ for all $A\in CO_1$, we have

$$\|\alpha\|_u\le\|\alpha\otimes\delta_0\|_u.$$

Thus, the map $\alpha\mapsto\alpha\otimes\delta_0$ is norm-preserving. □

We will identify $\mathcal{M}_1$ with its image in $\mathcal{M}_2$ under the embedding $\alpha\mapsto\alpha\otimes\delta_{\mathbf{0}}$. In particular, we will write α_0 for the zero measure in both $\mathcal{M}_1$ and $\mathcal{M}_2$.

Lemma 36. *Let $\alpha,\beta\in\mathcal{M}_1$, and let $\boldsymbol{a}\in(\mathbb{Z}_p^\times)^2$. Then*

$$(\alpha\otimes\beta)\circ\boldsymbol{a}=(\alpha\circ a_1)\otimes(\beta\circ a_2).$$

Proof. We have

$$
\begin{aligned}
((\widehat{\alpha \otimes \beta) \circ \boldsymbol{a}})(\boldsymbol{T}) &= (\widehat{\alpha \otimes \beta})(\boldsymbol{T}^{\wedge \boldsymbol{a}^{-1}}) \\
&= \widehat{\alpha}(T_1^{a_1^{-1}})\widehat{\beta}(T_2^{a_2^{-1}}) \\
&= (\widehat{\alpha \circ a_1})(T_1)(\widehat{\beta \circ a_2})(T_2) \\
&= \left((\alpha \circ \widehat{a_1) \otimes (}\beta \circ a_2)\right)(\boldsymbol{T}).
\end{aligned}
$$

The equality of power series gives the claimed equality of measures. ☐

Lemma 37. *Let $\alpha, \beta \in \mathcal{M}_1$. Suppose $g \in C(\mathbb{Z}_p^2, R)$ is such that $g(\boldsymbol{x}) = g_1(x_1)g_2(x_2)$ for some $g_1, g_2 \in C(\mathbb{Z}_p, R)$. Then*

$$
(\alpha \otimes \beta)_g = \alpha_{g_1} \otimes \beta_{g_2}.
$$

Proof. If $f \in LC(\mathbb{Z}_p^2, R)$, then $f(\boldsymbol{x}) = f_1(x_1)f_2(x_2)$ for some $f_1, f_2 \in LC(\mathbb{Z}_p, R)$. Lemma 35 gives

$$
\begin{aligned}
\int_{\mathbb{Z}_p^2} f(\boldsymbol{x})\, d(\alpha \otimes \beta)_g(\boldsymbol{x}) &= \int_{\mathbb{Z}_p^2} f(\boldsymbol{x}) g(\boldsymbol{x})\, d(\alpha \otimes \beta)(\boldsymbol{x}) \\
&= \left(\int_{\mathbb{Z}_p} f_1(x) g_1(x)\, d\alpha(x)\right)\left(\int_{\mathbb{Z}_p} f_2(x) g_2(x)\, d\beta(x)\right) \\
&= \left(\int_{\mathbb{Z}_p} f_1(x)\, d\alpha_{g_1}(x)\right)\left(\int_{\mathbb{Z}_p} f_2(x)\, d\beta_{g_2}(x)\right) \\
&= \int_{\mathbb{Z}_p^2} f(\boldsymbol{x})\, d(\alpha_{g_1} \otimes \beta_{g_2}).
\end{aligned}
$$

We obtain

$$
\int_{\mathbb{Z}_p^2} f(\boldsymbol{x})\, d(\alpha \otimes \beta)_g(\boldsymbol{x}) = \int_{\mathbb{Z}_p^2} f(\boldsymbol{x})\, d(\alpha_{g_1} \otimes \beta_{g_2})(\boldsymbol{x})
$$

for every $f \in C(\mathbb{Z}_p^2, R)$ by realizing f as a uniform limit of locally constant functions.

☐

Corollary 24. *Let $\alpha, \beta \in \mathcal{M}_1$. Suppose $A \in CO_2$ is such that $A = A_1 \times A_2$, with $A_1, A_2 \in CO_1$. Then*

$$(\alpha \otimes \beta)|_A = \alpha|_{A_1} \otimes \beta|_{A_2}.$$

Proof. Since $A = A_1 \times A_2$, we have that $g_A(\boldsymbol{x}) = g_{A_1}(x_1)g_{A_2}(x_2)$ for all $\boldsymbol{x} \in \mathbb{Z}_p^2$. The Corollary now follows immediately from Lemma 16 and Lemma 37. □

Lemma 38. *Let $\alpha, \beta \in \mathcal{M}_1$. Let $\boldsymbol{u} \in U^2$ be such that u_1, u_2 are topological generators of U. Then*

$$(\alpha \otimes \beta) \circ \varphi_{\boldsymbol{u}} = (\alpha \circ \varphi_{u_1}) \otimes (\beta \circ \varphi_{u_2}).$$

Proof. Let A be a polyball in $\mathbb{Z}_p^2$, and write $A = A_1 \times A_2$ for balls A_1, A_2 in $\mathbb{Z}_p$. Since

$$\varphi_{\boldsymbol{u}}(A) = \varphi_{u_1}(A_1) \times \varphi_{u_2}(A_2),$$

we readily obtain

$$\begin{aligned}
((\alpha \otimes \beta) \circ \varphi_{\boldsymbol{u}})\,(A) &= (\alpha \otimes \beta)(\varphi_{\boldsymbol{u}}(A)) \\
&= (\alpha \otimes \beta)(\varphi_{u_1}(A_1) \times \varphi_{u_2}(A_2)) \\
&= \alpha(\varphi_{u_1}(A_1))\beta(\varphi_{u_2}(A_2)) \\
&= (\alpha \circ \varphi_{u_1})(A_1)(\beta \circ \varphi_{u_2})(A_2) \\
&= ((\alpha \circ \varphi_{u_1}) \otimes (\beta \circ \varphi_{u_2}))\,(A).
\end{aligned}$$

Thus,

$$(\alpha \otimes \beta) \circ \varphi_{\boldsymbol{u}} = (\alpha \circ \varphi_{u_1}) \otimes (\beta \circ \varphi_{u_2})$$

by Lemma 7. □

Corollary 25. *Let $\alpha, \beta \in \mathcal{M}_1$. For each $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p} - \boldsymbol{2}$,*

$$\gamma^{(\boldsymbol{i})}(\alpha \otimes \beta) = \gamma^{(i_1)}(\alpha) \otimes \gamma^{(i_2)}(\beta)$$

$$\Gamma^{(\boldsymbol{i})}_{\alpha\otimes\beta}(\boldsymbol{s}) = \Gamma^{(i_1)}_{\alpha}(s_1)\Gamma^{(i_2)}_{\beta}(s_2)$$

$$G^{(\boldsymbol{i})}_{\alpha\otimes\beta}(\boldsymbol{T}) = G^{(i_1)}_{\alpha}(T_1)G^{(i_2)}_{\beta}(T_2).$$

□

Finally, suppose R is a discrete valuation ring, as in Section 2.5.

Lemma 39. *Let $\alpha, \beta \in \mathcal{M}_1$. Then $\mu(\alpha \otimes \beta) = \mu(\alpha) + \mu(\beta)$, and $L(\alpha \otimes \beta) = \{(\lambda(\alpha), \lambda(\beta))\}$, where $\lambda(\alpha), \lambda(\beta)$ are the unique univariate λ-invariants of α and β, respectively.*

Proof. This is a special case of Lemma 25. □

Example 26. *Let $\alpha \in \mathcal{M}_1$ be nonzero and let $s \in \mathbb{Z}_p$. In Example 21, we saw that $\mu(\delta_s) = 0$ and $\lambda(\delta_s) = 0$. Consequently,*

$$\mu(\alpha \otimes \delta_s) = \mu(\alpha),$$

and

$$L(\alpha \otimes \delta_s) = \{\lambda(\alpha)\boldsymbol{e}_1\}.$$

When $\alpha = \delta_t$ for some $t \in \mathbb{Z}_p$, we saw in Example 25 that $\delta_t \otimes \delta_s = \delta_{(t,s)}$. The above gives $\mu(\delta_{(t,s)}) = 0$ and $L(\delta_{(t,s)}) = \{\boldsymbol{0}\}$, as shown in Example 21. ♢

2.7 Pseudo-Polynomials

We continue to assume that R is a discrete valuation ring with valuation ord_R and uniformizer π. We will use the fact that

$$R[[\boldsymbol{T} - \boldsymbol{1}]] \cong \varprojlim R[[T_1 - 1]][[T_2 - 1]]/\langle T_2^{p^n} - 1\rangle$$

under the current assumptions on R.

Definition 30. *A (one-variable) pseudo-polynomial over R is a power series $F \in \Lambda_1$ which can be written in the form*

$$F(T) = \sum_{k=1}^{n} c_k T^{x_k},$$

where $c_k \in R$ and $x_k \in \mathbb{Z}_p$ for each $1 \le k \le n$.

Definition 31. *A pseudo-polynomial in T_2 over R is a power series $F \in \Lambda_2$ which can be written in the form*

$$F(T_1, T_2) = \sum_{k=1}^{n} g_k(T_1) T_2^{x_k},$$

where $g_k \in \Lambda_1$ and $x_k \in \mathbb{Z}_p$ for each $1 \le k \le n$. If the g_k are all of the form $g_k(T_1) = c_k T_1^{y_k}$ for $c_k \in R$ and $y_k \in \mathbb{Z}_p$, then F is called a (two-variable) pseudo-polynomial over R. We say that $\alpha \in \mathcal{M}_2$ is a pseudo-polynomial measure in T_2 if $\widehat{\alpha}$ is a pseudo-polynomial in T_2.

Let

$$F(T_1, T_2) = \sum_{k=1}^{n} g_k(T_1) T_2^{x_k}$$

be a pseudo-polynomial in T_2 over R. If $\alpha_k \in \mathcal{M}_1$ is the measure associated to g_k, then the measure $\alpha \in \mathcal{M}_2$ associated to F is

$$\alpha = \sum_{k=1}^{n} \alpha_k \otimes \delta_{x_k}.$$

If F is a two-variable pseudo-polynomial over R, then each $\alpha_k = c_k \delta_{y_k}$ for some $c_k \in R$ and $y_k \in \mathbb{Z}_p$. Then

$$\alpha = \sum_{k=1}^{n} (c_k \delta_{y_k}) \otimes \delta_{x_k} = \sum_{k=1}^{n} c_k \delta_{(y_k, x_k)}.$$

From the discussion in Appendix B, we see that the collection of all pseudo-polynomials in T_2 over R forms a subring of $\Lambda_2 = R[[T_1 - 1, T_2 - 1]]$.

Using the results from Section 2.6 and the examples from Section 2.4, we can describe the Γ-transform of a pseudo-polynomial in T_2 explicitly. Let

$$F(T_1, T_2) = \sum_{k=1}^{n} g_k(T_1) T_2^{x_k}$$

be a pseudo-polynomial in T_2 over R with associated measure

$$\alpha = \sum_{k=1}^{n} \alpha_k \otimes \delta_{x_k}.$$

Fix $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p-2}$. Then Corollary 25, Example 20, and Lemma 34 give

$$\begin{aligned}
\gamma^{(\boldsymbol{i})}(\alpha) &= \sum_{k=1}^{n} \gamma^{(i_1)}(\alpha_k) \otimes \gamma^{(i_2)}(\delta_{x_k}) \\
&= \sum_{k=1}^{n} \gamma^{(i_1)}(\alpha_k) \otimes \left(\sum_{\eta \in V} \eta^{i_2} \delta_{\eta^{-1} x_k} \right) \\
&= \sum_{k=1}^{n} \sum_{\eta \in V} \left(\eta^{i_2} \gamma^{(i_1)}(\alpha_k) \otimes \delta_{\eta^{-1} x_k} \right).
\end{aligned}$$

Now, $\eta^{-1} x_k \in U$ if and only if $x_k \in \mathbb{Z}_p^\times$ and $\eta = \omega(x_k)$. Consequently, Lemma 13 and Lemma 24 give

$$\begin{aligned}
\gamma^{(\boldsymbol{i})}(\alpha)|_{U^2} &= \sum_{k=1}^{n} \sum_{\eta \in V} \left(\eta^{i_2} \gamma^{(i_1)}(\alpha_k)|_U \otimes \delta_{\eta^{-1} x_k}|_U \right) \\
&= \sum_{\substack{1 \le k \le n \\ x_k \in \mathbb{Z}_p^\times}} \left(\omega^{i_2}(x_k) \gamma^{(i_1)}(\alpha_k)|_U \otimes \delta_{\omega^{i_2}(x_k^{-1}) x_k} \right).
\end{aligned}$$

Finally, Lemma 19 and Lemma 38 give

$$\begin{aligned}
\gamma^{(\boldsymbol{i})}(\alpha) \circ \varphi_{\boldsymbol{u}} &= \sum_{\substack{1 \le k \le n \\ x_k \in \mathbb{Z}_p^\times}} \left(\left(\omega^{i_2}(x_k) \gamma^{(i_1)}(\alpha_k) \circ \varphi_{u_1} \right) \otimes \left(\delta_{\omega^{i_2}(x_k^{-1}) x_k} \circ \varphi_{u_2} \right) \right) \\
&= \sum_{\substack{1 \le k \le n \\ x_k \in \mathbb{Z}_p^\times}} \left(\omega^{i_2}(x_k) \left(\gamma^{(i_1)}(\alpha_k) \circ \varphi_{u_1} \right) \otimes \delta_{\ell_{u_2}(x_k)} \right),
\end{aligned}$$

so

$$G_\alpha^{(\boldsymbol{i})}(\boldsymbol{T}) = \sum_{\substack{1 \le k \le n \\ x_k \in \mathbb{Z}_p^\times}} \omega^{i_2}(x_k) G_{\alpha_k}^{(i_1)}(T_1) T_2^{\ell_{u_2}(x_k)}.$$

Proposition 14. *Suppose $\alpha \in \mathcal{M}_2$ is a pseudo polynomial measure in T_2. If $\widehat{\alpha} \equiv 0 \pmod{T_2^{p^n} - 1}$, then for each $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p} - \boldsymbol{2}$,*

$$\widehat{\gamma^{(\boldsymbol{i})}(\alpha)} \equiv 0 \pmod{T_2^{p^n} - 1}$$

and

$$G_\alpha^{(\boldsymbol{i})} \equiv 0 \pmod{T_2^{p^{n-1}} - 1}.$$

Proof. Let

$$\alpha = \sum_{k=1}^{n} \alpha_k \otimes \delta_{x_k},$$

and write

$$\widehat{\alpha}(\boldsymbol{T}) = \sum_{k=1}^{n} g_k(T_1) T_2^{x_k},$$

so that $\widehat{\alpha}_k = g_k$. Suppose that $\widehat{\alpha} \equiv 0 \pmod{T_2^{p^n} - 1}$. Since

$$\widehat{\alpha}(\boldsymbol{T}) \equiv \sum_{a=0}^{p^n - 1} \sum_{\substack{1 \le k \le n \\ x_k \equiv a \pmod{p^n}}} g_k(T_1) T_2^a \pmod{T_2^{p^n} - 1},$$

we must have that

$$\sum_{\substack{1 \le k \le n \\ x_k \equiv a \pmod{p^n}}} g_k(T_1) = 0$$

for all $0 \le a < p^n$, as the T_2^a are distinct and Λ_1-linearly independent modulo the ideal $(T_2^{p^n} - 1)$ of $\Lambda_2 \cong \Lambda_1[[T_2 - 1]]$ (Appendix B). But then

$$\alpha_0 = \gamma^{(i_1)} \left(\sum_{\substack{1 \le k \le n \\ x_k \equiv a \pmod{p^n}}} \alpha_k \right) = \sum_{\substack{1 \le k \le n \\ x_k \equiv a \pmod{p^n}}} \gamma^{(i_1)}(\alpha_k)$$

for all $0 \le a < p^n$. Thus, for each $\eta \in V$,

$$\alpha_0 = \sum_{\substack{1 \le k \le n \\ x_k \equiv a \pmod{p^n}}} \eta^{i_2} \gamma^{(i_1)}(\alpha_k)$$

for all $0 \le a < p^n$. It follows that

$$\alpha_0 = \sum_{a=0}^{p^n-1} \left(\sum_{\substack{1 \le k \le n \\ x_k \equiv a \pmod{p^n}}} \eta^{i_2} \gamma^{(i_1)}(\alpha_k) \right) \otimes \delta_{\eta^{-1}a}.$$

We have

$$\begin{aligned}
\widehat{\gamma^{(\boldsymbol{i})}}(\alpha)(T_1, T_2) &= \sum_{k=1}^{n} \sum_{\eta \in V} \eta^{i_2} \widehat{\gamma^{(i_1)}(\alpha_k)}(T_1) T_2^{\eta^{-1}x_k} \\
&\equiv \sum_{a=0}^{p^n-1} \sum_{\eta \in V} \sum_{\substack{1 \le k \le n \\ x_k \equiv a \pmod{p^n}}} \eta^{i_2} \widehat{\gamma^{(i_1)}(\alpha_k)}(T_1) T_2^{\eta^{-1}a} \pmod{T_2^{p^n} - 1} \\
&\equiv \sum_{\eta \in V} \left(\sum_{a=0}^{p^n-1} \sum_{\substack{1 \le k \le n \\ x_k \equiv a \pmod{p^n}}} \eta^{i_2} \widehat{\gamma^{(i_1)}(\alpha_k)}(T_1) T_2^{\eta^{-1}a} \right) \pmod{T_2^{p^n} - 1} \\
&\equiv 0 \pmod{T_2^{p^n} - 1}.
\end{aligned}$$

On the other hand, we also have that

$$\alpha_0 = \sum_{\substack{1 \le k \le n \\ x_k \equiv a \pmod{p^n}}} \gamma^{(i_1)}(\alpha_k) \circ \varphi_{u_1}$$

for all $0 \le a < p^n$. Since $x_k \equiv a \pmod{p^n}$ implies $\omega(x_k) = \omega(a)$, we further obtain

$$\alpha_0 = \sum_{\substack{1 \le k \le n \\ x_k \equiv a \pmod{p^n}}} \omega(x_k) \left(\gamma^{(i_1)}(\alpha_k) \circ \varphi_{u_1} \right)$$

for all $0 \le a < p^n$.

But since $\ell_{u_2}(x) \equiv b \pmod{p^{n-1}}$ if and only if $\langle x \rangle \equiv u_2^b \pmod{p^n}$, we have that

$$\alpha_0 = \sum_{\substack{1 \le k \le n \\ \ell_{u_2}(x_k) \equiv b \pmod{p^{n-1}}}} \omega(x_k) \left(\gamma^{(i_1)}(\alpha_k) \circ \varphi_{u_1} \right)$$

for all $0 \leq b < p^{n-1}$. From this, it follows that

$$
\begin{aligned}
(\gamma^{(\boldsymbol{i})}\widehat{(\alpha)\circ}\varphi_{\boldsymbol{u}})(T_1,T_2) &= \sum_{\substack{1\leq k\leq n \\ x_k\in\mathbb{Z}_p^\times}} \omega^{i_2}(x_k)(\gamma^{(i_1)}\widehat{(\alpha_k)\circ}\varphi_{u_1})(T_1)T_2^{\ell_{u_2}(x_k)} \\
&\equiv \sum_{b=0}^{p^{n-1}-1}\left(\sum_{\substack{1\leq k\leq n \\ \ell_{u_2}(x_k)\equiv b \pmod{p^{n-1}}}} \omega(x_k)(\gamma^{(i_1)}\widehat{(\alpha_k)\circ}\varphi_{u_1})(T_1)\right)T_2^b \\
&\qquad\qquad\qquad\qquad \pmod{T_2^{p^{n-1}}-1} \\
&\equiv 0 \pmod{T_2^{p^{n-1}}-1}.
\end{aligned}
$$

□

Corollary 26. *Suppose (F_n) is a sequence in Λ_2 of pseudo-polynomials in T_2 such that $F_m \equiv F_n \pmod{T_2^{p^n}-1}$ for all $m \geq n$. For each $n \in \mathbb{N}$, let α_n be the measure associated to F_n. Let $F = \lim F_n$ and let α be the measure associated to F. Then for all $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p-2}$,*

$$
\widehat{\gamma^{(\boldsymbol{i})}(\alpha)} = \lim_n \widehat{\gamma^{(\boldsymbol{i})}(\alpha_n)}
$$

$$
G_\alpha^{(\boldsymbol{i})} = \lim_n G_{\alpha_n}^{(\boldsymbol{i})}.
$$

□

We will use the following one-variable result (see Rosenberg (2004) or Childress and Zinzer (2015)) to describe the invariants of a pseudo-polynomial in T_2 over R.

Theorem 5. *Let*

$$
F(T) = \sum_{k=1}^{n} c_k T^{x_k}
$$

be a nonzero pseudo-polynomial in Λ_1 with the x_k distinct elements of $\mathbb{Z}_p$. Put

$$
S = \{k : \mathrm{ord}_R(c_k) = \min\{\mathrm{ord}_R(c_i) : 1 \leq i \leq n\}\}
$$

and define

$$N = N(F) := \max\{\mathrm{ord}_p(x_k - x_l) : k, l \in S, k \neq l\}$$

(with $\max(\emptyset) = -1$*). Then*

$$\mu(F) = \min\{\mathrm{ord}_R(c_k) : 1 \leq k \leq n\}$$

and $\lambda(F) < p^{N+1}$.

Proof. Let $\alpha \in \mathcal{M}_1$ be the measure associated to F, so that

$$\alpha = \sum_{k=1}^{n} c_k \delta_{x_k}.$$

First, for any $n \in \mathbb{N}$ and any $A \in L_n$,

$$\begin{aligned} \mathrm{ord}_R(\alpha(A)) &= \mathrm{ord}_R\left(\sum_{k=1}^{n} c_k \delta_{x_k}(A)\right) \\ &= \mathrm{ord}_R\left(\sum_{\substack{1 \leq k \leq n \\ x_k \in A}} c_k\right) \\ &\geq \min\{\mathrm{ord}_R(c_k) : 1 \leq k \leq n\}. \end{aligned}$$

By Lemma 27,

$$\mu(F) \geq \min\{\mathrm{ord}_R(c_k) : 1 \leq k \leq n\}.$$

By the definition of N, the x_k for $k \in S$ are distinct modulo p^{N+1}. Fix any $s \in S$ and put $A = x_s + p^{N+1}\mathbb{Z}_p \in L_{N+1}$. Then $x_k \notin A$ for all $k \in S \setminus \{s\}$. Since

$\operatorname{ord}_R(c_k) < \operatorname{ord}_R(c_s)$ for all $k \notin S$, we have

$$\begin{aligned}\operatorname{ord}_R(\alpha(A)) &= \operatorname{ord}_R\left(\sum_{k=1}^{n} c_k \delta_{x_k}(A)\right)\\ &= \operatorname{ord}_R\left(\sum_{\substack{1\le k\le n\\ x_k\in A}} c_k\right)\\ &= \operatorname{ord}_R\left(c_s + \sum_{\substack{k\notin S\\ x_k\in A}} c_k\right)\\ &= \operatorname{ord}_R(c_s).\end{aligned}$$

By Lemma 27,

$$\mu(F) \le \min\{\operatorname{ord}_R(c_k) : 1 \le k \le n\},$$

so we must have

$$\mu(F) = \min\{\operatorname{ord}_R(c_k) : 1 \le k \le n\}.$$

Furthermore, Lemma 28 now applies to give $\lambda(F) < p^{N+1}$. □

Theorem 6. *Let*

$$F(\boldsymbol{T}) = \sum_{k=1}^{n} g_k(T_1) T_2^{x_k} \in \Lambda_2$$

be a nonzero pseudo-polynomial in T_2 *over* R *with the* x_k *distinct elements of* $\mathbb{Z}_p$*. Let*

$$S = \{k : \mu(g_k) = \min\{\mu(g_i) : 1 \le i \le n\}\}$$

and let

$$X = \left\{k \in S : \lambda(g_k) = \min_{i\in S}\{\lambda(g_i)\}\right\}.$$

Define

$$N = N(F) := \max\{\operatorname{ord}_p(x_k - x_j) : k, j \in X, j \ne k\}$$

(with $\max(\emptyset) = -1$*), and put*

$$\ell = \min\{\lambda(g_k) : k \in S\}.$$

Then

$$\mu(F) = \min\{\mu(g_k) : 1 \le k \le n\}$$

and F *is nonzero modulo the ideal* $J_{(\ell+1,p^{N+1})}(\pi^{\mu(F)+1})$*. Moreover, there is some* $\lambda \in L(F)$ *with* $\lambda \preccurlyeq_2 (\ell, p^{N+1}-1)$*.*

Proof. It suffices to consider the case where

$$0 = \min\{\mu(g_i) : 1 \le i \le n\}.$$

For $k \in X$, let c_k denote the ℓth coefficient of $g_k(T_1)$. By the definition of X and S, $\mathrm{ord}_R(c_k) = 0$ and

$$g_k(T_1) \equiv \left\{ \begin{array}{ll} c_k(T_1-1)^\ell & : k \in X \\ 0 & : k \notin X \end{array} \right] \pmod{\pi, (T_1-1)^{\ell+1}}.$$

Consequently,

$$\begin{aligned} F &\equiv \sum_{k\in X} c_k (T_1-1)^\ell T_2^{x_k} \pmod{\pi, (T_1-1)^{\ell+1}} \\ &\equiv (T_1-1)^\ell \sum_{k \in X} c_k T_2^{x_k} \pmod{\pi, (T_1-1)^{\ell+1}}. \end{aligned}$$

Put

$$G(T) = \sum_{k\in X} c_k T^{x_k} \in \Lambda_1 \setminus \{0\}.$$

By Theorem 5, $\mu(G) = 0$ and $\lambda(G) < p^{N+1}$. Certainly, $\mu((T-1)^\ell) = 0$ and $\lambda((T-1)^\ell) = \ell$. By Lemma 25,

$$(T_1-1)^\ell \sum_{k\in T} c_k T_2^{x_k} \not\equiv 0 \pmod{J_{\ell+1,p^{N+1}}(\pi)},$$

and the $(\ell, \lambda(G))$th coefficient of

$$(T_1 - 1)^\ell \sum_{k \in X} c_k T_2^{x_k} = (T_1 - 1)^\ell G(T_2)$$

is nonzero modulo π. Since

$$F \equiv (T_1 - 1)^\ell G(T_2) \pmod{J_{\ell+1, p^{N+1}}(\pi)},$$

this gives that $\mu(F) = 0$, and we have shown that the $(\ell, \lambda(G))$th coefficient of F is nonzero modulo π. Thus F is nonzero modulo the ideal $J_{(\ell+1, p^{N+1})}(\pi^{\mu(F)+1})$, and there exists $\lambda \in L(F)$ with

$$\lambda \preccurlyeq_2 (\ell, \lambda(G)) \preccurlyeq_2 (\ell, p^{N+1} - 1).$$

□

Corollary 27. *Let*

$$F(\boldsymbol{T}) = \sum_{k=1}^{n} g_k(T_1) T_2^{x_k} \in \Lambda_2$$

be a nonzero pseudo-polynomial in T_2 over R with the x_k distinct elements of $\mathbb{Z}_p$, and retain the notation from Theorem 6. Every $\boldsymbol{m} \in L(F)$ satisfies $\ell \leq m_1$ and $m_2 < p^{N+1}$.

Proof. It will suffice to consider the case $\mu(F) = 0$. Let $\boldsymbol{m} \in L(F)$. Then in particular,

$$F \not\equiv 0 \pmod{\langle \pi, (T_1 - 1)^{m_1+1} \rangle}.$$

However, by the definition of the sets S and X,

$$g_k(T_1) \equiv 0 \pmod{\langle \pi, (T_1 - 1)^\ell \rangle}$$

for all $1 \leq k \leq n$. It follows that

$$F \equiv 0 \pmod{\langle \pi, (T_1 - 1)^\ell \rangle}.$$

This gives $\ell \le m_1$. Furthermore, Theorem 6 gives that there is some $\lambda \in L(F)$ with $\lambda \preccurlyeq_2 (\ell, p^{N+1} - 1)$. Thus, if $m_2 \ge p^{N+1}$, then

$$\lambda \preccurlyeq_2 (\ell, p^{N+1} - 1) \prec_2 \boldsymbol{m},$$

contradicting that $\boldsymbol{m}$ and λ are incomparable with respect to $\preccurlyeq_2$.

□

Corollary 28. *Let*

$$G(\boldsymbol{T}) = \sum_{k=1}^{n} g_k(T_1) T_2^{x_k}$$

be a nonzero pseudo-polynomial in T_2 over R with the x_k distinct elements of $\mathbb{Z}_p$, and let $N = N(G) \in \mathbb{N}$ be as in Theorem 6. Let $F \in \Lambda_2$ be any power series with

$$F \equiv G \pmod{T_2^{p^{N+1}} - 1}.$$

Then $\mu(F) \le \mu(G)$.

Proof. It suffices to consider the case where $\mu(G) = 0$. For a contradiction, suppose $\mu(F) \ge 1$. Then

$$\begin{aligned} 0 &\equiv F \pmod{\pi} \\ &\equiv G \pmod{\langle \pi, T_2^{p^{N+1}} - 1 \rangle} \\ &\equiv G \pmod{J_{p^{N+1}\boldsymbol{e}_2}(\pi)}, \end{aligned}$$

but this contradicts Theorem 6 for G. □

Corollary 29. *Let*

$$G(\boldsymbol{T}) = \sum_{k=1}^{n} g_k(T_1) T_2^{x_k}$$

be a nonzero pseudo-polynomial in T_2 over R. Suppose $F \in \Lambda_2$ is a power series with

$$F \equiv G \pmod{T_2^{p^n} - 1}$$

and $\mu(F) = \mu(G)$. Let $\preccurlyeq$ be a linear extension of $\preccurlyeq_d$, and suppose that the second component of $\lambda_{\preccurlyeq}(F)$, $\lambda_{\preccurlyeq}(F)_2$, is such that $\lambda_{\preccurlyeq}(F)_2 < p^n$. Then G is nonzero modulo the ideal $I_{\lambda_{\preccurlyeq}(F)+\mathbf{1}}(\pi^{\mu(F)+1})$ and there is $\boldsymbol{m} \in L(G)$ with $\boldsymbol{m} \preccurlyeq_d \lambda_{\preccurlyeq}(F)$. In particular, $\lambda_{\preccurlyeq}(G) \preccurlyeq \lambda_{\preccurlyeq}(F)$.

Proof. It suffices to consider the case $\mu(F) = 0 = \mu(G)$. By definition, F is nonzero modulo the ideal $I_{\lambda_{\preccurlyeq}(F)+\mathbf{1}}(\pi)$. Since we have assumed $\lambda_{\preccurlyeq}(F)_2 < p^n$,

$$\langle T_2^{p^n} - 1 \rangle \subseteq \langle \pi, T_2^{p^n} - 1 \rangle = J_{p^n \boldsymbol{e}_2}(\pi) \subseteq J_{\lambda_{\preccurlyeq}(F)+\mathbf{1}}(\pi),$$

and we have that

$$F \equiv G \pmod{J_{\lambda_{\preccurlyeq}(F)+\mathbf{1}}(\pi)}.$$

Consequently, G is nonzero modulo the ideal $I_{\lambda_{\preccurlyeq}(F)+\mathbf{1}}(\pi)$. Since $\lambda_{\preccurlyeq}(F) \prec_d \lambda_{\preccurlyeq}(F) + \mathbf{1}$, and the $\lambda_{\preccurlyeq}(F)$th coefficient of F is nonzero modulo π, also the $\lambda_{\preccurlyeq}(F)$th coefficient of G is nonzero modulo π. Thus, there is $\boldsymbol{m} \in L(G)$ with $\boldsymbol{m} \preccurlyeq_d \lambda_{\preccurlyeq}(F)$, so that

$$\lambda_{\preccurlyeq}(G) \preccurlyeq \boldsymbol{m} \preccurlyeq \lambda_{\preccurlyeq}(F).$$

□

Our main application of Theorem 6 and its corollaries will be to certain convergent sequences of pseudo-polynomials in T_2 over R. Recall

$$R[[\boldsymbol{T} - \mathbf{1}]] \cong \varprojlim R[[T_1 - 1]][[T_2 - 1]]/\langle T_2^{p^n} - 1 \rangle$$

under the current assumptions on R. Suppose $(F_n)_{n \in \mathbb{N}}$ is a sequence of non-zero pseudo-polynomials in T_2 over R such that

$$F_m \equiv F_n \pmod{T_2^{p^{n+1}} - 1}$$

for all $n \leq m$. Then (F_n) converges to a unique power series $F \in \Lambda_2$ satisfying

$$F \equiv F_n \pmod{T_2^{p^{n+1}} - 1}$$

for all $n \in \mathbb{N}$.

Proposition 15. *Suppose $(F_n)_{n\in\mathbb{N}}$ is a sequence of non-zero pseudo-polynomials in T_2 over R such that*

$$F_m \equiv F_n \pmod{T_2^{p^{n+1}} - 1}$$

for all $n \leq m$, and put $F = \lim F_n$. Suppose $N(F_n) \leq n$ for all $n \in \mathbb{N}$, where $N(F_n)$ is as in Theorem 6. The sequence $(\mu(F_n))$ in $\mathbb{N}$ is eventually constant and equal to $\mu(F)$. For each linear extension $\preccurlyeq$ of $\preccurlyeq_2$, the sequence $(\lambda_{\preccurlyeq}(F_n))$ in $\mathbb{N}^2$ is eventually bounded with respect to $\preccurlyeq$ by $\lambda_{\preccurlyeq}(F)$.

Proof. Fix a linear extension $\preccurlyeq$ of $\preccurlyeq_2$. Corollary 28 gives that $(\mu(F_n)_n$ is a decreasing sequence in $\mathbb{N}$ and that $\mu(F) \leq \mu(F_n)$ for all $n \in \mathbb{N}$. It suffices to consider the case $\mu(F) = 0$. The sequence $(\mu(F_n))_n$ is eventually constant, and we fix $m \in \mathbb{N}$ with $\mu(F_n) = \mu(F_m)$ for all $n \geq m$. Enlarging m if necessary, we further assume that $\lambda_{\preccurlyeq}(F)_2 < p^{m+1}$.

Suppose $\mu(F_m) > 0$, so that $F_m \equiv 0 \pmod{\pi}$. Then

$$\begin{aligned} F_m &\equiv F \pmod{\langle \pi, T_2^{p^{m+1}} - 1\rangle} \\ &\equiv 0 \pmod{\langle \pi, (T_2 - 1)^{p^{m+1}}\rangle}. \end{aligned}$$

But this is impossible since $\mu(F) = 0$ and $\lambda_{\preccurlyeq}(F)_2 < p^{m+1}$. Consequently, $\mu(F) = 0$, so that $\mu(F_n) = \mu(F)$ for all $n \geq m$. We may now apply Lemma 29 to obtain $\lambda_{\preccurlyeq}(F_n) \preccurlyeq \lambda_{\preccurlyeq}(F)$ for all $n \geq m$. ☐

CHAPTER 3

AN APPLICATION TO YAGER'S TWO-VARIABLE p-ADIC L-FUNCTION

Finally, we give an application of some of the results of Chapter 2 to the objects appearing in the work of Yager (1982). In this chapter, we borrow and modify slightly the notation from Yager (1982), which we review below. For the theory of elliptic curves, we refer to Silverman (1986); for the theory of elliptic curves with complex multiplication, we refer especially to Chapter II of Silverman (1994) (see also Lang (1987) and Shimura (1971)).

Throughout this chapter, we view all global fields as equipped with fixed embeddings into their completions. We will identify these fields with their images under such embeddings, and thus view a global field as a subfield of each of its completions. For this reason, we make no specific reference to the particular embeddings.

Fix an imaginary quadratic field K with class number one, discriminant $-d_K$, and ring of integers $\mathcal{O}_K$. Let $W_K = \mathcal{O}_K^\times$ denote the set of roots of unity in K, and note that $|W_K|$ is either 2, 4, or 6. We denote complex conjugation on $\mathbb{C}$ by $x \mapsto \overline{x}$.

Let E/K be an elliptic curve with endomorphism ring isomorphic to $\mathcal{O}_K$. Let S be the (finite) set consisting of the rational primes 2, 3, and all q for which E has bad reduction at a prime of K above q. Fix a Weierstrass model

$$E : y^2 = 4x^3 - g_2x - g_3$$

with $g_2, g_3 \in \mathcal{O}_K$ and with the discriminant $g_2^2 - 27g_3^2$ of this model divisible only by primes of K lying above primes in S. $\wp(z)$ will denote the Weierstrass function associated with this model, and L will be the period lattice of $\wp$. We may choose an element $\Omega_\infty \in \mathbb{C}$ such that $L = \Omega_\infty \mathcal{O}_K$.

We write x and y for the coordinate functions on E. Then a point P on E will be written in affine coordinates as $P = (x(P), y(P))$. We denote by O the point at infinity on E. The group law on E will be denoted simply by $+ : E \times E \to E$, and the inverse operation will be denoted by $- : E \to E$. Given a field F containing K, we write $E(F)$ for the group of F-valued points of E.

For $z \in \mathbb{C}$, set $\xi(z) = (\wp(z), \wp'(z))$, so that $\xi : \mathbb{C}/L \to E(\mathbb{C})$ is an analytic isomorphism. We identify $\mathcal{O}_K$ with $\mathrm{End}(E)$ by identifying $\alpha \in \mathcal{O}_K$ with the endomorphism given by $\xi(z) \mapsto \xi(\alpha z)$, which we will also write as $P \mapsto \alpha P$. Note that this association yields the normalized isomorphism $\mathcal{O}_K \to \mathrm{End}(E)$ as in Silverman (1994) Chapter II, and the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}/L & \xrightarrow{z \mapsto \alpha z} & \mathbb{C}/L \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \xi} \\ E(\mathbb{C}) & \xrightarrow[\xi(z) \mapsto \xi(\alpha z)]{P \mapsto \alpha P} & E(\mathbb{C}) \end{array}$$

Figure 1. The Endomorphism Associated to α

For $\alpha \in \mathcal{O}_K$, let $E_\alpha \subseteq E(\mathbb{C})$ denote the kernel of the endomorphism associated to α. Then the coordinates of the points in E_α are all algebraic over K. For an integral ideal $\mathfrak{a}$ of K, We also put

$$E_{\mathfrak{a}} = \bigcap_{a \in \mathfrak{a}} E_a,$$

and note that $E_{\mathfrak{a}} = E_\alpha$ if $\mathfrak{a} = \alpha \mathcal{O}_K$. Furthermore, $|E_{\mathfrak{a}}| = N\mathfrak{a}$, where $N\mathfrak{a}$ denotes the absolute norm of $\mathfrak{a}$; see Silverman (1994). As usual, $K(E_\alpha)$ denotes the field extension obtained by adjoining to K the coordinates of all points in E_α.

Let ψ be the Grossencharacter of E over K, and $\mathfrak{f}$ its conductor. Then there is an integral ideal $\mathfrak{m}$ of K, divisible by $\mathfrak{f}$, and such that ψ is a homomorphism from the group of fractional ideals of K which are relatively prime to $\mathfrak{m}$ taking values in $K^\times$ (de Shalit (1987), Chapter II.1). In fact, for an integral ideal $\mathfrak{a}$ of K prime to $\mathfrak{f}$, $\psi(\mathfrak{a}) \in \mathcal{O}_K$ and $\mathfrak{a} = \psi(\mathfrak{a})\mathcal{O}_K$. Let f be a fixed generator of $\mathfrak{f}$.

Fix a prime $p \notin S$ which splits in K, and fix a prime $\mathfrak{p}$ of K lying above p. Then $\mathfrak{p}$ has residue degree one, and $p\mathcal{O}_K = \mathfrak{p}\mathfrak{p}^*$ with $\mathfrak{p} \neq \mathfrak{p}^*$. We set $\pi = \psi(\mathfrak{p})$ and $\pi^* = \psi(\mathfrak{p}^*)$, so that $\mathfrak{p} = \pi\mathcal{O}_K$ and $\mathfrak{p}^* = \pi^*\mathcal{O}_K$.

For $n \in \mathbb{N}$, let $F_n = K(E_{\pi^{*n+1}})$ and for $\boldsymbol{n} \in \mathbb{N}^2$, let $K_{\boldsymbol{n}} = F_{n_2}(E_{\pi^{n_1+1}})$. From the theory of complex multiplication, F_{n_2} and $K_{\boldsymbol{n}}$ are Galois extensions of K, with

$$\mathrm{Gal}(K_{\boldsymbol{n}}/K) \cong (\mathbb{Z}/p^{n_1+1}\mathbb{Z})^\times \times (\mathbb{Z}/p^{n_2+1}\mathbb{Z})^\times$$

and

$$\mathrm{Gal}(F_{n_2}/K) \cong (\mathbb{Z}/p^{n_2+1}\mathbb{Z})^\times.$$

Moreover, $K_{\boldsymbol{n}}/F_{n_2}$ is totally ramified at the primes above $\mathfrak{p}$, while $\mathfrak{p}$ is unramified in F_{n_2}/K. Let r_n denote the number of primes of F_n above $\mathfrak{p}$. There is $N \in \mathbb{N}$ such that $r_n = r_0 p^n$ for $n < N$ and $r_n = r_0 p^N$ for $n \geq N$ (see de Shalit (1987), Chapter II.1 and Silverman (1994), Chapter II). For N as above, we will fix a prime $\mathfrak{p}_N$ of F_N above $\mathfrak{p}$. For each $n \in \mathbb{N}$, $\mathfrak{p}_n$ will denote the unique prime lying either above or below $\mathfrak{p}_N$, according to whether $n < N$ or $n \geq N$. For $\boldsymbol{n} \in \mathbb{N}^2$, we let $\mathfrak{p}_{\boldsymbol{n}}$ denote the unique prime of $K_{\boldsymbol{n}}$ lying above $\mathfrak{p}_{n_2}$.

Let $\boldsymbol{n} \in \mathbb{N}^2$. For any prime ϖ of F_{n_2} above $\mathfrak{p}$, we let $\Phi_{n_2,\varpi}$ denote the completion of F_{n_2} at ϖ and $\Xi_{\boldsymbol{n},\varpi}$ the completion of $K_{\boldsymbol{n}}$ at the unique prime $\mathfrak{p}_\varpi$ of $K_{\boldsymbol{n}}$ above ϖ. We let $\mathcal{I}_{n_2,\varpi}$ denote the ring of integers of $\Phi_{n_2,\varpi}$, and we also denote by ϖ the maximal ideal of $\mathcal{I}_{n_2,\varpi}$. We will denote the maximal ideal of the ring of integers of $\Xi_{\boldsymbol{n},\varpi}$ also by $\mathfrak{p}_\varpi$. When $\varpi = \mathfrak{p}_{n_2}$, we may simply write Φ_{n_2}, $\mathcal{I}_{n_2}$, and $\Xi_{\boldsymbol{n}}$. Let $K_\mathfrak{p}$ denote the completion of K at $\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ its ring of integers. Then $K_\mathfrak{p} \cong \mathbb{Q}_p$ and $\mathcal{O}_\mathfrak{p} \cong \mathbb{Z}_p$.

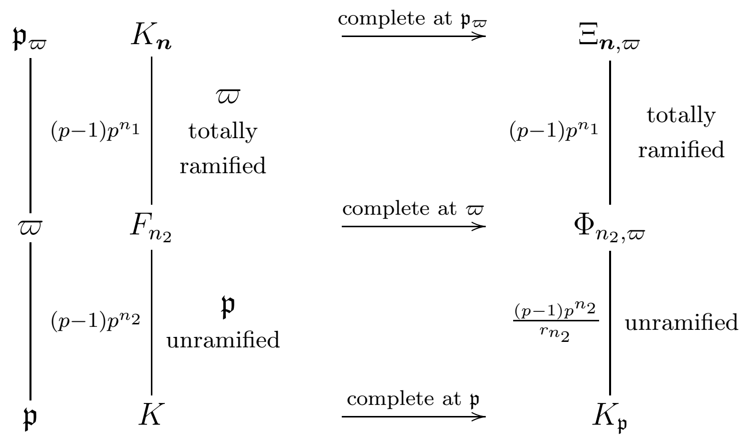

Figure 2. The Local Picture

Let

$$K_\infty = \bigcup_{\boldsymbol{n}\in\mathbb{N}^2} K_{\boldsymbol{n}}$$

$$F_\infty = \bigcup_{n\in\mathbb{N}} F_n$$

$$\Phi_\infty = \bigcup_{n\in\mathbb{N}} \Phi_n$$

$$\mathcal{I}_\infty = \bigcup_{n\in\mathbb{N}} \mathcal{I}_n \subseteq \Phi_\infty.$$

Let $\hat{\mathcal{I}}_\infty$ denote the completion of $\mathcal{I}_\infty$. Let $\mathfrak{p}_\infty$ denote the maximal ideal of $\hat{\mathcal{I}}_\infty$. Then $\hat{\mathcal{I}}_\infty$ is a closed subring of the completion of the ring of integers of $K_\mathfrak{p}^{\mathrm{ur}}$, the maximal unramified extension of $K_\mathfrak{p}$. Let $\varphi = \left(\frac{\mathfrak{p}}{F_\infty/K}\right)$ denote the Artin symbol for $\mathfrak{p}$ in F_∞/K. Then φ induces the Frobenius automorphism for $\Phi_\infty/K_\mathfrak{p}$.

Put $G_\infty = \mathrm{Gal}(K_\infty/K)$. Let

$$E_{\pi^\infty} = \bigcup_{n\in\mathbb{N}} E_{\pi^{n+1}}$$

$$E_{\pi^{*\infty}} = \bigcup_{n\in\mathbb{N}} E_{\pi^{*n+1}}.$$

Let $\kappa_1 : G_\infty \to \mathbb{Z}_p^\times$ and $\kappa_2 : G_\infty \to \mathbb{Z}_p^\times$ be the characters giving the action of G_∞ on E_{π^∞} and $E_{\pi^{*\infty}}$, respectively. One important thing to note is that for $\sigma \in G_\infty$ and $\alpha \in \mathcal{O}_K$ with $\sigma(u) = \alpha u$ for all $u \in E_{\pi^{*n+1}}$, $\kappa_2(\sigma) \in \mathbb{Z}_p^\times$ is given modulo p^{n+1} by an integer lying in the coset represented by α in $\mathcal{O}_K/\mathfrak{p}^{*n+1}$. Any such integer lies in the coset represented by $\overline{\alpha}$ in $\mathcal{O}_K/\mathfrak{p}^{n+1}$. Since we have identified $\mathcal{O}_\mathfrak{p}$ with $\mathbb{Z}_p$, the integer representatives of $\kappa_2(\sigma) \pmod{p^{n+1}}$ and $\overline{\alpha} \pmod{\mathfrak{p}^{n+1}}$ are congruent modulo p^{n+1} (see Yager (1982), page 415).

Let $\Gamma = \mathrm{Gal}(K_\infty/K_{\mathbf{0}})$. Then $G_\infty = \Gamma \times \Delta$, where Δ is the product of two cyclic groups of order $p-1$ and may be identified with $\mathrm{Gal}(K_{\mathbf{0}}/K)$. The characters κ_1 and κ_2 yield an isomorphism $G_\infty \cong (\mathbb{Z}_p^\times)^2$ via $\sigma \mapsto (\kappa_1(\sigma), \kappa_2(\sigma))$. From this, it follows that $\Gamma \cong \mathbb{Z}_p^2$.

For $\boldsymbol{n} \in \mathbb{N}^2$, we define the rings

$$\boldsymbol{\Xi}_{\boldsymbol{n}} = \prod_{\varpi} \Xi_{\boldsymbol{n},\varpi}$$
$$\boldsymbol{\Phi}_{n_2} = \prod_{\varpi} \Phi_{n_2,\varpi},$$

where both products are taken over all primes ϖ of F_{n_2} lying above $\mathfrak{p}$. We embed $K_{\boldsymbol{n}}$ into $\boldsymbol{\Xi}_{\boldsymbol{n}}$ and F_{n_2} into $\boldsymbol{\Phi}_{n_2}$ via the diagonal embedding. G_∞ acts on these rings via its action on the rings of adeles of $K_{\boldsymbol{n}}$ and F_{n_2}. That is to say, given $(x_\varpi)_\varpi$ in $\boldsymbol{\Xi}_{\boldsymbol{n}}$ or $\boldsymbol{\Phi}_{n_2}$, if $(x_{k,\varpi})_k$ is a Cauchy sequence in $K_{\boldsymbol{n}}$ or F_{n_2} which converges to x_ϖ in $\Xi_{\boldsymbol{n},\varpi}$ or $\Phi_{n_2,\varpi}$, then the $\sigma(\varpi)$ component of $((x_\varpi)_\varpi)^\sigma$ is the limit of the Cauchy sequence $(x_{k,\varpi}^\sigma)_k$ in $\Xi_{\boldsymbol{n},\sigma(\varpi)}$ or $\Phi_{n_2,\sigma(\varpi)}$. The following diagrams commute for each $\boldsymbol{n} \in \mathbb{N}^2$:

$$\begin{array}{ccc} K_{\boldsymbol{n}} & \xrightarrow{\text{diag}} & \boldsymbol{\Xi}_{\boldsymbol{n}} \\ \sigma \downarrow & & \downarrow \sigma \\ K_{\boldsymbol{n}} & \xrightarrow[\text{diag}]{} & \boldsymbol{\Xi}_{\boldsymbol{n}} \end{array}$$

$$\begin{array}{ccc} F_{n_2} & \xrightarrow{\text{diag}} & \boldsymbol{\Phi}_{n_2} \\ \sigma \downarrow & & \downarrow \sigma \\ F_{n_2} & \xrightarrow[\text{diag}]{} & \boldsymbol{\Phi}_{n_2} \end{array}$$

Figure 3. Commutative Squares for the Galois Action

Here the horizontal maps are the diagonal embeddings and the vertical maps are the action of $\sigma \in G_\infty$.

There are also natural norm and trace maps on the $\boldsymbol{\Xi}_{\boldsymbol{n}}$ and $\boldsymbol{\Phi}_{n_2}$ arising from the norm and trace maps on the adele rings of $K_{\boldsymbol{n}}$ and F_{n_2}. Precisely, fix $\boldsymbol{n} \in \mathbb{N}^2$. For each fixed prime ϖ of F_{n_2} lying above $\mathfrak{p}$, and any $\boldsymbol{m} \in \mathbb{N}^2$ with $\boldsymbol{n} \preccurlyeq_2 \boldsymbol{m}$, the set of primes ϖ' of F_{m_2} lying above ϖ is finite. For each prime ϖ' of F_{m_2} lying above ϖ, let $N^{\boldsymbol{m},\boldsymbol{n}}_{\varpi',\varpi}$ and $T^{\boldsymbol{m},\boldsymbol{n}}_{\varpi',\varpi}$ denote the local field norm and trace maps $\Xi_{\boldsymbol{m},\varpi'} \to \Xi_{\boldsymbol{n},\varpi}$ and $N^{m_2,n_2}_{\varpi',\varpi}$ and $T^{m_2,n_2}_{\varpi',\varpi}$ denote the local field norm and trace maps $\Phi_{m_2,\varpi'} \to \Phi_{n_2,\varpi}$. Now given $(x_{\varpi'})_{\varpi'} \in \boldsymbol{\Xi}_{\boldsymbol{m}}$, we define $N((x_{\varpi'})_{\varpi'}) \in \boldsymbol{\Xi}_{\boldsymbol{n}}$ by

$$N((x_{\varpi'})_{\varpi'}) = \left(\prod_{\varpi' \mid \varpi} N^{\boldsymbol{m},\boldsymbol{n}}_{\varpi',\varpi}(x_{\varpi}) \right)_{\varpi}$$

and $T((x_{\varpi'})_{\varpi'}) \in \boldsymbol{\Xi}_{\boldsymbol{n}}$ by

$$T((x_{\varpi'})_{\varpi'}) = \left(\sum_{\varpi' \mid \varpi} T^{\boldsymbol{m},\boldsymbol{n}}_{\varpi',\varpi}(x_{\varpi}) \right)_{\varpi} .$$

Similarly, for $(x_{\varpi'})_{\varpi'} \in \mathbf{\Phi}_{m_2}$, we define $N((x_{\varpi'})_{\varpi'}) \in \mathbf{\Phi}_{n_2}$ by

$$N((x_{\varpi'})_{\varpi'}) = \left(\prod_{\varpi' | \varpi} N^{m_2,n_2}_{\varpi',\varpi}(x_{\varpi}) \right)_{\varpi}$$

and $T((x_{\varpi'})_{\varpi'}) \in \mathbf{\Phi}_{n_2}$ by

$$T((x_{\varpi'})_{\varpi'}) = \left(\sum_{\varpi' | \varpi} T^{m_2,n_2}_{\varpi',\varpi}(x_{\varpi}) \right)_{\varpi} .$$

The following diagrams commute for each $\boldsymbol{n} \preccurlyeq_2 \boldsymbol{m}$:

$$\begin{array}{ccc} K_{\boldsymbol{m}} & \xrightarrow{\text{diag}} & \boldsymbol{\Xi}_{\boldsymbol{m}} \\ \downarrow & & \downarrow \\ K_{\boldsymbol{n}} & \xrightarrow[\text{diag}]{} & \boldsymbol{\Xi}_{\boldsymbol{n}} \end{array}$$

$$\begin{array}{ccc} F_{m_2} & \xrightarrow{\text{diag}} & \mathbf{\Phi}_{m_2} \\ \downarrow & & \downarrow \\ F_{n_2} & \xrightarrow[\text{diag}]{} & \mathbf{\Phi}_{n_2} \end{array}$$

Figure 4. Commutative Squares for the Norm and Trace Maps

Here the horizontal maps are the diagonal embeddings, the left vertical map is the Global field norm or trace, and the right vertical map is the norm or trace map described above.

We let $U'_{\boldsymbol{n},\varpi}$ denote the group of units of $\Xi_{\boldsymbol{n},\varpi}$, and $U_{\boldsymbol{n},\varpi}$ denote the subgroup of $U'_{\boldsymbol{n},\varpi}$ of units congruent to 1 modulo $\mathfrak{p}_{\varpi}$. We set

$$\boldsymbol{U}'_{\boldsymbol{n}} = \prod_{\varpi} U'_{\boldsymbol{n},\varpi}$$

$$\boldsymbol{U}_{\boldsymbol{n}} = \prod_{\varpi} U_{\boldsymbol{n},\varpi},$$

where the products extend over all primes ϖ of F_{n_2} above $\mathfrak{p}$. Finally, let $\boldsymbol{U}'_\infty$ denote the projective limit of the $\boldsymbol{U}'_{\boldsymbol{n}}$ and $\boldsymbol{U}_\infty$ the projective limit of the $\boldsymbol{U}_{\boldsymbol{n}}$, with these projective limits defined with respect to the norm maps on the $\boldsymbol{\Xi}_{\boldsymbol{n}}$.

For $n \in \mathbb{N}$, we put

$$\boldsymbol{\mathcal{I}}_n[[X]] = \left(\prod_{\varpi} \mathcal{I}_{n,\varpi} \right) [[X]],$$

where the product extends over all primes ϖ of F_n above $\mathfrak{p}$. Then $\boldsymbol{\mathcal{I}}_n[[X]]$ is a subring of $\boldsymbol{\Phi}_n[[X]]$; we let G_∞ act on $\boldsymbol{\mathcal{I}}_n[[X]]$ by acting coefficient-wise on a power series via the action inherited from the action on $\boldsymbol{\Phi}_n$. We likewise use the norm and trace maps on the $\boldsymbol{\Phi}_n$ to define norm and trace maps on the $\boldsymbol{\mathcal{I}}_n[[X]]$ by defining these maps coefficient-wise. We will denote by $N_{m,n}$ and $T_{m,n}$, respectively, the norm and trace maps $\boldsymbol{\mathcal{I}}_m[[X]] \to \boldsymbol{\mathcal{I}}_n[[X]]$ when $n \leq m$. For each prime ϖ of F_n above $\mathfrak{p}$, the natural projection $\Phi_{\boldsymbol{n}} \to \Phi_{n,\varpi}$ yields a natural surjective ring homomorphism $\boldsymbol{\mathcal{I}}_n[[X]] \to \mathcal{I}_{n,\varpi}[[X]]$. The image of $g \in \boldsymbol{\mathcal{I}}_n[[X]]$ under this homomorphism will be called the ϖ-component of g. Note that $g \in \boldsymbol{\mathcal{I}}_n[[X]]$ is uniquely determined by the collection of its ϖ-components.

We may view the Weierstrass model for E over $\mathcal{O}_\mathfrak{p}$ and thus also view E as an elliptic curve over $K_\mathfrak{p}$. The function $t = -2x/y$ is a local uniformizing parameter at O on E. Let $\tilde{E}$ denote the reduction of E modulo $\mathfrak{p}$, and let $E_1(K_\mathfrak{p})$ the kernel of reduction modulo $\mathfrak{p}$. Let $\hat{E}$ denote the formal completion of E at O with respect to the parameter t. The formal group law $\hat{E}$ is defined over $\mathcal{O}_K$; it is an (absolute) Lubin-Tate formal group law when viewed over $\mathcal{O}_\mathfrak{p}$ (de Shalit (1987), Chapter II 1.10). We denote addition on $\hat{E}$ by $[+]_{\hat{E}}$ and let $\hat{E}(\mathfrak{p})$ denote the group $(\mathfrak{p}, [+]_{\hat{E}})$ (the "$\mathfrak{p}$-valued points" of $\hat{E}$). Then the map

$$(x(P), y(P)) \mapsto t(P) = -2x(P)/y(P)$$

is a group isomorphism $E_1(K_{\mathfrak{p}}) \to \hat{E}(\mathfrak{p})$ and $\hat{E}$ is said to give the kernel of reduction modulo $\mathfrak{p}$ on E. There is a power series $a(t) \in 1 + t\mathcal{O}_{\mathfrak{p}}[[t]]$ such that the coordinate functions x and y have the t-expansions

$$x(t) = t^{-2}a(t), \quad y(t) = -2t^{-3}a(t).$$

We may take z as the parameter of the additive formal group $\mathbb{G}_a$. Then $\varepsilon : \mathbb{G}_a \to \hat{E}$ given by

$$\varepsilon(z) = -2\mathfrak{p}(z)/\mathfrak{p}(z) = -2x/y = t$$

is the exponential map for $\hat{E}$ and is defined over $K_{\mathfrak{p}}$. Let $\lambda : \hat{E} \to \mathbb{G}_a$ be the logarithm of $\hat{E}$, i.e., the inverse of ε, defined over $K_{\mathfrak{p}}$. It is well-known that in this case, $\lambda'(X)$ is an invertible power series in $\mathbb{Z}_p[[X]]$ (Hazewinkel (1978), I.5.8 or Iwasawa (1986), IV.1). As usual, for $n \in \mathbb{N}$, $[\pi^{n+1}]$ will denote the endomorphism of $\hat{E}$ associated to π^{n+1}, and we denote the kernel of $[\pi^{n+1}]$ by $\hat{E}_{\pi^{n+1}}$, which we may identify with $E_{\pi^{n+1}}$. Finally, let

$$T_\pi = \lim_{\leftarrow} \hat{E}_{\pi^{n+1}},$$

with the projection maps defining the inverse limit given by powers of $[\pi]$ on $\hat{E}$. Then T_π is a rank one $\mathbb{Z}_p$-module, and we fix a basis (v_n) of T_π (Hazewinkel (1978), VI.32.1).

Let $\mathbb{G}_m$ denote the multiplicative formal group. It is well known that the formal group laws $\mathbb{G}_m$ and $\hat{E}$ are isomorphic when viewed over the completion of the ring of integers of $K_{\mathfrak{p}}^{\mathrm{ur}}$ (see Hazewinkel (1978), Chapter I.8.3, de Shalit (1987), Chapter I, or Iwasawa (1986), Chapter IV). Yager (1982) identifies a particular choice of isomorphism $\eta : \mathbb{G}_m \to \hat{E}$ which is defined over $\hat{\mathcal{I}}_\infty$. Then $\eta(X) \in \hat{\mathcal{I}}_\infty[[X]]$ is of the form

$$\eta(X) = \exp(\Omega_{\mathfrak{p}}\lambda(X)) - 1 = \Omega_{\mathfrak{p}}X + \cdots ,$$

where $\Omega_{\mathfrak{p}} \in \hat{\mathcal{I}}_\infty^\times$. We let $\iota \in \hat{\mathcal{I}}_\infty[[X]]$ denote formal group isomorphism $\hat{E} \to \mathbb{G}_m$ which is the inverse of η (see page 429 of Yager (1982) for a precise construction of $\Omega_{\mathfrak{p}}$ via the Weil pairing).

Fix $\beta = (\beta_{\boldsymbol{n}}) \in \boldsymbol{U}'_\infty$. For fixed $n_2 \in \mathbb{N}$ and ϖ a prime of F_{n_2} above $\mathfrak{p}$, the element $(\beta_{(n_1,n_2),\varpi})_{n_1}$ is a norm-coherent sequence of units in the local division tower of the $\Xi_{(n_1,n_2),\varpi}/\Phi_{n_2,\varpi}$ for $n_1 \in \mathbb{N}$. By Coleman (1979) (see also de Shalit (1987), Chapter I.2), there is a unique power series $c_{n_2,\varpi,\beta} \in \mathcal{I}_{n_2,\varpi}[[X]]^\times$ satisfying

$$\beta_{(n_1,n_2),\varpi} = c_{n_2,\varpi,\beta}^{\varphi^{-n_1}}(v_{n_1})$$

for all $n_1 \in \mathbb{N}$. We also have the functional equation

$$(c_{n_2,\varpi,\beta}^{\varphi} \circ [\pi])(X) = \prod_{\eta \in \hat{E}_\pi} c_{n_2,\varpi,\beta}\left(X[+]_{\hat{E}}\eta\right).$$

We take this opportunity to remark that the functional equation gives the following equivalence of power series modulo $\mathfrak{p}_\infty$:

$$(c_{n_2,\varpi,\beta}(X))^p \equiv \prod_{\eta \in \hat{E}_\pi} c_{n_2,\varpi,\beta}\left(X[+]_{\hat{E}}\eta\right) \pmod{\mathfrak{p}_\infty}.$$

For each $n \in \mathbb{N}$, we let $c_{n,\beta}(X) \in \boldsymbol{\mathcal{I}}_n[[X]]$ denote the element with ϖ-component equal to $c_{n,\varpi,\beta}(X)$ for each prime ϖ of F_n above $\mathfrak{p}$; by a slight abuse of language, we call $c_{n,\beta}(X) \in \boldsymbol{\mathcal{I}}_n[[X]]$ the nth Coleman power series of $\beta \in \boldsymbol{U}'_\infty$. If $\boldsymbol{n} \preccurlyeq_2 \boldsymbol{m}$ with $n_1 = m_1$, then for each prime ϖ of F_{n_2} above $\mathfrak{p}$,

$$\prod_{\varpi'|\varpi} N_{\varpi',\varpi}^{\boldsymbol{m},\boldsymbol{n}}(c_{m_2,\varpi',\beta}^{\varphi^{-m_1}}(v_{m_1})) = \beta_{\boldsymbol{n},\varpi},$$

the product extending over all primes ϖ' of F_{m_2} lying above ϖ. This gives

$$\prod_{\varpi'|\varpi}\left(\left(\prod_{\sigma \in \mathrm{Gal}(\Phi_{m_2,\varpi'}/\Phi_{n_2,\varpi})} c_{m_2,\varpi',\beta}^{\sigma}\right)^{\varphi^{-m_1}}(v_{m_1})\right) = \beta_{\boldsymbol{n},\varpi}.$$

By the uniqueness of Coleman power series, we find

$$c_{n,\beta}(X) = N_{m,n}(c_{m,\beta}(X))$$

whenever $n \leq m$ (see Yager (1982) for a more thorough treatment).

Recall that $\lambda'(X) \in \mathbb{Z}_p[[X]]^\times \subseteq \mathcal{I}_{n,\varpi}[[X]]^\times$ for each $n \in \mathbb{N}$ and each prime ϖ of F_n above $\mathfrak{p}$. Given $n \in \mathbb{N}$ and ϖ a prime of F_n above $\mathfrak{p}$, we put

$$g_{n,\varpi,\beta}(X) = \left(\frac{1}{\lambda'(X)}\frac{d}{dX}\right)\log c_{n,\varpi,\beta}(X) = \frac{c'_{n,\varpi,\beta}(X)}{\lambda'(X)c_{n,\varpi,\beta}(X)},$$

and we let $g_{n,\beta} \in \boldsymbol{\mathcal{I}}_n[[X]]$ denote the element with ϖ-component $g_{n,\varpi,\beta}(X)$.

Lemma 40 (Lemma 3 of Yager (1982))**.** *Let $0 \leq n \leq m$. For each $\beta \in \boldsymbol{U}'_\infty$,*

$$g_{n,\beta}(X) = T_{m,n}\left(g_{m,\beta}(X)\right).$$

Moreover, $g_{n,\beta}$ satisfies the functional equation

$$\pi g^{\varphi}_{n,\beta}([\pi](X)) = \sum_{\eta \in \hat{E}_\pi} g_{n,\beta}(X[+]_{\hat{E}}\eta).$$

□

For the upcoming application, we need to record one important fact (see Yager (1982)). Let $\beta \in \boldsymbol{U}'_\infty$. Then for each $\boldsymbol{n} \in \mathbb{N}^2$ and each prime ϖ of F_{n_2} above $\mathfrak{p}$, we may write

$$\beta_{\boldsymbol{n},\varpi} = \omega(\beta_{\boldsymbol{n},\varpi})\langle\beta_{\boldsymbol{n},\varpi}\rangle,$$

where $\omega(\beta_{\boldsymbol{n},\varpi})$ is a root of unity in $\Phi_{n_2,\varpi}$ and $\langle\beta_{\boldsymbol{n},\varpi}\rangle \in U_{\boldsymbol{n},\varpi}$. Let $\langle\beta_{\boldsymbol{n}}\rangle$ denote the element of $\boldsymbol{U}_{\boldsymbol{n}}$ with ϖ-component equal to $\langle\beta_{\boldsymbol{n},\varpi}\rangle$. Then $(\langle\beta_{\boldsymbol{n}}\rangle)_{\boldsymbol{n}}$ corresponds to an element of $\boldsymbol{U}_\infty$, which we denote by $\langle\beta\rangle$, and we have that

$$g_{n_2,\beta}(X) = g_{n_2,\langle\beta\rangle}(X)$$

for all $n_2 \in \mathbb{N}$.

The key constructive step in Yager (1982) involves a method for assembling the power series $g_{n,\beta}(X)$ for $\beta \in \boldsymbol{U}'_\infty$ and $n \in \mathbb{N}$ into a single two-variable power series with coefficients in $\hat{\mathcal{I}}_\infty$. The trace compatibility described in Lemma 40 makes such an elementary construction possible. For $\beta \in \boldsymbol{U}'_\infty$ and $n \geq 0$, consider the power series

$$G_{n,\beta}(X_1, X_2) = \sum_{\sigma \in \mathrm{Gal}(F_n/K)} \left(g^\sigma_{n,\beta}(X_1)\right)_{\mathfrak{p}_n} (1 + X_2)^{k_2(\sigma)} \in \mathcal{I}_n[[X_1, X_2]] \subseteq \hat{\mathcal{I}}_\infty[[X_1, X_2]],$$

where $\left(g^\sigma_{n,\beta}(X_1)\right)_{\mathfrak{p}_n}$ denotes the $\mathfrak{p}_n$-component of $g_{n,\beta}$ under the action of any element $\theta \in G_\infty$ whose restriction to F_n is σ, and where $k_2(\sigma)$ is the unique integer $0 \leq k_2(\sigma) < p^{n+1}$ satisfying

$$\kappa_2(\theta) \equiv k_2(\sigma) \pmod{p^{n+1}}$$

for any $\theta \in G_\infty$ whose restriction to F_n is σ. Note that if $\theta_1, \theta_2 \in G_\infty$ both restrict to σ on F_n, then $\theta_1\theta_2^{-1}$ is trivial on F_n, and hence on $E_{\pi^{*n+1}}$. By the above remarks,

$$\kappa_2(\theta_1\theta_2^{-1}) \equiv 1 \pmod{p^{n+1}},$$

and since $\kappa_2(\theta_1), \kappa_2(\theta_2) \in \mathbb{Z}_p^\times$, we have that $k_2(\sigma)$ is well-defined. Moreover, the $k_2(\sigma)$ for $\sigma \in \mathrm{Gal}(F_n/K)$ are distinct modulo p^{n+1}. Since $g_{n,\beta} \in \boldsymbol{\mathcal{I}}_n[[X]]$, $\left(g^\sigma_{n,\beta}(X_1)\right)_{\mathfrak{p}_n}$ is well-defined for each $\sigma \in \mathrm{Gal}(F_n/K)$.

From the trace compatibility of the $g_{n,\beta}$ (Lemma 40), we have that in $\hat{\mathcal{I}}_\infty[[X_1, X_2]]$,

$$G_{m,\beta}(X_1, X_2) \equiv G_{n,\beta}(X_1, X_2) \pmod{(1 + X_2)^{p^{n+1}} - 1}$$

for all $m \geq n$.

Theorem 7 (Theorem 5 of Yager (1982)). *For each $\beta \in \boldsymbol{U}'_\infty$, there is a unique power series $g_\beta \in \hat{\mathcal{I}}_\infty[[X_1, X_2]]$ such that*

$$g_\beta(X_1, X_2) \equiv \sum_{\sigma \in \mathrm{Gal}(F_n/K)} \left(g^\sigma_{n,\beta}(X_1)\right)_{\mathfrak{p}_n} (1 + X_2)^{k_2(\sigma)} \pmod{(1 + X_2)^{p^{n+1}} - 1}$$

for all $n \in \mathbb{N}$. Moreover, g_β satisfies the functional equation

$$\pi g_\beta\left([\pi](X_1), (1 + X_2)^{\kappa_2(\varphi)^{-1}} - 1\right) = \sum_{\eta \in \hat{E}_\pi} g_\beta(X_1[+]_{\hat{E}}\eta, X_2).$$

□

For $\beta \in \boldsymbol{U}'_\infty$, we define

$$h_\beta(X_1, X_2) = g_\beta(\iota(X_1), X_2) \in \hat{\mathcal{I}}_\infty[[X_1, X_2]].$$

Then

$$h_\beta(X_1, X_2) \equiv \sum_{\sigma \in \mathrm{Gal}(F_n/K)} \left(g^\sigma_{n,\beta}(\iota(X_1))\right)_{\mathfrak{p}_n} (1 + X_2)^{k_2(\sigma)} \pmod{(1 + X_2)^{p^{n+1}} - 1}$$

for all $n \in \mathbb{N}$.

We will set $X_1 = T_1 - 1$ and $X_2 = T_2 - 1$, and view the above power series as elements of $\hat{\mathcal{I}}_\infty[[T_1 - 1, T_2 - 1]] = \hat{\mathcal{I}}_\infty[[\boldsymbol{T}] - \boldsymbol{1}]$. When we do this, we will write $\iota(T_1)$ instead of $\iota(T_1 - 1)$ to denote the power series $\iota \in \hat{\mathcal{I}}_\infty[[T_1 - 1]]$.

The power series h_β thus obtained satisfies

$$h_\beta(\boldsymbol{T}) \equiv \sum_{\sigma \in \mathrm{Gal}(F_n/K)} \left(g^\sigma_{n,\beta}(\iota(T_1))\right)_{\mathfrak{p}_n} T_2^{k_2(\sigma)} \pmod{T_2^{p^{n+1}} - 1}$$

for all $n \in \mathbb{N}$ with the $k_2(\sigma) \in \mathbb{Z}_p^\times$ distinct modulo p^{n+1} but not modulo p^n. We will apply the results of Section 2.7 to h_β. Let $\alpha_\beta \in \mathcal{M}_2(\hat{\mathcal{I}}_\infty)$ denote the measure associated to h_β and, for each $n \in \mathbb{N}$, $\alpha_{\beta_n} \in \mathcal{M}_2(\hat{\mathcal{I}}_\infty)$ the measure associated to

$$\sum_{\sigma \in \mathrm{Gal}(F_n/K)} \left(g^\sigma_{n,\beta}(\iota(T_1))\right)_{\mathfrak{p}_n} T_2^{k_2(\sigma)}.$$

For $n \in \mathbb{N}$, and $\sigma \in \mathrm{Gal}(F_n/K)$, let $\nu_{n,\sigma} \in \mathcal{M}_1(\hat{\mathcal{I}}_\infty)$ be the measure associated to $\left(g^\sigma_{n,\beta}(\iota(T_1))\right)_{\mathfrak{p}_n}$. We wish to study the Γ-transforms of the measure α_β. Fix $\boldsymbol{u} \in U^2$ as in Section 2.4. As shown in Section 2.7, for each $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p-2}$,

$$G^{(\boldsymbol{i})}_{\alpha_\beta}(\boldsymbol{T}) \equiv G^{(\boldsymbol{i})}_{\alpha_{\beta_n}}(\boldsymbol{T}) \pmod{T_2^{p^n} - 1},$$

and $\mu(\Gamma^{(\boldsymbol{i})}_{\alpha_\beta}) \le \mu(\Gamma^{(\boldsymbol{i})}_{\alpha_{\beta_n}})$ for all $n \in \mathbb{N}$, with $\mu(\Gamma^{(\boldsymbol{i})}_{\alpha_\beta}) = \mu(\Gamma^{(\boldsymbol{i})}_{\alpha_{\beta_n}})$ for n sufficiently large. Also

$$G^{(\boldsymbol{i})}_{\alpha_{\beta_n}}(\boldsymbol{T}) = \sum_{\sigma \in \mathrm{Gal}(F_n/K)} \omega^{i_2}(k_2(\sigma)) G^{(i_1)}_{\nu_{n,\sigma}}(T_1) T_2^{\ell_{u_2}(k_2(\sigma))}.$$

Of course, $\omega^{i_2}(k_2(\sigma)) \in \mathbb{Z}_p^\times$ for all $\sigma \in \mathrm{Gal}(F_n/K)$ and the $\ell_{u_2}(k_2(\sigma))$ are distinct modulo p^n. By Theorem 6, we have

$$\begin{aligned} \mu(\Gamma^{(\boldsymbol{i})}_{\alpha_{\beta_n}}) &= \min_{\sigma \in \mathrm{Gal}(F_n/K)} \{\mu(\omega^{i_2}(k_2(\sigma))\Gamma^{(i_1)}_{\nu_{n,\sigma}})\} \\ &= \min_{\sigma \in \mathrm{Gal}(F_n/K)} \{\mu(\Gamma^{(i_1)}_{\nu_{n,\sigma}})\}. \end{aligned}$$

Finally,

$$\mu(\Gamma^{(\boldsymbol{i})}_{\alpha_\beta}) = \min_n \left\{ \min_{\sigma \in \mathrm{Gal}(F_n/K)} \{\mu(\Gamma^{(i_1)}_{\nu_{n,\sigma}})\} \right\}.$$

3.1 Elliptic Units

The objects appearing in this section are those used in Sections 4 and 5 of Yager (1982). Similar constructions can be found throughout the literature, and we mention in particular the descriptions contained in Bernardi *et al.* (1984), Cassou-Noguès (1981), Coates and Goldstein (1983), Section 5 of Coates and Wiles (1977), Coates and Wiles (1978), Yager (1984), Lichtenbaum (1980), and Rubin (1999).

For an integral ideal $\mathfrak{a}$ of K, let $\mathfrak{a}^{-1}L$ denote the lattice $\Omega_\infty \mathfrak{a}^{-1}$. Consider the following elliptic function for L,

$$\Theta(z, \mathfrak{a}) = \frac{\Delta(L)^{N\mathfrak{a}}}{\Delta(\mathfrak{a}^{-1}L)} \prod_l (\wp(z) - \wp(l))^{-6},$$

where the product above is over any set $\{l\}$ of the nonzero cosets of $\mathfrak{a}^{-1}L/L$, and where $\Delta(\cdot)$ is the discriminant function for a lattice. The number $c_L(\mathfrak{a}) = \Delta(L)^{N\mathfrak{a}}\Delta(\mathfrak{a}^{-1}L)^{-1}$ is a unit in K (de Shalit (1987)). For a point $P \in E(\mathbb{C})$, we also put

$$\Theta(P, \mathfrak{a}, E) = c_L(\mathfrak{a}) \prod_{Q \in E_{\mathfrak{a}} \setminus \{O\}} (x(P) - x(Q))^{-6}.$$

Then

$$\Theta(\xi(z), \mathfrak{a}, E) = \Theta(z, \mathfrak{a}).$$

As in Lichtenbaum (1980) (Corollary 2.6), $\Theta(P, \mathfrak{a}, E)$ has a pole of order 12 at $P = Q$ for each $Q \in E_{\mathfrak{a}} \setminus \{O\}$, a zero of order $12(N\mathfrak{a} - 1)$ at $P = O$, and no other zeros or poles. This observation will be very useful in the next section.

For $\boldsymbol{n} \in \mathbb{N}^2$, let $\mathcal{R}_{\boldsymbol{n}}$ and $\mathcal{R}_{n_2}$ denote the ray class fields of K modulo $\mathfrak{f}\mathfrak{p}^{n_1+1}\mathfrak{p}^{*n_2+1}$ and $\mathfrak{f}\mathfrak{p}^{*n_2+1}$, respectively. Then $F_{n_2} \subseteq \mathcal{R}_{n_2}$, $K_{\boldsymbol{n}} \subseteq \mathcal{R}_{\boldsymbol{n}}$, and $K(E_f) \subseteq \mathcal{R}_{n_2} \subseteq \mathcal{R}_{\boldsymbol{n}}$.

Let $\rho_{n_2} = \Omega_\infty / f\pi^{*n_2+1} \in \mathbb{C}$, and note that $Q_{n_2} = \xi(\rho_{n_2})$ is a primitive $\mathfrak{f}\mathfrak{p}^{*n_2+1}$-division point on E. Furthermore, $\pi^*\rho_{n_2} = \rho_{n_2-1}$ and $\pi^*Q_{n_2} = Q_{n_2-1}$ for all $n_2 \in \mathbb{N}$ (by the normalized identification of $\mathcal{O}_K$ with $\mathrm{End}(E)$). Let B_{n_2} be any set of integral ideals of K prime to $\mathfrak{f}\mathfrak{p}^*$ and such that

$$\mathrm{Gal}(\mathcal{R}_{n_2}/F_{n_2}) = \left\{ \left(\frac{\mathfrak{b}}{\mathcal{R}_{n_2}/K} \right) : \mathfrak{b} \in B_{n_2} \right\}$$

as an equality of sets. Given an ideal $\mathfrak{b} \in B_{n_2}$, we put

$$\sigma_{\mathfrak{b}} = \left(\frac{\mathfrak{b}}{\mathcal{R}_{n_2}/K} \right).$$

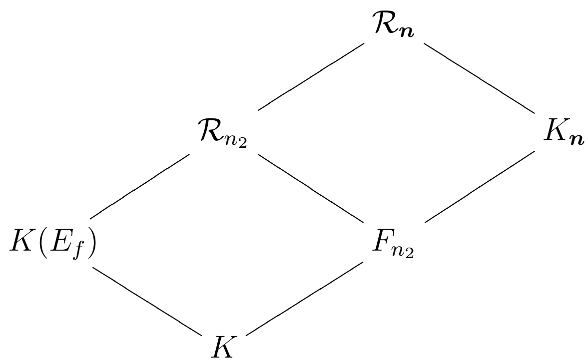

Figure 5. The Ray Class Field Tower

Since each $\mathfrak{b} \in B_{n_2}$ is prime to $\mathfrak{f}\mathfrak{p}^*$, we have

$$Q_{n_2}^{\sigma_{\mathfrak{b}}} = \xi(\rho_{n_2})^{\sigma_{\mathfrak{b}}} = \xi(\psi(\mathfrak{b})\rho_{n_2}) = \psi(\mathfrak{b})Q_{n_2}$$

by the definition of grossencharacter (de Shalit (1987) II.1.3) and the normalization of $\mathcal{O}_K \cong \mathrm{End}(E)$. If $\mathfrak{a}$ is an integral ideal of K prime to $6pf$, we put

$$\Lambda_{n_2}(z, \mathfrak{a}) = \prod_{\mathfrak{b} \in B_{n_2}} \Theta(z + \psi(\mathfrak{b})\rho_{n_2}, \mathfrak{a}).$$

We also put

$$\Lambda_{n_2}(P, \mathfrak{a}, E) = \prod_{\mathfrak{b} \in B_{n_2}} \Theta(P + Q^{\sigma_{\mathfrak{b}}}, \mathfrak{a}, E) = \prod_{\sigma \in \mathrm{Gal}(\mathcal{R}_{n_2}/F_{n_2})} \Theta(P + Q^{\sigma}, \mathfrak{a}, E),$$

so that

$$\Lambda_{n_2}(\xi(z), \mathfrak{a}, E) = \Lambda_{n_2}(z, \mathfrak{a}).$$

Lemma 41 (Lemma 7 of Yager (1982))**.** *$\Lambda_{n_2}(z, \mathfrak{a})$ is a rational function of $\wp(z)$ and $\wp'(z)$ with coefficients in F_{n_2} and is independent of the set B_{n_2}.* □

For an element $\sigma \in G_\infty$, we let $\Lambda_{n_2}^{\sigma}$ be the rational function of $\wp(z)$ and $\wp'(z)$ obtained by applying σ to the coefficients of the rational function $\Lambda_{n_2}(z, \mathfrak{a})$ of $\wp(z)$

and $\wp'(z)$. Lemma 41 allows us to view $\Lambda_{n_2}(z, \mathfrak{a})$ as a rational function on E with coefficients in $\Phi_{n_2,\varpi}$ for each prime ϖ of F_{n_2} lying above $\mathfrak{p}$.

We let I denote the set of integral ideals of K which are prime to $6pf$, and let $\mathcal{S}$ be the set of all functions $\mathfrak{n} : I \to \mathbb{Z}$ for which $\mathfrak{n}(\mathfrak{a}) = 0$ for all but finitely many $\mathfrak{a} \in I$ and

$$\sum_{\mathfrak{a}\in I}(N\mathfrak{a} - 1)\mathfrak{n}(\mathfrak{a}) = 0.$$

Given a function $\mathfrak{n} \in \mathcal{S}$, we define

$$\Lambda_{n_2}(z; \mathfrak{n}) = \prod_{\mathfrak{a}\in I} \Lambda_{n_2}(z, \mathfrak{a})^{\mathfrak{n}(\mathfrak{a})}$$

and

$$\Lambda_{n_2}(P; \mathfrak{n}, E) = \prod_{\mathfrak{a}\in I} \Lambda_{n_2}(P, \mathfrak{a}, E)^{\mathfrak{n}(\mathfrak{a})}.$$

Recall that π^* is a unit in $\mathcal{O}_{\mathfrak{p}} \cong \mathbb{Z}_p$, the completion of $\mathcal{O}_K$ at $\mathfrak{p}$. Thus, for each $n \in \mathbb{N}$, we may choose $\varepsilon_n \in \mathcal{O}_K$ with

$$\varepsilon_n\pi^* \equiv 1 \pmod{\mathfrak{p}^{n+1}}.$$

Since $\varepsilon : \mathbb{G}_a \to \hat{E}$ is a formal group isomorphism, for each $n \in \mathbb{N}$, we may choose τ_n for which $v_n = \varepsilon(\tau_n)$ (Hazewinkel (1978) Chapter VI.35). Observe further that for $m \in \mathbb{N}$,

$$[\pi^{*-(m+1)}](v_n) = \varepsilon(\varepsilon_n^{m+1}\tau_n).$$

For each $\mathfrak{n} \in \mathcal{S}$, the element $\Lambda_{n_2}(\varepsilon_{n_1}^{n_2+1}\tau_{n_1}; \mathfrak{n})$ is a unit in $K_{\boldsymbol{n}}$. The group of all such units is called the group of elliptic units of $K_{\boldsymbol{n}}$ and is denoted $C'_{\boldsymbol{n}}$. The group $C'_{\boldsymbol{n}}$ is stable under the action of G_∞ (see Yager (1982), section 4).

Lemma 42 (Corollary 9 in Yager (1982))**.** *For* $\mathfrak{n} \in \mathcal{S}$*, put*

$$e_{\boldsymbol{n}}(\mathfrak{n}) = \Lambda_{n_2}^{\varphi^{-n_1}}(z; \mathfrak{n})\big|_{z=\varepsilon_{n_1}^{n_2+1}\tau_{n_1}}.$$

Then $e(\mathfrak{n}) = (e_{\boldsymbol{n}}(\mathfrak{n})) \in \boldsymbol{U}'_\infty$. □

We may embed the group $C'_{\boldsymbol{n}}$ diagonally into $\boldsymbol{U}'_{\boldsymbol{n}}$. Then the $C'_{\boldsymbol{n}}$ are norm-compatible subgroups with respect to the norm maps on the $\boldsymbol{\Xi}_{\boldsymbol{n}}$, and we let C'_∞ denote the projective limit with respect to the norm maps (see Yager (1982)). For all $\mathfrak{n} \in \mathcal{S}$, $e(\mathfrak{n}) \in C'_\infty$.

Now for fixed $\mathfrak{n} \in \mathcal{S}$, $\Lambda_{n_2}(z; \mathfrak{n})$ is a rational function of $\wp(z)$ and $\wp'(z)$ with coefficients in F_{n_2} without a pole at $z = 0$. Thus, $\Lambda_{n_2}(z; \mathfrak{n})$ has a power series expansion in $F_{n_2}[[z]]$. Via the diagonal embedding and working formally with power series, we may view $\Lambda_{n_2}(\pi^{*-(n_2+1)}\lambda(X); \mathfrak{n})$ as an element of $\boldsymbol{\Phi}_{n_2}[[X]]$. Note that via the identification of F_{n_2} with a subfield of each $\Phi_{n_2,\varpi}$, as an element of $\boldsymbol{\Phi}_{n_2}[[X]]$, all components of $\Lambda_{n_2}(\pi^{*-(n_2+1)}\lambda(X); \mathfrak{n})$ are equal. We will not distinguish notationally between the power series $\Lambda_{n_2}(\pi^{*-(n_2+1)}\lambda(X); \mathfrak{n})$ viewed as an element of $\boldsymbol{\Phi}_{n_2}[[X]]$ and a component of $\Lambda_{n_2}(\pi^{*-(n_2+1)}\lambda(X); \mathfrak{n})$.

Theorem 8 (Theorem 10 of Yager (1982))**.** *For each* $\mathfrak{n} \in \mathcal{S}$*,*

$$\Lambda_{n_2}(\pi^{*-(n_2+1)}\lambda(X); \mathfrak{n}) = c_{n_2,e(\mathfrak{n})}(X) \in \boldsymbol{\mathcal{I}}_{n_2}[[X]].$$

□

We now turn to $g_{n_2,e(\mathfrak{n})}(X) \in \boldsymbol{\mathcal{I}}_{n_2}[[X]]$. As above, all of the components of $g_{n_2,e(\mathfrak{n})}(X)$ are given by

$$\frac{1}{\lambda'(X)} \frac{d}{dX} \log \Lambda_{n_2}(\pi^{*-(n_2+1)}\lambda(X); \mathfrak{n}).$$

Again, we do not distinguish notationally between the power series $g_{n_2,e(\mathfrak{n})}(X) \in \boldsymbol{\mathcal{I}}_{n_2}[[X]]$ and its components. As discussed in Yager (1982), the Galois action commutes with the operator $(\lambda'(X))^{-1}\frac{d}{dX}\log$. Since the Galois action commutes with the diagonal embedding, for $\sigma \in \mathrm{Gal}(F_n/K)$, all of the components of $g^{\sigma}_{n_2,e(\mathfrak{n})}(X)$ are equal.

We wish to consider the two-variable power series $h_{e(\mathfrak{n})}(\boldsymbol{T}) \in \hat{\mathcal{I}}_\infty[[\boldsymbol{T}]]$, so that

$$h_{e(\mathfrak{n})}(\boldsymbol{T}) \equiv \sum_{\sigma \in \mathrm{Gal}(F_n/K)} g^\sigma_{n,e(\mathfrak{n})}(\iota(T_1)) T_2^{k_2(\sigma)} \pmod{T_2^{p^{n+1}} - 1}$$

for all $n \in \mathbb{N}$.

We now need to examine the one-variable Γ-transforms of the measures associated to the power series $g^\sigma_{n,e(\mathfrak{n})}(\iota(T_1))$. In terms of the parameter z, we have that

$$g^\sigma_{n,e(\mathfrak{n})}(z) = \frac{d}{dz} \log \Lambda^\sigma_{n,e(\mathfrak{n})}(\pi^{*-(n+1)} z).$$

Fix $\sigma \in \mathrm{Gal}(F_n/K)$; then there is an integral ideal $\mathfrak{c}$ of K, prime to $\mathfrak{f}\mathfrak{p}^*$, for which σ is the restriction of $\left(\frac{\mathfrak{c}}{\mathcal{R}_n/K}\right) = \sigma_{\mathfrak{c}}$ to F_n. Then for an ideal $\mathfrak{a} \in I$, we have

$$\Lambda^\sigma_n(z, \mathfrak{a}) = \prod_{\mathfrak{b} \in B_n} \Theta(z + \psi(\mathfrak{b})(\psi(\mathfrak{c})\rho_n), \mathfrak{a}).$$

As a function on the curve, we have

$$\Lambda^\sigma_n(P, \mathfrak{a}, E) = \prod_{\tau \in \mathrm{Gal}(R_n/F_n)} \Theta(P + (Q_n^{\sigma_{\mathfrak{c}}})^\tau, \mathfrak{a}),$$

and this last equation is independent of the choice of ideal $\mathfrak{c}$ of K. Thus, $\Lambda^\sigma_n(z; \mathfrak{n})$ is obtained by replacing the primitive $\mathfrak{f}\mathfrak{p}^{*n+1}$-division point Q_n with a Galois conjugate, which we denote hereafter by Q_n^σ.

We have that

$$\frac{d}{dz} \log \Lambda^\sigma_n(\pi^{*-(n+1)} z; \mathfrak{n})$$

is a rational function of $\wp(z)$ and $\wp'(z)$ with coefficients in F_n. As in Chapter 2.4.9 of de Shalit (1987),

$$\frac{1}{\lambda'(t)} \frac{d}{dt} \log \Lambda^\sigma_n(\pi^{*-(n+1)} \lambda(t); \mathfrak{n}) \in \hat{\mathcal{I}}_\infty[[t]]$$

is the t-expansion at O of

$$\frac{d}{dz} \log \Lambda^\sigma_n(\pi^{*-(n+1)} z; \mathfrak{n}).$$

Thus, for each $n \in \mathbb{N}$, $g^{\sigma}_{n,e(\mathfrak{n})}(\iota(T_1))$ is of the form $R(\iota(T_1))$ for a rational function R on E whose Laurent expansion in $t = -2x/y$ is an element of $\hat{\mathcal{I}}_\infty[[t]]$. This setting has been studied in the works of Gillard (1987) and Schneps (1987) independently (see Goldstein (1986) for a summary of these results). We have the following result:

Theorem 9 (Theorem 1 in Schneps (1987), Théorèm 2.9 in Gillard (1987))**.** *Suppose $\alpha \in \mathcal{M}_1(\hat{\mathcal{I}}_\infty)$ has associated power series $\widehat{\alpha}$ of the form $R(\iota(T_1))$ for a rational function R on E whose Laurent expansion in t is an element of $\hat{\mathcal{I}}_\infty[[t]]$. For each $0 \le i \le p-2$,*

$$\mu(\Gamma^{(i)}_{\alpha}) = \mu\left(\sum_{v \in W_K} \omega^i(v)\alpha^* \circ v\right).$$

$\square$

In the setting described in Theorem 9, it is useful to move between the measure α, its associated power series, and the associated rational function R on the curve. The operations on measures and power series outlined in Section 2.4 can then be described in terms of the associated rational function, and a complete description can be found in any of Gillard (1981), Gillard (1985), Gillard (1987), or Schneps (1987). We will simply cite these results as needed.

For a function $\mathfrak{n} \in \mathcal{S}$, we define the sets

$$Y_{\mathfrak{n}} = \{\mathfrak{a} \in I : \mathfrak{n}(\mathfrak{a}) \neq 0\}$$

$$Z_{\mathfrak{n}} = \{\mathfrak{a} \in Y_{\mathfrak{n}} : \mathfrak{n}(\mathfrak{a}) \not\equiv 0 \pmod{p}\}$$

$$\mathcal{L}_{\mathfrak{n}} = \{R : R \text{ is an } \mathfrak{a}\text{-division point for some } \mathfrak{a} \in Y_{\mathfrak{n}}\}$$

$$\mathscr{L}_{\mathfrak{n}} = \{R : R \text{ is an } \mathfrak{a}\text{-division point for some } \mathfrak{a} \in Z_{\mathfrak{n}}\}.$$

We now view E over Φ_∞, and let $\tilde{E}$ denote the reduction of E modulo $\mathfrak{p}_\infty$. Recall that all $\mathfrak{a} \in I$ are prime to $6p\mathfrak{f}$, so are prime to $\mathfrak{p}_\infty$. Hereafter, we fix a function $\mathfrak{n} \in \mathcal{S}$ for which $Z_{\mathfrak{n}} \neq \emptyset$ and such that the elements of $Z_{\mathfrak{n}}$ are pairwise coprime. As a consequence, reduction modulo $\mathfrak{p}_\infty$ is injective on the set $\mathscr{L}_\mathfrak{n}$ (Silverman (1986), Proposition 3.1 in Chapter VII). Any such choice of $\mathfrak{n} \in \mathcal{S}$ will be called "good."

Let α denote the measure associated to the rational function R on E given by

$$\frac{d}{dz} \log \Lambda_n(z; \mathfrak{n}).$$

By Schneps (1987), the rational function associated to $\alpha \circ \pi^{*n+1}$ is given by

$$\frac{d}{dz} \log \Lambda_n(\pi^{*-(n+1)} z; \mathfrak{n}),$$

which is the rational function associated to the measure corresponding to $g_{n,e(\mathfrak{n})}(\iota(T_1))$. But by Corollary 15,

$$(\alpha \circ \pi^{*n+1})^* = \alpha^* \circ \pi^{*n+1},$$

from which it follows from Lemma 10 and Lemma 12 that

$$\begin{aligned}\sum_{v \in W_K} \omega^{i_1}(v)(\alpha \circ \pi^{*n+1})^* \circ v &= \sum_{v \in W_K} \omega^{i_1}(v)(\alpha^* \circ \pi^{*n+1}) \circ v \\ &= \left(\sum_{v \in W_K} \omega^{i_1}(v) \alpha^* \circ v \right) \circ \pi^{*n+1}.\end{aligned}$$

In light of Theorem 9 and Lemma 30, if

$$0 = \mu \left(\sum_{v \in W_K} \omega^{i_1}(v) \alpha^* \circ v \right),$$

then the μ-invariant of the $\Gamma^{(i_1)}$-transform of the measure associated to $g_{n,e(\mathfrak{n})}(\iota(T_1))$ also vanishes, and thus the μ-invariant of the $\Gamma^{(i_1,i_2)}$-transform of the measure associated to $h_{e(\mathfrak{n})}(\boldsymbol{T})$ vanishes for all $0 \leq i_2 \leq p-2$.

The following three lemmas employ the techniques from Schneps (1987) to show that

$$0 = \mu \left(\sum_{v \in W_K} \omega^{i}(v) \alpha^* \circ v \right).$$

For the first two lemmas, we could also employ the results from Section 3.2 of Gillard (1987) (see also Section 2.1 of Gillard (1985)).

Lemma 43. *For α as above, $\mu(\alpha) = 0$.*

Proof. We use the explicitly given rational function on E and exhibit its poles on the reduced curve $\tilde{E}$. As a function on E, we have

$$\frac{d}{dz}\log\Lambda_n(z;\mathfrak{n}) = \sum_{\mathfrak{a}\in Y_\mathfrak{n}} -6\mathfrak{n}(\mathfrak{a}) \sum_{\sigma\in \mathrm{Gal}(\mathcal{R}_n/F_n)} \sum_{R\in E_\mathfrak{a}\setminus\{O\}} \frac{-2y(P+Q_n^\sigma)}{x(P+Q_n^\sigma)-x(R)}.$$

From the remarks at the beginning of this section, the poles of this function (if they exist) are all simple and must come from the points $-Q_n^\sigma$ for $\sigma \in \mathrm{Gal}(\mathcal{R}_n/F_n)$ and $R - Q_n^\sigma$ for $\sigma \in \mathrm{Gal}(R_n/F_n)$ and $R \in \mathcal{L}_\mathfrak{n}$. The residue at a pole $-Q_n^\sigma$ is

$$\sum_{\mathfrak{a}\in Y_\mathfrak{n}} 12\mathfrak{n}(\mathfrak{a})(N\mathfrak{a}-1) = 0,$$

so there are no poles at the $-Q_n^\sigma$. The residue at a pole $R - Q_n^\sigma$ is $-12\mathfrak{n}(\mathfrak{a})$. Modulo $\mathfrak{p}_\infty$, this residue is 0 for $\mathfrak{a} \notin Z_\mathfrak{n}$. However, since $p > 3$ and $\mathfrak{n}(\mathfrak{a}) \not\equiv 0 \pmod{\mathfrak{p}_\infty}$ for $\mathfrak{a} \in Z_\mathfrak{n}$, $R - Q_n^\sigma$ is a pole of the reduced function on $\tilde{E}$ for each $R \in \mathscr{L}_\mathfrak{n}$ and each $\sigma \in \mathrm{Gal}(\mathcal{R}_n/F_n)$. The point $R - Q_n^\sigma$ is a primitive $\mathfrak{a}\mathfrak{f}\mathfrak{p}^{*n+1}$-division point, and reduction modulo $\mathfrak{p}_\infty$ is injective on this set. Consequently, each $R - Q_n^\sigma$ gives a distinct pole for the function on $\tilde{E}$, so the reduced rational function on $\tilde{E}$ is nonzero. This gives $\mu(\alpha) = 0$. □

Lemma 44. *For α as above, $\mu(\alpha|_{p\mathbb{Z}_p}) > 0$. Consequently, $\mu(\alpha^*) = 0$.*

Proof. Let $\sigma_\mathfrak{p} = \left(\frac{\mathfrak{p}}{\mathcal{R}_n/K}\right)$, and note that $\sigma_\mathfrak{p}$ is the restriction of φ to $\mathcal{R}_n$. Let us briefly return to the power series $g_{n,e(\mathfrak{n})}(\iota(T_1))$, which corresponds to the measure $\alpha \circ \pi^{*n+1}$. Since $\iota : \mathbb{G}_m \to \hat{E}$ is a formal group isomorphism, the functional equation for $g_{n,e(\mathfrak{n})}$ gives that the power series associated to $(\alpha \circ \pi^{*n+1})|_{p\mathbb{Z}_p}$ is

$$\begin{aligned}\frac{1}{p}\sum_{\zeta^p=1} g_{n,e(\mathfrak{n})}(\iota(\zeta T_1)) &= \frac{1}{p}\sum_{\eta\in\hat{E}_\pi} g_{n,e(\mathfrak{n})}(\iota(T_1)[+]_{\hat{E}}\eta) \\ &= \frac{\pi}{p} g^{\varphi}_{n,e(\mathfrak{n})}([\pi](\iota(T_1))).\end{aligned}$$

The rational function associated to this last power series is

$$\frac{\pi}{p}\frac{d}{dz}\log\Lambda_n^{\sigma_{\mathfrak{p}}}(\pi^{*-(n+1)}\pi z;\mathfrak{n}),$$

and since $(\alpha\circ\pi^{*n+1})|_{p\mathbb{Z}_p}=\alpha|_{p\mathbb{Z}_p}\circ\pi^{*n+1}$, Lemma 12 gives that the rational function associated to $\alpha|_{p\mathbb{Z}_p}$ is

$$\frac{\pi}{p}\frac{d}{dz}\log\Lambda_n^{\sigma_{\mathfrak{p}}}(\pi z;\mathfrak{n}).$$

The discussion after Theorem 8 gives that the function on E associated to $\alpha|_{p\mathbb{Z}_p}$ is

$$\frac{\pi}{p}\sum_{\mathfrak{a}\in Y_{\mathfrak{n}}}-6\mathfrak{n}(\mathfrak{a})\sum_{S\in E_\pi}\sum_{\sigma\in\mathrm{Gal}(\mathcal{R}_n/F_n)}\sum_{R\in E_{\mathfrak{a}}\backslash\{O\}}\frac{-2y(\pi P+Q_n^{\sigma_{\mathfrak{p}}\sigma})}{x(\pi P+Q_n^{\sigma_{\mathfrak{p}}\sigma})-x(R)}.$$

But $Q_n^{\sigma_{\mathfrak{p}}}=\psi(\mathfrak{p})Q_n=\pi Q_n$ and multiplication by π is a bijection $E_{\mathfrak{a}}\to E_{\mathfrak{a}}$. Therefore, the function on E associated to $\alpha|_{p\mathbb{Z}_p}$ is

$$\frac{\pi}{p}\sum_{\mathfrak{a}\in Y_{\mathfrak{n}}}-6\mathfrak{n}(\mathfrak{a})\sum_{S\in E_\pi}\sum_{\sigma\in\mathrm{Gal}(\mathcal{R}_n/F_n)}\sum_{R\in E_{\mathfrak{a}}\backslash\{O\}}\frac{-2y(\pi(P+Q_n^{\sigma}))}{x(\pi(P+Q_n^{\sigma}))-x(\pi R)}.$$

The poles of this function come from the points $S-Q_n^\sigma$ and $R+S-Q_n^\sigma$ for $S\in E_\pi$, $\sigma\in\mathrm{Gal}(\mathcal{R}_n/F_n)$, and $R\in\mathcal{L}_{\mathfrak{n}}$. Since all the $S\in E_\pi$ reduce to zero modulo $\mathfrak{p}_\infty$, the $S-Q_n^\sigma$ for fixed $\sigma\in\mathrm{Gal}(\mathcal{R}_n/F_n)$ and varying $S\in E_\pi$ all reduce to the same point on $\tilde{E}$, and the $R+S-Q_n^\sigma$ for fixed $\sigma\in\mathrm{Gal}(\mathcal{R}_n/F_n)$ and $R\in\mathcal{L}_{\mathfrak{n}}$ and varying $S\in E_\pi$ all reduce to the same point on $\tilde{E}$. Moreover, for fixed $\sigma\in\mathrm{Gal}(\mathcal{R}_n/F_n)$, the residues at the poles $S-Q_n^\sigma$ are all equal, and for fixed $\sigma\in\mathrm{Gal}(\mathcal{R}_n/F_n)$ and $R\in\mathcal{L}_{\mathfrak{n}}$, the residues at the poles $R+S-Q_n^\sigma$ are all equal. Consequently, each pole of the reduced function on $\tilde{E}$ associated to $\alpha|_{p\mathbb{Z}_p}$ has residue which is a multiple of p, so the reduced function must be identically zero. This gives $\mu(\alpha|_{p\mathbb{Z}_p})>0$. Since we showed in Lemma 43 that $\mu(\alpha)=0$, Corollary 20 gives $\mu(\alpha^*)=0$. □

Lemma 45. *For α as above and for any $0\le i\le p-2$,*

$$0=\mu\left(\sum_{v\in W_K}\omega^i(v)\alpha^*\circ v\right).$$

Proof. We have

$$\widehat{\alpha^*} = \widehat{\alpha} - \widehat{\alpha|_{p\mathbb{Z}_p}},$$

so Lemma 43 and Lemma 44 together give that the poles of the reduced rational function associated to α^* on $\tilde{E}$ and the poles of the rational function associated to α on $\tilde{E}$ are the same. Thus, the poles of the reduced function associated to α^* on $\tilde{E}$ are of the form $R - Q_n^\sigma$ for $R \in \mathscr{L}_\mathfrak{n}$ and $\sigma \in \mathrm{Gal}(\mathcal{R}_n/F_n)$. For each $R \in \mathscr{L}_\mathfrak{n}$, define the set

$$P_R = \{R - Q_n^\sigma : \sigma \in \mathrm{Gal}(\mathcal{R}_n/F_n)\},$$

and for each $v \in W_K$, define

$$vP_R = \{v(R - Q_n^\sigma) : \sigma \in \mathrm{Gal}(\mathcal{R}_n/F_n)\}.$$

The orbit of Q_n under $\mathrm{Gal}(\mathcal{R}_n/F_n)$ lies in one congruence class modulo W_K, so the vP_R are pairwise disjoint sets for fixed R and varying $v \in W_K$. Suppose we have an equality of sets $vP_{R_1} = vP_{R_2}$ for $R_1, R_2 \in \mathscr{L}_\mathfrak{n}$. Since multiplication by v is a bijection $E_\mathfrak{a} \to E_\mathfrak{a}$ for each $\mathfrak{a} \in I$, R_1 and R_2 must be $\mathfrak{a}$-division points for the same $\mathfrak{a} \in Z_\mathfrak{n}$. However,

$$f\pi^{*n+1}v(R_1 - Q_n^\sigma) = f\pi^{*n+1}v(R_1)$$

and

$$f\pi^{*n+1}v(R_2 - Q_n^\sigma) = f\pi^{*n+1}v(R_2)$$

for all $\sigma \in \mathrm{Gal}(\mathcal{R}_n/F_n)$. Since multiplication by $f\pi^{*n+1}v$ is a bijection $E_\mathfrak{a} \to E_\mathfrak{a}$, we must have $R_1 = R_2$. This gives that the poles of the reduced rational function on $\tilde{E}$ associated to $\alpha^* \circ v$ are given by the sets vP_R for $R \in \mathscr{L}_\mathfrak{n}$, and these poles are all distinct.

Suppose now that

$$v_1(R_1 - Q_n^{\sigma_1}) = v_2(R_2 - Q_n^{\sigma_2})$$

for some $v_1, v_2 \in W_K$, with $v_1 \neq v_2$, $R_1, R_2 \in \mathscr{L}_{\mathfrak{n}}$, and $\sigma_1, \sigma_2 \in \mathrm{Gal}(\mathcal{R}_n/F_n)$. Say R_1 is an (α_1)-division point and R_2 is an (α_2)-division point. Then $\alpha_1\alpha_2$ is prime to $6pf$, and we have that

$$v_1\alpha_1\alpha_2(Q_n^{\sigma_1}) = \alpha_1\alpha_2 v_1(R_1 - Q_n^{\sigma_1}) = \alpha_1\alpha_2 v_2(R_2 - Q_n^{\sigma_2}) = v_2\alpha_1\alpha_2(Q_n^{\sigma_2}).$$

This is impossible since the orbit of Q_n under $\mathrm{Gal}(\mathcal{R}_n/F_n)$ lies in one congruence class modulo W_K. Consequently, the sets of poles for the reduced rational functions on $\tilde{E}$ associated to the measures $\alpha^* \circ v$ for $v \in W_K$ are pairwise disjoint.

We have that all of the poles of the reduced rational function on $\tilde{E}$ associated to

$$\sum_{v \in W_K} \omega^i(v)\alpha^* \circ v$$

are given by the vP_R for $v \in W_K$ and $R \in \mathscr{L}_{\mathfrak{n}}$. As in Lemma 43, the residues at these poles are all nonzero modulo $\mathfrak{p}_\infty$, and reduction modulo $\mathfrak{p}_\infty$ is injective on the collection of all the vP_R. From these facts, we conclude that the reduced rational function on the curve $\tilde{E}$ associated to

$$\sum_{v \in W_K} \omega^i(v)\alpha^* \circ v$$

is nonzero, and this gives

$$0 = \mu\left(\sum_{v \in W_K} \omega^i(v)\alpha^* \circ v\right).$$

□

Finally, we record our main result.

Theorem 10. *Let $\mathfrak{n} : I \to \mathbb{Z}$ be good, and let $\alpha \in \mathcal{M}_2(\hat{\mathcal{I}}_\infty)$ denote the measure associated to $h_{e(\mathfrak{n})} \in \hat{\mathcal{I}}[[\boldsymbol{T}]]$. Then for each $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p} - \boldsymbol{2}$, $\mu(\Gamma_\alpha^{(\boldsymbol{i})}) = 0$.* □

We denote by $\bar{\psi}$ the character given by $\bar{\psi}(\mathfrak{a}) = \overline{\psi(\mathfrak{a})}$ whenever $(\mathfrak{a}, \mathfrak{f}) = 1$. For $k > 1$, let

$$L(\bar{\psi}^k, s) = \sum_{(\mathfrak{a},\mathfrak{f})=1} \frac{\bar{\psi}^k(\mathfrak{a})}{(N\mathfrak{a})^s}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > k/2 + 1$. Then $L(\bar{\psi}^k, s)$ may be analytically continued to all of $\mathbb{C}$ (e.g. Silverman (1994), Chapter II or Miyake (1989)). For $k > j \geq 0$, we put

$$L_\infty(\bar{\psi}^{k+j}, k) = \left(1 - \frac{\psi^{k+j}(\mathfrak{p})}{(N\mathfrak{p})^{j+1}}\right)\left(1 - \frac{\bar{\psi}^{k+j}(\mathfrak{p}^*)}{(N\mathfrak{p}^*)^k}\right)\left(\frac{\tau}{\sqrt{d_K}}\right)^j \Omega_\infty^{-(k+j)} L(\bar{\psi}^{k+j}, k)$$

(note that τ in the equation above is the real number $6.283185\ldots$). See Yager (1982) for more details.

We now describe how to construct Yager's two variable p-adic L-function using the two-variable objects from the previous section. One of the main results in Yager (1982) is the following.

Theorem 11 (Theorem 29 of Yager (1982))**.** *Let* $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p} - \boldsymbol{2}$. *There is a power series* $\mathcal{G}^{(\boldsymbol{i})}(\boldsymbol{T}) \in \hat{\mathcal{I}}_\infty[[\boldsymbol{T}]]$ *such that for all integers* $k_1 > -k_2 \geq 0$ *with* $k_j \equiv i_j \pmod{p-1}$ *for* $j = 1, 2$,

$$\mathcal{G}^{(\boldsymbol{i})}(\boldsymbol{u}^{\wedge \boldsymbol{k}}) = (k_1 - 1)! \Omega_{\mathfrak{p}}^{k_2 - k_1} L_\infty(\bar{\psi}^{k_1 - k_2}, k_1).$$

□

Given $\beta \in \boldsymbol{U}_\infty$, let α_β be the measure associated to h_β. For $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p} - \boldsymbol{2}$, Consider the power series $G_{\alpha_\beta}^{(i_1-1,-i_2)}(\boldsymbol{T})$ associated to the $\Gamma^{(i_1-1,-i_2)}$-transform of α_β, or what amounts to the same, the $\Gamma^{(i_1-1,-i_2)}$-transform of α_β^* (where we use the least non-negative residues of $i_1 - 1$ and $-i_2$ modulo $p-1$). Now set

$$\mathcal{G}_\beta^{(\boldsymbol{i})}(\boldsymbol{T}) = G_{\alpha_\beta}^{(i_1-1,-i_2)}(u_1^{-1}T_1 - 1, T_2^{-1} - 1) \in \hat{\mathcal{I}}_\infty[[\boldsymbol{T}]].$$

It is crucial to note that the change of variables $T_1 \to u_1^{-1}T_1$ and $T_2 \to T_2^{-1}$ cannot increase the divisibility by $\mathfrak{p}_\infty$ of an element of $\hat{\mathcal{I}}_\infty[[\boldsymbol{T}]]$. Indeed, take $F \in \hat{\mathcal{I}}_\infty[[\boldsymbol{T}]]$. Dividing through by an appropriate power of a generator of $\mathfrak{p}_\infty$, we may assume $\mu(F) = 0$. Lemma 30 gives that

$$F(T_1, T_2^{-1}) \equiv F(T_1, T_2) \pmod{\mathfrak{p}_\infty}$$

in this case. On the other hand, since $u_1 \in U$, $u_1^{-1}T_1 \equiv T_1 \pmod{\mathfrak{p}_\infty}$, we have

$$F(u_1^{-1}T_1, T_2) \equiv F(T_1, T_2) \pmod{\mathfrak{p}_\infty}.$$

When $\boldsymbol{i} \neq \boldsymbol{0}$ and $\boldsymbol{i} \neq \boldsymbol{1}$, the ideal of $\hat{\mathcal{I}}_\infty[[\boldsymbol{T}]]$ generated by $\mathcal{G}^{(\boldsymbol{i})}$ is also generated by one of the $\mathcal{G}^{(\boldsymbol{i})}_{\langle e(\mathfrak{n})\rangle}$ for an appropriate choice of the function $\mathfrak{n} : I \to \mathbb{Z}$ depending on $\boldsymbol{i}$ (see Lemma 28 of Yager (1982) for a suitable choice of the function $\mathfrak{n}$). In any case, we have

$$\mu(\mathcal{G}^{(\boldsymbol{i})}) = \mu(\mathcal{G}^{(\boldsymbol{i})}_{\langle e(\mathfrak{n})\rangle}) = \mu(G^{(i_1-1,-i_2)}_{\alpha_{\langle e(\mathfrak{n})\rangle}})$$

for the appropriate choice of the function $\mathfrak{n} : I \to \mathbb{Z}$. Each of these functions $\mathfrak{n} : I \to \mathbb{Z}$ used in Yager (1982) satisfies the conditions required in the proofs of Lemmas 43, 44, and 45 in the previous section. Recalling finally that $h_\beta = h_{\langle\beta\rangle}$ for all $\beta \in \boldsymbol{U}'_\infty$, we obtain

Theorem 12. *For $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p} - \boldsymbol{2}$, let $\mathcal{G}^{(\boldsymbol{i})}(\boldsymbol{T}) \in \hat{\mathcal{I}}_\infty[[\boldsymbol{T}]]$ be Yager's interpolating power series in Theorem 11. Then for each $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p} - \boldsymbol{2}$ with $\boldsymbol{i} \neq \boldsymbol{0}, \boldsymbol{1}$, $\mu(\mathcal{G}^{(\boldsymbol{i})}) = 0$.* □

When $\boldsymbol{i} = \boldsymbol{0}$, there is a choice of a good $\mathfrak{n}$ similar to the function given in Lemma 28 of Yager (1982) such that $\mathcal{G}^{(\boldsymbol{0})}_{\langle e(\mathfrak{n})\rangle}(\boldsymbol{T})$ differs from $(T_2 - 1)\mathcal{G}^{(\boldsymbol{0})}(\boldsymbol{T})$ by a unit power series when both of these power series are considered modulo the ideal $\langle \mathfrak{p}_\infty^2, T_1^p - 1, T_2^p - 1\rangle$ of $\hat{\mathcal{I}}_\infty[[\boldsymbol{T}]]$. Similarly, when $\boldsymbol{i} = \boldsymbol{1}$, there is a choice of a good $\mathfrak{n}$ similar to the function given in Lemma 28 of Yager (1982) such that $\mathcal{G}^{(\boldsymbol{1})}_{\langle e(\mathfrak{n})\rangle}(\boldsymbol{T})$ differs from $(1-u+(T_1-1))\mathcal{G}^{(\mathbf{1})}(\boldsymbol{T})$ by a unit power series when both of these power series are considered modulo the ideal $\langle \mathfrak{p}_\infty^2, T_1^p-1, T_2^p-1\rangle$ of $\hat{\mathcal{I}}_\infty[[\boldsymbol{T}]]$. The constructions of these good functions can easily be modified to produce congruences modulo the ideal $\langle \mathfrak{p}_\infty^{n+1}, T_1^{p^n}-1, T_2^{p^n}-1\rangle$ for any $n \in \mathbb{N}$, and hence congruences modulo the ideal $\langle \mathfrak{p}_\infty, T_1^{p^n}-1, T_2^{p^n}-1\rangle = \langle \mathfrak{p}_\infty, (T_1-1)^{p^n}, (T_2-1)^{p^n}\rangle$. In particular, choosing $n \in \mathbb{N}$ so that $L(\mathcal{G}^{\boldsymbol{i}}_{\langle e(\mathfrak{n})\rangle}) \cap [\boldsymbol{p}^{\wedge n\mathbf{1}}-\mathbf{1}] \neq \emptyset$ gives the following.

Theorem 13. *For $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p}-\mathbf{2}$, let $\mathcal{G}^{(\boldsymbol{i})}(\boldsymbol{T}) \in \hat{\mathcal{I}}_\infty[[\boldsymbol{T}]]$ be Yager's interpolating power series in Theorem 11. For $\boldsymbol{i}=\mathbf{0}$ and $\boldsymbol{i}=\mathbf{1}$, $\mu(\mathcal{G}^{(\boldsymbol{i})})=0$.* □

We remark that Gillard (1987) obtains $\mu(\mathcal{G}^{(\boldsymbol{i})}) = 0$ for each $\boldsymbol{i} \neq \mathbf{0}$ (Section 3.6 of Gillard (1987)). Gillard's approach involves considering the power series $\mathcal{G}^{(\boldsymbol{i})}(\boldsymbol{T})|_{T_2=1}$, which can be identified with the image of $\mathcal{G}^{(\boldsymbol{i})}$ under the natural projection

$$\hat{\mathcal{I}}_\infty[[\boldsymbol{T}]] \to \hat{\mathcal{I}}_\infty[[\boldsymbol{T}]]/\langle T_2-1\rangle \cong \hat{\mathcal{I}}_\infty[[T_1-1]].$$

Using a result of Wintenberger (1981) and the class field theory considerations in the following section, Gillard obtains an equality between the μ-invariant of $\mathcal{G}^{(\boldsymbol{i})}(\boldsymbol{T})|_{T_2=1}$ and the μ-invariant of a certain Iwasawa series (which is known to vanish) considered in Gillard (1987).

3.3 The Two-Variable Main Conjecture and Questions of Class Group Growth

Recall that

$$G_\infty = \mathrm{Gal}(K_\infty/K) \cong \Gamma \times \Delta,$$

where $\Gamma = \mathrm{Gal}(K_\infty/K_{\mathbf{0}}) \cong \mathbb{Z}_p^2$ and $\Delta \cong \mathrm{Gal}(K_{\mathbf{0}}/K)$ is a product of two cyclic groups of order $p-1$. Denote by χ_1 and χ_2, respectively, the restrictions of the characters κ_1

and κ_2 to Δ. If M is a $\mathbb{Z}_p[\Delta]$-module and $\boldsymbol{i} \in \mathbb{N}^2$, we denote by $M^{(\boldsymbol{i})}$ the submodule of M on which Δ acts by $\chi_1^{i_1}\chi_2^{i_2}$, so that

$$M = \bigoplus_{\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p-2}} M^{(\boldsymbol{i})}.$$

We let $\Lambda = \mathbb{Z}_p[[\boldsymbol{X}]] = \mathbb{Z}_p[[X_1, X_2]]$.

For $\boldsymbol{n} \in \mathbb{N}$, recall that $C'_{\boldsymbol{n}}$ denotes the group of elliptic units in $K_{\boldsymbol{n}}$. We let $\mathcal{E}_{\boldsymbol{n}}$ denote the group of units of $K_{\boldsymbol{n}}$ which are congruent to 1 modulo every prime ideal of $K_{\boldsymbol{n}}$ above $\mathfrak{p}$, and put $C_{\boldsymbol{n}} = C'_{\boldsymbol{n}} \cap \mathcal{E}_{\boldsymbol{n}}$. We then embed $\mathcal{E}_{\boldsymbol{n}}$ and $C_{\boldsymbol{n}}$ into $\boldsymbol{U}_{\boldsymbol{n}}$ via the diagonal map and denote by $\bar{\mathcal{E}}_{\boldsymbol{n}}$ and $\bar{C}_{\boldsymbol{n}}$ the closures of $\mathcal{E}_{\boldsymbol{n}}$ and $C_{\boldsymbol{n}}$ in $\boldsymbol{U}_{\boldsymbol{n}}$. Finally, we put

$$\mathcal{E}_\infty = \lim_{\leftarrow} \bar{\mathcal{E}}_{\boldsymbol{n}}$$

$$C_\infty = \lim_{\leftarrow} \bar{C}_{\boldsymbol{n}}$$

with the inverse limits defined by the norm maps on the $\boldsymbol{\Xi}_{\boldsymbol{n}}$.

Recall that for $n \in \mathbb{N}$, $\mathrm{Gal}(K_{n\boldsymbol{1}}/K_{\boldsymbol{0}}) \cong (\mathbb{Z}/p^n\mathbb{Z})^2$. For $n \in \mathbb{N}$, let A_n denote the p-part of the ideal class group of $K_{n\boldsymbol{1}}$, and put

$$A_\infty = \lim_{\leftarrow} A_n,$$

where the inverse limit is defined in terms of the norm maps on ideal class groups. Let M_∞ be the maximal abelian p-extension of K_∞ which is unramified outside of the primes above $\mathfrak{p}$, and set $X_\infty = \mathrm{Gal}(M_\infty/K_\infty)$.

Now U_∞, $\mathcal{E}_\infty$, C_∞, X_∞, and A_∞ are all Λ-modules, as are $U_\infty^{(\boldsymbol{i})}$, $\mathcal{E}_\infty^{(\boldsymbol{i})}$, $C_\infty^{(\boldsymbol{i})}$, $X_\infty^{(\boldsymbol{i})}$, and $A_\infty^{(\boldsymbol{i})}$ for each $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p-2}$. Moreover, for each $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p-2}$, $U_\infty^{(\boldsymbol{i})}$, $\mathcal{E}_\infty^{(\boldsymbol{i})}$, $C_\infty^{(\boldsymbol{i})}$, $X_\infty^{(\boldsymbol{i})}$, and $A_\infty^{(\boldsymbol{i})}$ are finitely generated Λ-modules, and $A_\infty^{(\boldsymbol{i})}$, $\mathcal{E}_\infty^{(\boldsymbol{i})}/C_\infty^{(\boldsymbol{i})}$, $\boldsymbol{U}_\infty^{(\boldsymbol{i})}/C_\infty^{(\boldsymbol{i})}$, and $X_\infty^{(\boldsymbol{i})}$ are all torsion Λ-modules.

Class field theory gives the following exact sequence:

$$0 \to \mathcal{E}_\infty/C_\infty \to \boldsymbol{U}_\infty/C_\infty \to X_\infty \to A_\infty \to 0$$

Figure 6. The Fundamental Exact Sequence

We also have the following exact sequence for each $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p-2}$:

$$0 \to \mathcal{E}_\infty^{(\boldsymbol{i})}/C_\infty^{(\boldsymbol{i})} \to \boldsymbol{U}_\infty^{(\boldsymbol{i})}/C_\infty^{(\boldsymbol{i})} \to X_\infty^{(\boldsymbol{i})} \to A_\infty^{(\boldsymbol{i})} \to 0$$

(see Rubin (1991)).

There are well-known structure theorems for finitely generated, torsion Λ-modules (see Bourbaki (1972), Chapter VII or Serre (1960)). For a finitely generated, torsion Λ-module M, there exists a Λ-module homomorphism

$$M \to \bigoplus_{i=1}^{t} \Lambda/P_i^{r_i}$$

whose kernel and cokernel have localization (0) at each height one prime ideal of Λ (such a homomorphism is called a pseudo-isomorphism). In the above decomposition, the P_i are height one prime ideals of Λ and the r_i are positive integers. The principal ideal

$$\prod_{i=1}^{t} P_i^{r_i}$$

of Λ is called the characteristic ideal of M, and any generator of the characteristic ideal is called a characteristic power series of M. The characteristic ideal of M is a well-defined invariant of the Λ-module M, which we denote by $\mathrm{char}(M)$, and a characteristic power series of M is well-defined up to a unit in Λ. Characteristic ideals are multiplicative in exact sequences of Λ-modules. That is to say, if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact, then $\mathrm{char}(M_2) = \mathrm{char}(M_1)\mathrm{char}(M_3)$ (see Cuoco and Monsky (1981), Perrin-Riou (1984)).

In the setting currently under consideration, the two-variable main conjecture takes the following form:

Theorem 14 (Two-Variable Main Conjecture)**.** *For each* $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p-2}$*,*

$$\mathrm{char}(A_\infty^{(\boldsymbol{i})}) = \mathrm{char}(\mathcal{E}_\infty^{(\boldsymbol{i})}/C_\infty^{(\boldsymbol{i})}) \quad \textit{and} \quad \mathrm{char}(X_\infty^{(\boldsymbol{i})}) = \mathrm{char}(\boldsymbol{U}_\infty^{(\boldsymbol{i})}/C_\infty^{(\boldsymbol{i})}).$$

□

The main conjectures of Iwasawa theory for imaginary quadratic fields were proved in Rubin (1991) (see also Rubin (1994)). The second main theorem in Yager (1982) regards the Λ-module structure of $\boldsymbol{U}_\infty^{(\boldsymbol{i})}/C_\infty^{(\boldsymbol{i})}$.

Theorem 15 (Theorem 30 in Yager (1982))**.** *Let* $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p-2}$*. There is an element* $G^{(\boldsymbol{i})} \in \Lambda$ *which generates the same ideal in* $\hat{\mathcal{I}}_\infty[[X_1, X_2]]$ *as* $\mathcal{G}^{(\boldsymbol{i})}(X_1, X_2)$*. Moreover,* $\mathrm{char}(\boldsymbol{U}_\infty^{(\boldsymbol{i})}/C_\infty^{(\boldsymbol{i})}) = G^{(\boldsymbol{i})}\Lambda$*.* □

Let $K_{\mathbb{Z}_p}$ denote the composite of all $\mathbb{Z}_p$-extension of K. Then $K_{\mathbb{Z}_p}/K$ is the unique $\mathbb{Z}_p^2$-extension of K (Greenberg (1973), Washington (1997) Chapter 13), and we have that $K_\infty = K_{\boldsymbol{0}}K_{\mathbb{Z}_p}$. We let K_n denote the intermediate field in $K_{\mathbb{Z}_p}/K$ with $\mathrm{Gal}(K_n/K) \cong (\mathbb{Z}/p^n\mathbb{Z})^2$. For each $n \in \mathbb{N}$, $K_{n\boldsymbol{1}} = K_{\boldsymbol{0}}K_n$. Let B_n denote the p-part of the ideal class group of K_n.

Define the integers $e_n(K_\infty/K_{\boldsymbol{0}})$ and $e_n(K)$ by $p^{e_n(K_\infty/K_{\boldsymbol{0}})} = |A_n|$ and $p^{e_n(K)} = |B_n|$. We have the following theorem regarding the growth rate of these integers (see Cuoco and Monsky (1981)).

Theorem 16. *There exist non-negative integers* $m_0(K_\infty/K_{\mathbf{0}})$, $m_0(K)$, $l_0(K_\infty/K_{\mathbf{0}})$, *and* $l_0(K)$ *such that*

$$e_n(K_\infty/K_{\mathbf{0}}) = (m_0(K_\infty/K_{\mathbf{0}})p^n + l_0(K_\infty/K_{\mathbf{0}})n + O(1))p^n$$

and

$$e_n(K) = (m_0(K)p^n + l_0(K)n + O(1))p^n.$$

□

We note that a refinement of the above growth formulas appears in Monsky (1989). We also have $m_0(K) \le m_0(K_\infty/K_{\mathbf{0}})$, as in Chapter 13 of Lang (1990). The integer $m_0(K_\infty/K_{\mathbf{0}})$ appearing in Theorem 16 is simply the μ-invariant of a characteristic power series for the Λ-module A_∞ (see Cuoco and Monsky (1981)). For any finitely generated torsion Λ-module M, we let $m_0(M)$ denote the μ-invariant of a characteristic power series for M. We have

$$m_0(M) = \sum_{\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p-2}} m_0(M^{(\boldsymbol{i})}).$$

Given Theorem 30 of Yager (1982), we have shown:

Corollary 30. *For all* $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p-2}$, $m_0(X^{(\boldsymbol{i})}) = 0$. □

In fact, class field theory gives the exact sequence

$$0 \to \boldsymbol{U}_\infty^{(\boldsymbol{i})}/\mathcal{E}_\infty^{(\boldsymbol{i})} \to X_\infty^{(\boldsymbol{i})} \to A_\infty^{(\boldsymbol{i})} \to 0$$

(see Rubin (1988)). Since

$$m_0(\boldsymbol{U}_\infty^{(\boldsymbol{i})}/\mathcal{E}_\infty^{(\boldsymbol{i})}) + m_0(A_\infty^{(\boldsymbol{i})}) = m_0(X_\infty^{(\boldsymbol{i})}),$$

we further obtain:

Corollary 31. *For all* $\boldsymbol{i} \preccurlyeq_2 \boldsymbol{p-2}$, $m_0(A_\infty^{(\boldsymbol{i})}) = 0$. *Consequently,* $m_0(K) = 0$. □

Cuoco (1980) obtains $m_0(K) = 0$ from class field theory considerations as a consequence of the Ferrero-Washington theorem for $K/\mathbb{Q}$. We have obtained this same result from measure theoretic considerations. An interesting consequence of this fact is that only finitely many $\mathbb{Z}_p$-extensions of K may have nonzero μ-invariant (see Cuoco (1980)).

REFERENCES

Amice, Y., "Mesures p-adiques", Séminaire Delange-Pisot-Poitou. Théorie des nombres **6**, 2, 1–6 (1964-1965).

Amice, Y., "Duals", in "Proceedings of the Conference on p-adic Analysis (Nijmegen, 1978)", pp. 1–15 (1978).

Bernardi, D., C. Goldstein and N. Stephens, "Notes p-adiques sur les courbes elliptiques", J. Reine Angew. **351**, 129–170 (1984).

Bojanic, R., "A simple proof of Mahler's theorem on approximation of continuous functions of a p-adic variable by polynomials", J. Num. Theory **6**, 412–415 (1974).

Bourbaki, N., *Commutative Algebra*, Elements of mathematics (Addison Wesley, 1972).

Cassou-Noguès, P., "p-adic L-functions for elliptic curves with complex multiplication I", Compositio Math. **42**, 31–56 (1981).

Childress, N., "λ-invariants and Γ-transforms", Manuscripta Math. **64**, 359–375 (1989).

Childress, N., "p-adic numbers", Course offered at Arizona State University (2012).

Childress, N. and S. Zinzer, "On λ-invariants associated to linear extensions of the product ordering on $\mathbb{N}^d$", J. Num. Theory **147**, 478–489 (2015).

Coates, J. and C. Goldstein, "Some remarks on the main conjecture for elliptic curves with complex multiplication", Amer. J. Math. **103**, 2, 337–366 (1983).

Coates, J. and R. Sujatha, *Cyclotomic fields and zeta values*, Springer Monographs in Mathematics (Springer, 2006).

Coates, J. and A. Wiles, "On the conjecture of Birch and Swinnerton-Dyer", Invent. Math. **39**, 223–251 (1977).

Coates, J. and A. Wiles, "On p-adic L-functions and elliptic units", J. Austral. Math. Soc. (Series A) **26**, 1–25 (1978).

Coleman, R. F., "Division values in local fields", Invent. Math. **53**, 91–116 (1979).

Cuoco, A. A., "The growth of Iwasawa invariants in a family", Compositio Math. **41**, 425–437 (1980).

Cuoco, A. A. and P. Monsky, "Class numbers in $\mathbb{Z}_p^d$-extensions", Math. Ann. **255**, 235–258 (1981).

de Shalit, E., *Iwasawa Theory of Elliptic Curves with Complex Multiplication*, vol. 3 of *Perspectives in Mathematics* (Academic Press, 1987).

Dickson, L. E., "Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors", Amer. J. Math. **35**, 413–422 (1913).

Dickson, L. E., *History of the theory of numbers. Vol. I: Divisibility and primality*, chap. IX (Chelsea Publishing Co., 1966).

Folland, G. B., *Real analysis : modern techniques and their applications*, Pure and applied mathematics (Wiley, 1999), second edn.

Frolich, A., *Formal Groups*, vol. 74 of *Lecture Notes in Mathematics* (Springer-Verlag, 1968).

Gillard, R., "Unités elliptiques et fonctions L p-adiques", Compositio Math. **42**, 57–88 (1981).

Gillard, R., "Fonctions L p-adiques des corps quadratiques imaginaires et de leurs extensions abéliennes", J. Reine Angew. Math. **358**, 76–91 (1985).

Gillard, R., "Transformation de Mellin-Leopoldt des fonctions elliptiques", J. Num. Theory **25**, 379–393 (1987).

Goldstein, C., "L'invariant μ des fonctions L p-adiques des courbes elliptiques à multiplication complexe est nul (d'après L. Schneps et R. Gillard)", in "Séminaire de théorie des nombres, Paris 1984–85", edited by C. Goldstein, vol. 63 of *Progr. Math.*, pp. 23–32 (Birkhäuser Boston, 1986).

Gouvêa, F. Q., *p-adic Numbers* (Springer, 1997).

Greenberg, R., "The Iwasawa invariants of Γ-extensions of a fixed number field", Amer. J. Math. **95**, 204–214 (1973).

Grothendieck, A., "Produits tensoriels topologiques et espaces nucléaires", Séminaire Bourbaki **2**, 193–200 (1951-1954).

Hazewinkel, M., *Formal groups and applications* (Academic Press, 1978).

Iwasawa, K., *Local class field theory*, Oxford Mathematical Monographs (Oxford University Press, 1986).

Kida, Y., "The λ-invariants of p-adic measures on $\mathbb{Z}_p$ and $1+q\mathbb{Z}_p$", Sci. Rep. Kanazawa Univ. **30**, 273–282 (1986).

Koblitz, N., *p-adic Numbers, p-adic Analysis, and Zeta-Functions*, vol. 58 of *Graduate Texts in Mathematics* (Springer, 1984), second edn.

Lang, S., *Elliptic Functions*, vol. 112 of *Graduate Texts in Mathematics* (Springer-Verlag, 1987), second edn.

Lang, S., *Cyclotomic Fields I and II*, vol. 121 of *Graduate Texts in Mathematics* (Springer-Verlag, 1990).

Lichtenbaum, S., "On p-adic L-functions associated to elliptic curves", Invent. Math. **56**, 1, 19–55 (1980).

Lucas, E., "Théorie des fonctions numériques simplement périodiques. [continued]", Amer. J. Math. **1**, 3, 197–240 (1878).

Miyake, T., *Modular Forms*, chap. 3 (Springer-Verlag, 1989).

Monna, A., *Analyse Non-Archimédienne* (Springer-Verlag, 1970).

Monsky, P., "Fine estimates for the growth of e_n in $\mathbb{Z}_p^d$-extensions", in "Algebraic Number Theory–in honor of K. Iwasawa", vol. 17 of *Adv. Stud. Pure Math*, pp. 309–330 (Academic Press, 1989).

Perrin-Riou, B., "Arithmétique des courbes elliptiques et théorie d'Iwasawa", Mémoires de la Société Mathématique de France **17**, 1–130 (1984).

Petalas, C. G. and A. K. Katsaras, "Tensor products of p-adic measures", in "Advances in p-adic and non-Archimedean analysis", edited by M. Berz and K. Shamseddine, vol. 508 of *Contemporary Mathematics*, pp. 187–199 (American Mathematical Society, 2010).

Robert, A., *A course in p-adic analysis*, vol. 198 of *Graduate Texts in Mathematics* (Springer, 2000).

Rosenberg, S. J., *On some conjectures in Mazur's deformation theory with supplementary results on p-adic L-functions*, Ph.D. thesis, Ohio State University (1996).

Rosenberg, S. J., "On the Iwasawa invariants of the Γ-transform of a rational function", J. Num. Theory **109**, 89–95 (2004).

Rubin, K., "On the main conjecture of Iwasawa theory for imaginary quadratic fields", Invent. Math. **93**, 701–713 (1988).

Rubin, K., "The "main conjectures" of Iwasawa theory for imaginary quadratic fields", Invent. Math. **103**, 25–68 (1991).

Rubin, K., "More "main conjectures" for imaginary quadratic fields", in "Elliptic Curves and Related Topics", vol. 4 of *CRM Proc. Lecture Notes*, pp. 23–28 (American Math. Soc., Providence, RI, 1994).

Rubin, K., “Elliptic curves with complex multiplication and the conjecture of Birch and Swinnerton-Dyer”, in “Arithmetic theory of elliptic curves (Cetraro, 1997)”, vol. 1716 of *Lecture Notes in Math.*, pp. 167–234 (Springer, 1999).

Satoh, J., “Iwasawa λ-invariants of Γ-transforms”, J. Num. Theory **41**, 98–101 (1992).

Schikhof, W. H., *Ultrametric Calculus: An Introduction to p-adic Analysis*, vol. 4 (Cambridge University Press, 2007).

Schneider, P., *Nonarchimedean Functional Analysis*, Springer Monographs in Mathematics (Springer, 2002).

Schneps, L., “On the μ-invariant of p-adic L-functions attached to elliptic curves with complex multiplication”, J. Num. Theory **25**, 20–33 (1987).

Serre, J.-P., “Classes des corps cyclotomiques”, Séminaire Bourbaki **5**, 83–93 (1958-1960).

Serre, J.-P., “Endomorphismes complètement continus des espaces de Banach p-adiques”, Publications Mathématiques de l’IHÉS **12**, 69–85 (1962).

Shimura, G., *Introduction to the Arithmetic Theory of Automorphic Functions*, vol. 11 of *Publications of the Mathematical Society of Japan* (Iwanami Shoten and Princeton University Press, 1971).

Silverman, J. H., *The Arithmetic of Elliptic Curves*, vol. 106 of *Graduate Texts in Mathematics* (Springer-Verlag, 1986).

Silverman, J. H., *Advanced Topics in the Arithmetic of Elliptic Curves*, vol. 151 of *Graduate Texts in Mathematics* (Springer-Verlag, 1994).

Sinnott, W., “On the μ-invariant of the Γ-transform of a rational function”, Invent. Math. **75**, 273–282 (1984).

Sinnott, W., “Γ-transforms of rational function measures on $\mathbb{Z}_S$”, Invent. Math. **89**, 139–157 (1987a).

Sinnott, W., “On the power series attached to p-adic L-functions”, J. reine agnew. Math. **382**, 22–34 (1987b).

Stanley, R. P., *Enumerative Combinatorics, Volume I*, vol. 49 of *Cambridge Studies in Advanced Mathematics* (Cambridge University Press, 2012), second edn.

Szpilrajn, E., “Sur l’extension de l’ordre partiel”, Fund. Math. **16**, 1, 386–389 (1930).

van Rooij, A., *Non-Archimedean Functional Analysis*, vol. 51 of *Monographs and textbooks in pure and applied mathematics* (Marcel Dekker, 1978).

Washington, L. C., *Introduction to Cyclotomic Fields*, vol. 83 of *Graduate Texts in Mathematics* (Springer-Verlag, 1997).

Wintenberger, J.-P., "Structure Galoisienne de limites projectives d'unites locales", Compositio Math. **42**, 89–103 (1981).

Yager, R. I., "On two variable p-adic L-functions", Ann. Math. **115**, 411–449 (1982).

Yager, R. I., "p-adic measures on Galois groups", Invent. Math. **76**, 331–343 (1984).

APPENDIX A

TOPOLOGICAL PREREQUISITES

We collect several important topological properties of $\mathbb{Z}_p$ without proof. Along the way, we also record relevant algebraic properties of $\mathbb{Z}_p$ (for proofs of these results, consult any of Gouvêa (1997), Koblitz (1984), Robert (2000), or Schikhof (2007)).

The ring $\mathbb{Z}_p$ of p-adic integers is the completion of the ring of rational integers $\mathbb{Z}$ with respect to the p-adic topology. To define the p-adic topology on $\mathbb{Z}$, we begin with the p-adic valuation ord_p and its associated absolute value $|\cdot|_p$ (we will often suppress reference to the fixed prime p in the notation of both ord_p and $|\cdot|_p$ when there is no fear of confusion). By unique factorization in $\mathbb{Z}$, any element $x \in \mathbb{Z} \setminus \{0\}$ may be written as $x = p^{\mathrm{ord}(x)}y$ with $\mathrm{ord}(x) \in \mathbb{N}$ and $p \nmid y$. We set $\mathrm{ord}(0) = \infty$ and adopt the usual conventions for the symbol ∞: $\infty > x$ for all $x \in \mathbb{R}$ and $\infty + x = \infty$ for all $x \in \mathbb{R}$. Then ord is a discrete non-archimedean valuation on $\mathbb{Z}$, i.e. a function $\mathbb{Z} \to \mathbb{R}_+ \cup \{\infty\}$ satisfying

1. $\mathrm{ord}(x) = \infty$ if and only if $x = 0$.
2. $\mathrm{ord}(xy) = \mathrm{ord}(x) + \mathrm{ord}(y)$ for any $x, y \in \mathbb{Z}$.
3. $\mathrm{ord}(x + y) \geq \min\{\mathrm{ord}(x), \mathrm{ord}(y)\}$ for any $x, y \in \mathbb{Z}$. Equality holds if and only if $\mathrm{ord}(x) \neq \mathrm{ord}(y)$.

For $x \in \mathbb{Z}$, the (normalized) p-adic absolute value of x is $|x|_p = p^{-\mathrm{ord}(x)}$, where we adopt the convention that $p^{-\infty} = 0$. Then $|\cdot|_p$ is a discrete non-archimedean absolute value on $\mathbb{Z}$, i.e. a function $\mathbb{Z} \to [0, \infty)$ satisfying

1. $|x|_p = 0$ if and only if $x = 0$.
2. $|xy|_p = |x|_p|y|_p$ for any $x, y \in \mathbb{Z}$.
3. $|x + y|_p \leq \max\{|x|_p, |y|_p\}$ for any $x, y \in \mathbb{Z}$. Equality holds if and only if $|x|_p \neq |y|_p$.

We define the p-adic metric d on $\mathbb{Z}$ by $\mathsf{d}(x,y) = |x-y|_p$ for any $x, y \in \mathbb{Z}$; the above properties of $|\cdot|_p$ guarantee that d is a metric (in fact an ultrametric). The metric topology induced by d is the p-adic topology on $\mathbb{Z}$. Note that $\mathsf{d}(x,y) \leq p^{-n}$ if and only if $x \equiv y \pmod{p^n}$, so that the p-adic metric encodes congruences modulo powers of p via distance. When endowed with the p-adic topology, $\mathbb{Z}$ becomes a topological ring; that is to say, multiplication and addition are continuous maps $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, where $\mathbb{Z} \times \mathbb{Z}$ is given the product topology, and the inversion map $\mathbb{Z} \to \mathbb{Z}$ given by $x \mapsto -x$ is continuous.

The ring of p-adic integers, denoted $\mathbb{Z}_p$, is the completion of $\mathbb{Z}$ with respect to the p-adic topology. The p-adic valuation ord and p-adic absolute value $|\cdot|_p$ extend uniquely to $\mathbb{Z}_p$ and also give the metric topology on $\mathbb{Z}_p$ by $\mathsf{d}(x,y) = |x-y|_p$. Then $\mathbb{Z}_p$ is a topological ring which is complete with respect to the p-adic topology. $\mathbb{Z}$ is naturally identified with a dense (proper) subring of $\mathbb{Z}_p$, and we view $\mathbb{Z}$ as a subring of $\mathbb{Z}_p$ in this way. In fact, $\mathbb{N}$ is a dense subset of $\mathbb{Z}_p$ and any element $x \in \mathbb{Z}_p$ may be written uniquely as a power series in p:

$$x = \sum_{n \geq 0} a_n p^n$$

where the $a_n \in \{0, 1, 2, \ldots, p-1\}$ are called the p-adic digits of x. The above representation is called the p-adic expansion of $x \in \mathbb{Z}_p$, and if x_n denotes the nth partial sum of the p-adic expansion of x, then the sequence $(x_n)_n$ converges to x in $\mathbb{Z}_p$. Using this representation, an element $x \in \mathbb{Z}_p$ is invertible if and only if $x_0 \neq 0$, i.e. if and only if $x \notin p\mathbb{Z}_p$. $\mathbb{Z}_p$ is a discrete valuation ring with maximal ideal $p\mathbb{Z}_p$. Any element $x \in \mathbb{Z}_p$ may be written uniquely as $x = p^{\mathrm{ord}(x)} y$ with $y \in \mathbb{Z}_p^\times$.

In $\mathbb{Z}_p$, the collection of sets $\{p^n\mathbb{Z}_p : n \geq 0\}$ gives a neighborhood base at 0. Thus, sets of the form $x + p^n\mathbb{Z}_p$ with $x \in \mathbb{Z}_p$ and $n \in \mathbb{N}$ form a basis for the topology on $\mathbb{Z}_p$. Note that any translate $x + p^n\mathbb{Z}_p$ is none other than the "open ball" of radius

$p^{-(n+1)}$ centered at x:

$$B_{p^{-(n+1)}}(x) = \{y \in \mathbb{Z}_p : |x - y| < p^{-(n+1)}\}.$$

Note, however, that $x + p^n\mathbb{Z}_p$ is also the "closed ball" of radius p^{-n} centered at x:

$$\bar{B}_{p^{-n}}(x) = \{y \in \mathbb{Z}_p : |x - y| \leq p^{-n}\}.$$

For this reason, we refer to sets of the form $x + p^n\mathbb{Z}_p$ with $x \in \mathbb{Z}_p$ and $n \in \mathbb{N}$ simply as balls. The union of two non-disjoint balls in $\mathbb{Z}_p$ is equal to one of the two balls. Any point of a ball may be used as its center; that is to say, for any $y \in x + p^n\mathbb{Z}_p$,

$$x + p^n\mathbb{Z}_p = y + p^n\mathbb{Z}_p.$$

For $x, y \in \mathbb{Z}_p$, we write $x \equiv y \pmod{p^n\mathbb{Z}_p}$ if $x - y = p^n z$ for some $z \in \mathbb{Z}_p$. Then $x \equiv y \pmod{p^n}$ if and only if $x - y \in p^n\mathbb{Z}_p$, and this occurs if and only if $|x - y|_p \leq p^{-n}$.

For $n \in \mathbb{N}$, the nth level of $\mathbb{Z}_p$ is $L_n = \{s + p^n\mathbb{Z}_p : s \in \mathbb{Z}_p\}$, the collection of all balls of radius p^{-n}. Of course, $|L_n| = p^n$, and the $a + p^n\mathbb{Z}_p$ for $0 \leq a < p^n$ are all the distinct elements of L_n. For fixed n, we obtain a partition of $\mathbb{Z}_p$ by balls in L_n:

$$\mathbb{Z}_p = \bigsqcup_{a=0}^{p^n-1} a + p^n\mathbb{Z}_p = \bigcup_{A \in L_n} A.$$

It follows that balls in $\mathbb{Z}_p$ are both open and closed ("clopen"). Even more is true, any ball $x + p^n\mathbb{Z}_p$ can be partitioned into finitely many balls:

$$x + p^n\mathbb{Z}_p = \bigsqcup_{a=0}^{p^{m-n}-1} (x + ap^n) + p^m\mathbb{Z}_p$$

for any $m > n$. This fact implies that $\mathbb{Z}_p$ is totally bounded, hence compact. Furthermore, $\mathbb{Z}_p$ is totally disconnected, in the sense that its only non-trivial connected subsets are singletons $\{x\}$.

For $x \in \mathbb{Z}_p$, the translation map $\tau_x : \mathbb{Z}_p \to \mathbb{Z}_p$ given by $\tau_x : y \mapsto x + y$ is an isometric isomorphism. Since $|a|_p \leq 1$ for all $a \in \mathbb{Z}_p$, the multiplication by a map $m_a : \mathbb{Z}_p \to \mathbb{Z}_p$ given by $m_a(x) = ax$ is a contraction mapping. For $a \neq 0$, the map m_a is always an isomorphism $\mathbb{Z}_p \to a\mathbb{Z}_p$.

Balls in $\mathbb{Z}_p$ are compact, being the continuous (and, in fact, isomorphic) image of the compact set $\mathbb{Z}_p$. A set which is both compact and open will be called "compact open". Any compact open subset of $\mathbb{Z}_p$ can in fact be written as a finite disjoint union of balls from L_n for a suitable choice of n. From this, it follows immediately that the compact open subsets of $\mathbb{Z}_p$ form an algebra of sets. That is to say, if CO_1 is the collection of all compact open subsets of $\mathbb{Z}_p$ then

1. $\mathbb{Z}_p \in CO_1$.
2. If $A \in CO_1$, then $A^c := \mathbb{Z}_p \setminus A \in CO_1$.
3. If $A, B \in CO_1$, then $A \cup B \in CO_1$ and $A \cap B \in CO_1$.

Compact open subsets of $\mathbb{Z}_p$ will play a fundamental role in the p-adic measure theory developed in Chapter 2.

$\mathbb{Z}_p$ is an integral domain of characteristic 0, and we let $\mathbb{Q}_p$ denote its field of fractions, called the field of p-adic numbers. We extend the p-adic valuation ord and the p-adic absolute value $|\cdot|_p$ to $\mathbb{Q}_p$ by $\mathrm{ord}_p(x/y) = \mathrm{ord}_p(x) - \mathrm{ord}_p(y)$ and $|x/y|_p = |x|_p/|y|_p$. We likewise extend the p-adic metric d to $\mathbb{Q}_p$ via the p-adic absolute value. Then $\mathbb{Q}_p$ is a complete topological field with respect to the p-adic absolute value (the operation of taking multiplicative inverses is a continuous map $\mathbb{Q}_p^\times \to \mathbb{Q}_p^\times$, where $\mathbb{Q}_p^\times$ is given the subspace topology, and addition, multiplication, and the process of taking additive inverses are likewise continuous). The subspace topology on $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ is the p-adic topology on $\mathbb{Z}_p$ as defined above.

In fact, the p-adic valuation and absolute value can be extended uniquely to any algebraic field extension of $\mathbb{Q}_p$. In particular, the p-adic valuation and absolute value may be extended uniquely to an algebraic closure of $\mathbb{Q}_p$, and then extended uniquely to the topological completion of this field.

We will also need to study the group of units $\mathbb{Z}_p^\times$ of the topological ring $\mathbb{Z}_p$, endowed with the subspace topology. Since $\mathbb{Z}_p$ is a local ring and $p\mathbb{Z}_p$ its maximal ideal,

$$\mathbb{Z}_p^\times = \mathbb{Z}_p \setminus p\mathbb{Z}_p.$$

Thus, for a ball $x + p^n\mathbb{Z}_p$ with $n \geq 1$, we have

$$(x + p^n\mathbb{Z}_p) \cap \mathbb{Z}_p^\times = \begin{cases} x + p^n\mathbb{Z}_p & : x \notin p\mathbb{Z}_p \\ \emptyset & : x \in p\mathbb{Z}_p \end{cases}$$

Consequently, balls of the form $x + p^n\mathbb{Z}_p$ with $x \notin p\mathbb{Z}_p$ form a basis for the subspace topology on $\mathbb{Z}_p^\times$. Since $\mathbb{Z}_p^\times \subseteq \mathbb{Q}_p^\times$ and inversion in $\mathbb{Q}_p^\times$ maps $\mathbb{Z}_p^\times$ into $\mathbb{Z}_p^\times$, $\mathbb{Z}_p^\times$ is in fact a topological group under multiplication.

We have that $\mathbb{Z}_p^\times \cong V \times U$, where V is the set of $(p-1)$th roots of unity in $\mathbb{Z}_p$ and $U = 1 + p\mathbb{Z}_p$. We let ω and $\langle \cdot \rangle$ denote the projections onto the first and second components of this decomposition, respectively. Then each $x \in \mathbb{Z}_p^\times$ may be uniquely written in the form $\omega(x)\langle x \rangle$ with $\omega(x) \in V$ and $\langle x \rangle \in U$.

The multiplicative subgroup $U = 1 + p\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$ will play a particularly important role in Chapter 2. Note that U is itself a topological group under multiplication when endowed with the subspace topology. In fact, U is topologically cyclic, and if $u \in U$ is a fixed topological generator, then the map $\varphi_u : \mathbb{Z}_p \to U$ given by $\varphi_u : x \mapsto u^x$ is a topological group isomorphism. The inverse of this map is $\varphi_u^{-1} : U \to \mathbb{Z}_p$, given by $\varphi_u^{-1}(y) = \frac{\log y}{\log u}$, where log is the p-adic logarithm. We note here that if $x \equiv y \pmod{p^n}$, then $u^x \equiv u^y \pmod{p^{n+1}}$. This follows because

$$u^{p^n} - 1 = \sum_{k=1}^{p^n} \binom{p^n}{k} (u-1)^k.$$

In particular, this shows that φ_u maps L_n into L_{n+1} for each $n \in \mathbb{N}$. We may also view φ_u^{-1} as a map on all of $\mathbb{Z}_p^\times$ by setting $\varphi_u^{-1}(y) = \varphi_u^{-1}(\langle y \rangle)$. Then $\varphi_u^{-1}(x) \equiv \varphi_u^{-1}(y) \pmod{p^n\mathbb{Z}_p}$ if and only if $\langle x \rangle \equiv \langle y \rangle \pmod{p^{n+1}\mathbb{Z}_p}$.

Much of our study will concern $\mathbb{Z}_p^d$, the product ring of d copies of $\mathbb{Z}_p$. We endow $\mathbb{Z}_p^d$ with the product topology by taking the p-adic topology on each copy of $\mathbb{Z}_p$ in the product. The collection of all sets of the form

$$\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d = \prod_{i=1}^{d} a_i + p^{n_i}\mathbb{Z}_p$$

with $\boldsymbol{n} \in \mathbb{N}^d$ and $\boldsymbol{a} \in \mathbb{Z}_p^d$ is thus a basis for the topology on $\mathbb{Z}_p^d$. We will call sets of the above form "polyballs" in $\mathbb{Z}_p^d$. Since $\mathbb{Z}_p$ is a metric space, the topology on $\mathbb{Z}_p^d$ is also metric, and is equivalent to the "max metric" defined by

$$\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|_d := \max\{|x_i - y_i|_p : 1 \le i \le d\}.$$

We will view $\mathbb{Z}_p^d$ as endowed with this particular metric, making $\mathbb{Z}_p^d$ is a topological ring.

Unlike the case with balls in $\mathbb{Z}_p$, two polyballs in $\mathbb{Z}_p^d$ may have non-empty intersection which is properly contained in each polyball in the intersection. However, note that if $\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d \cap \boldsymbol{b} + \boldsymbol{p}^{\wedge \boldsymbol{m}}\mathbb{Z}_p^d \neq \emptyset$, then $a_i + p^{n_i}\mathbb{Z}_p \cap b_i + p^{m_i}\mathbb{Z}_p \neq \emptyset$ for each i. In the lattice $\mathbb{N}^d$, let $\boldsymbol{s} = \sup(\boldsymbol{n}, \boldsymbol{m})$ and $\boldsymbol{t} = \inf(\boldsymbol{n}, \boldsymbol{m})$, so that $s_i = \max\{n_i, m_i\}$ and $t_i = \min\{n_i, m_i\}$ for all $1 \le i \le d$. Since $a_i + p^{n_i}\mathbb{Z}_p \cap b_i + p^{m_i}\mathbb{Z}_p \neq \emptyset$, we must have

$$a_i \equiv b_i \pmod{p^{t_i}}$$

for each $1 \leq i \leq d$, and we let $0 \leq c_i < p^{t_i}$ denote this common residue. Also, at least one of the following must hold:

$$a_i + p^{n_i}\mathbb{Z}_p = a_i + p^{s_i}\mathbb{Z}_p$$

$$b_i + p^{m_i}\mathbb{Z}_p = b_i + p^{s_i}\mathbb{Z}_p.$$

We let $0 \leq d_i < p^{t_i}$ be the unique integer with

$$a_i \equiv d_i \pmod{p^{s_i}}$$

in the first case and

$$b_i \equiv d_i \pmod{p^{s_i}}$$

in the second case (if both cases hold, then $a_i \equiv b_i \pmod{p^{s_i}}$). Then

$$\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d \cap \boldsymbol{b} + \boldsymbol{p}^{\wedge \boldsymbol{m}}\mathbb{Z}_p^d = \boldsymbol{d} + \boldsymbol{p}^{\wedge \boldsymbol{s}}\mathbb{Z}_p^d$$

$$\boldsymbol{a} + \boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d \cup \boldsymbol{b} + \boldsymbol{p}^{\wedge \boldsymbol{m}}\mathbb{Z}_p^d = \boldsymbol{c} + \boldsymbol{p}^{\wedge \boldsymbol{t}}\mathbb{Z}_p^d.$$

By Tychonoff's theorem, polyballs in $\mathbb{Z}_p^d$ are compact sets in the product topology, hence are compact open sets in $\mathbb{Z}_p^d$. If O is any compact open subset of $\mathbb{Z}_p^d$, then it can be written as a finite union of polyballs. In light of the above remarks, we can then write O as a finite disjoint union of polyballs in $\mathbb{Z}_p^d$. As with balls in $\mathbb{Z}_p$, the collection of all compact open subsets of $\mathbb{Z}_p^d$, denoted CO_d, forms an algebra of sets.

The multiplicative group $(\mathbb{Z}_p^\times)^d$ will play an important role in Chapter 2. For $\boldsymbol{a} \in (\mathbb{Z}_p^\times)^d$, we write $\boldsymbol{a}^{-1}$ for the inverse of $\boldsymbol{a}$ in $(\mathbb{Z}_p^\times)^d$. Of course, $\boldsymbol{a}^{-1}$ is the element

$$\boldsymbol{a}^{-1} = (a_1^{-1}, \ldots, a_d^{-1}) = \boldsymbol{a}^{\wedge(-\boldsymbol{1})}.$$

As in Chapter 2, let $(R, \|\cdot\|_R)$ be a normed $\mathbb{Z}_p$-algebra such that the structure map $\mathbb{Z}_p \to R$ is injective, $\|\cdot\|_R$ is non-archimedean, and R is complete with respect to the topology induced by $\|\cdot\|_R$.

In what follows, it will be helpful to use the language of nets (see Folland (1999), Chapter 4 or Schneider (2002), Chapter 1.7).

Definition 32. *A directed set is a pair $(A, \precsim)$ where A is a set and $\precsim$ is a binary relation on A such that*

- *$a \precsim a$ for all $a \in A$.*
- *if $a \precsim b$ and $b \precsim c$, then $a \precsim c$.*
- *For any $a, b \in A$, there is $c \in A$ such that $a \precsim c$ and $b \precsim c$.*

Definition 33. *A net in a set X is a mapping $a \mapsto x_a$ from a directed set A into X. We denote such a mapping by $(x_a)_{a \in A}$, or by (x_a) when A is understood, and we say that (x_a) is indexed by A.*

The directed set we will use most often is $\mathbb{N}^d$ with the product order $\preccurlyeq_d$; we make a definition for convenience of reference.

Definition 34. *A d-net in a set X is a net in X indexed by the directed set $(\mathbb{N}^d, \preccurlyeq_d)$.*

Definition 35. *A d-net $(x_{\boldsymbol{n}})$ in a metric space (X, d) converges to $x \in X$ if for all $\varepsilon > 0$ there is $\boldsymbol{k} \in \mathbb{N}^d$ such that $\mathsf{d}(x_{\boldsymbol{n}}, x) < \varepsilon$ for all $\boldsymbol{k} \preccurlyeq_d \boldsymbol{n}$.*

Definition 36. *A d-net $(x_{\boldsymbol{n}})$ in a metric space (X, d) is Cauchy if for all $\varepsilon > 0$ there is $\boldsymbol{k} \in \mathbb{N}^d$ such that $\mathsf{d}(x_{\boldsymbol{n}}, x_{\boldsymbol{m}}) < \varepsilon$ for all $\boldsymbol{k} \preccurlyeq_d \boldsymbol{n}, \boldsymbol{m}$.*

A metric space is complete if and only if every Cauchy net converges. Our applications will require a study of d-nets in R.

Recall from Definition 11 that a d-net $(x_{\boldsymbol{n}})$ in R is bounded if

$$\sup_{\boldsymbol{n}\in\mathbb{N}^d}\{\|x_{\boldsymbol{n}}\|_R\} < \infty.$$

For $\boldsymbol{n} \in \mathbb{N}^d$ and $\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge\boldsymbol{n}} - \boldsymbol{1}$, we let $g_{\boldsymbol{a},\boldsymbol{n}} : \mathbb{Z}_p^d \to R$ denote the characteristic function of the polyball $\boldsymbol{a} + \boldsymbol{p}^{\wedge\boldsymbol{n}}\mathbb{Z}_p^d$. Since $\boldsymbol{a} + \boldsymbol{p}^{\wedge\boldsymbol{n}}\mathbb{Z}_p^d$ is both open and closed in $\mathbb{Z}_p^d$, $g_{\boldsymbol{a},\boldsymbol{n}} : \mathbb{Z}_p^d \to R$ is continuous.

Definition 37. *A function $f : \mathbb{Z}_p^d \to R$ is called locally constant if there is some $\boldsymbol{n} \in \mathbb{N}^d$ and elements $r_{\boldsymbol{a}} \in R$ for $\boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge\boldsymbol{n}} - \boldsymbol{1}$ such that*

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{a}\preccurlyeq_d\boldsymbol{p}^{\wedge\boldsymbol{n}}-\boldsymbol{1}} r_{\boldsymbol{a}} g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}).$$

In this case, we say f factors through the level $L_{\boldsymbol{n}}$.

Proposition 16. *Let $f : \mathbb{Z}_p^d \to R$ be a continuous function. Then f is the uniform limit of a d-net of locally constant functions $\mathbb{Z}_p^d \to R$.*

Proof. Note first that f is actually uniformly continuous since $\mathbb{Z}_p^d$ is compact. For each $\boldsymbol{n} \in \mathbb{N}^d$, put

$$f_{\boldsymbol{n}}(\boldsymbol{x}) = \sum_{\boldsymbol{a}\preccurlyeq_d\boldsymbol{p}^{\wedge\boldsymbol{n}}-\boldsymbol{1}} f(\boldsymbol{a}) g_{\boldsymbol{a},\boldsymbol{n}}(\boldsymbol{x}).$$

Then $(f_{\boldsymbol{n}})$ is a d-net of locally constant functions. Let $\varepsilon > 0$ and fix $\delta > 0$ such that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}_p^d$ and $\|\boldsymbol{x} - \boldsymbol{y}\|_d < \delta$ imply

$$\|f(\boldsymbol{x}) - f(\boldsymbol{y})\|_R < \varepsilon.$$

Choose $N \in \mathbb{N}$ such that $p^{-N} < \delta$ and set $\boldsymbol{N} = (N, N, \ldots, N) \in \mathbb{N}^d$. Now let $\boldsymbol{n} \in \mathbb{N}^d$ with $\boldsymbol{N} \preccurlyeq_d \boldsymbol{n}$ and let $\boldsymbol{x} \in \mathbb{Z}_p^d$. We have $f_{\boldsymbol{n}}(\boldsymbol{x}) = f(\boldsymbol{a}_{\boldsymbol{x}})$ for the unique $\boldsymbol{a}_{\boldsymbol{x}} \in \mathbb{N}^d$

satisfying $\boldsymbol{a_x} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{n}} - \mathbf{1}$ and $\boldsymbol{x} \equiv \boldsymbol{a_x} \pmod{\boldsymbol{p}^{\wedge \boldsymbol{n}}\mathbb{Z}_p^d}$. But now

$$\|\boldsymbol{x} - \boldsymbol{a_x}\|_d \leq p^{-N} < \delta,$$

from which we obtain

$$\|f_{\boldsymbol{n}}(\boldsymbol{x}) - f(\boldsymbol{x})\|_R = \|f(\boldsymbol{a_x}) - f(\boldsymbol{x})\|_R < \varepsilon.$$

Thus $f_{\boldsymbol{n}} \to f$ uniformly. ☐

For $x \in \mathbb{Z}_p$ and $n \in \mathbb{N}$, we define

$$\binom{x}{n} = \begin{cases} \frac{x(x-1)\cdots(x-n+1)}{n!} & : n > 0 \\ 1 & : n = 0 \end{cases}.$$

For fixed $n \in \mathbb{N}$, the map $x \mapsto \binom{x}{n}$ is a continuous function $\mathbb{Z}_p \to \mathbb{Q}_p$ (it is a polynomial). For $x \in \mathbb{Z}_p$, write $x = \lim_{i\to\infty} n_i$, where each $n_i \in \mathbb{N}$. Then $\binom{n_i}{n} \in \mathbb{Z}$ for all i, and by continuity, $\binom{n_i}{n} \to \binom{x}{n}$. Therefore, we actually have $\binom{x}{n} \in \mathbb{Z}_p$. For $\boldsymbol{n} \in \mathbb{N}^d$ and $\boldsymbol{x} \in \mathbb{Z}_p^d$, we define

$$\binom{\boldsymbol{x}}{\boldsymbol{n}} = \prod_{i=1}^{d} \binom{x_i}{n_i}.$$

Then for $\boldsymbol{n} \in \mathbb{N}^d$, the function $\boldsymbol{x} \mapsto \binom{\boldsymbol{x}}{\boldsymbol{n}}$ is continuous $\mathbb{Z}_p^d \to \mathbb{Z}_p$.

Lemma 46. *Let $(a_{\boldsymbol{n}})$ be a d-net in R converging to 0. Then the function*

$$g(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{N}^d} a_{\boldsymbol{n}} \binom{\boldsymbol{x}}{\boldsymbol{n}}$$

is continuous $\mathbb{Z}_p^d \to R$.

Proof. First, for each $\boldsymbol{x} \in \mathbb{Z}_p^d$, $\binom{\boldsymbol{x}}{\boldsymbol{n}} \in \mathbb{Z}_p$, so that

$$\left\| a_{\boldsymbol{n}} \binom{\boldsymbol{x}}{\boldsymbol{n}} \right\|_R \leq \|a_{\boldsymbol{n}}\|_R \to 0.$$

Thus, the series

$$\sum_{\boldsymbol{n}\in\mathbb{N}^d} a_{\boldsymbol{n}}\binom{\boldsymbol{x}}{\boldsymbol{n}}$$

converges in R for each $\boldsymbol{x}\in\mathbb{Z}_p^d$, and g is a well-defined function $\mathbb{Z}_p^d\to R$.

For each $\boldsymbol{n}\in\mathbb{N}^d$, set

$$g_{\boldsymbol{n}}(\boldsymbol{x})=\sum_{\boldsymbol{k}\preccurlyeq_d\boldsymbol{n}} a_{\boldsymbol{n}}\binom{\boldsymbol{x}}{\boldsymbol{k}}.$$

Then $(g_{\boldsymbol{n}})$ is a d-net in $C(\mathbb{Z}_p^d,R)$. For each $\boldsymbol{n}\in\mathbb{N}^d$, we have that

$$\begin{aligned}\|g(\boldsymbol{x})-g_{\boldsymbol{n}}(\boldsymbol{x})\|_R&=\left\|\sum_{\boldsymbol{k}\not\preccurlyeq_d\boldsymbol{n}} a_{\boldsymbol{k}}\binom{\boldsymbol{x}}{\boldsymbol{k}}\right\|_R\\&\le\sup_{\boldsymbol{k}\not\preccurlyeq_d\boldsymbol{n}}\left\{\left\|a_{\boldsymbol{k}}\binom{\boldsymbol{x}}{\boldsymbol{k}}\right\|_R\right\}\\&\le\sup_{\boldsymbol{k}\not\preccurlyeq_d\boldsymbol{n}}\left\{\|a_{\boldsymbol{k}}\|_R\right\}.\end{aligned}$$

But the d-net

$$\left(\sup_{\boldsymbol{k}\not\preccurlyeq_d\boldsymbol{n}}\{\|a_{\boldsymbol{k}}\|_R\}\right)_{\boldsymbol{n}}$$

converges to 0 in $\mathbb{R}$ since $(a_{\boldsymbol{n}})$ converges to 0 in R. Therefore, $g_{\boldsymbol{n}}\to g$ uniformly, and it follows that g is continuous $\mathbb{Z}_p^d\to R$. □

Let $f\in C(\mathbb{Z}_p,R)$. For each $n\in\mathbb{N}$, define

$$\Delta^n f(x)=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}f(x+k)\in C(\mathbb{Z}_p,R).$$

Observe that $\Delta^0 f(x)=f(x)$.

Lemma 47. *For each $n\in\mathbb{N}$, $\Delta^n: C(\mathbb{Z}_p,R)\to C(\mathbb{Z}_p,R)$ is R-linear. Moreover, $\Delta^k(\Delta^n f(x))=\Delta^{n+k}f(x)$ for all $k\in\mathbb{N}$ and all $f\in C(\mathbb{Z}_p,R)$.*

Proof. Let $n \in \mathbb{N}$. Let $f, g \in C(\mathbb{Z}_p, R)$ and $r \in R$. Then

$$\begin{aligned}
\Delta^n(f+g)(x) &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (f+g)(x+k) \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (f(x+k) + g(x+k)) \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k) + \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} g(x+k) \\
&= \Delta^n f(x) + \Delta^n g(x)
\end{aligned}$$

and

$$\begin{aligned}
\Delta^n(rf)(x) &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (rf)(x+k) \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} r f(x+k) \\
&= r \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k) \\
&= r \Delta^n f(x).
\end{aligned}$$

It will suffice to establish the second claim for $k = 1$, and arbitrary $n \in \mathbb{N}$, as the general result follows from this case and a standard induction argument. Note that there is nothing to show if $n = 0$, so we assume $n > 0$. Using the R-linearity of Δ, we have

$$\begin{aligned}
\Delta(\Delta^n f(x)) &= \Delta \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k) \right) \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \Delta f(x+k) \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (-f(x+k) + f(x+k+1))
\end{aligned}$$

Recall the Pascal relation (see Appendix C)

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1},$$

valid for all $1 \leq k$. This gives

$$\begin{aligned}
\Delta^{n+1} f(x) &= \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{n+1-k} f(x+k) \\
&= \binom{n+1}{0} (-1)^{n+1} f(x) + \sum_{k=1}^{n+1} \left(\binom{n}{k} + \binom{n}{k-1} \right) (-1)^{n+1-k} f(x+k) \\
&= -\binom{n}{0} (-1)^{n} f(x) + \sum_{k=1}^{n+1} \binom{n}{k} (-1)^{n+1-k} f(x+k) \\
&\qquad\qquad + \sum_{k=1}^{n+1} \binom{n}{k-1} (-1)^{n-(k-1)} f(x+k) \\
&= -\binom{n}{0} (-1)^{n} f(x) - \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} f(x+k) \\
&\qquad\qquad + \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k+1) \\
&= -\sum_{i=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k) + \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k+1) \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (-f(x+k) + f(x+k+1)).
\end{aligned}$$

Thus, $\Delta(\Delta^n f(x)) = \Delta^{n+1} f(x)$. □

The following lemma will be used in the proof of Mahler's theorem (See Bojanic (1974)).

Lemma 48. *For each $n, m \in \mathbb{N}$,*

$$\Delta^n f(x) = \sum_{j=0}^{m} \binom{m}{j} \Delta^{n+j} f(x-m).$$

Proof. Fix $n, m \in \mathbb{N}$. We have

$$\begin{aligned}
\sum_{j=0}^{m} \binom{m}{j} \Delta^j f(x-m) &= \sum_{j=0}^{m} \binom{m}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} f(x-m+i) \\
&= \sum_{j=0}^{m} \sum_{i=0}^{m} \binom{m}{j} \binom{j}{i} (-1)^{j-i} f(x-m+i) \\
&= \sum_{i=0}^{m} \sum_{j=0}^{m} \binom{m}{j} \binom{j}{i} (-1)^{j-i} f(x-m+i) \\
&= \sum_{i=0}^{m} (-1)^{-i} f(x-m+i) \sum_{j=0}^{m} \binom{m}{j} \binom{j}{i} (-1)^{j}.
\end{aligned}$$

But (see Appendix C)

$$\sum_{j=0}^{m} \binom{m}{j} \binom{j}{i} (-1)^{j} = \begin{cases} (-1)^m & : i = m \\ 0 & : i < m \end{cases}.$$

Consequently,

$$\begin{aligned}
\sum_{j=0}^{m} \binom{m}{j} \Delta^j f(x-m) &= \sum_{i=0}^{m} (-1)^{-i} f(x-m+i) \sum_{j=0}^{m} \binom{m}{j} \binom{j}{i} (-1)^{j} \\
&= (-1)^{-m} (-1)^m f(x) \\
&= f(x).
\end{aligned}$$

Therefore, Lemma 47 gives

$$\begin{aligned}
\Delta^n f(x) &= \Delta^n \left(\sum_{j=0}^{m} \binom{m}{j} \Delta^j f(x-m) \right) \\
&= \sum_{j=0}^{m} \binom{m}{j} \Delta^n (\Delta^j f(x-m)) \\
&= \sum_{j=0}^{m} \binom{m}{j} \Delta^{n+j} f(x-m).
\end{aligned}$$

□

Theorem 17 (Mahler's Theorem)**.** *Let $f : \mathbb{Z}_p \to R$ be a continuous function. For $n \in \mathbb{N}$, put*

$$a_n(f) = \Delta^n f(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k) \in R.$$

Then $a_n(f) \to 0$ and f can be written uniquely in the form

$$f(x) = \sum_{n \geq 0} a_n(f) \binom{x}{n}.$$

Proof. For uniqueness, suppose there are two sequences (a_n) and (b_n) which converge to 0 and such that

$$\sum_{n \geq 0} a_n \binom{x}{n} = \sum_{n \geq 0} b_n \binom{x}{n}$$

for all $x \in \mathbb{Z}_p$ (note that the two series thus converge for every $x \in \mathbb{Z}_p$). If (a_n) and (b_n) are distinct sequences, then there is a minimal index k for which $a_k \neq b_k$. For any $x \in \mathbb{Z}_p$, we have

$$\begin{aligned} 0 &= \sum_{n \geq 0} a_n \binom{x}{n} - \sum_{n \geq 0} b_n \binom{x}{n} \\ &= \sum_{n \geq 0} (a_n - b_n) \binom{x}{n} \\ &= \sum_{n \geq k} (a_n - b_n) \binom{x}{n}. \end{aligned}$$

In particular, for $x = k$, we have

$$0 = (a_k - b_k) \binom{k}{k} = a_k - b_k,$$

a contradiction.

To show that $a_n(f) \to 0$, we closely follow the proof given in Bojanic (1974). Since $\mathbb{Z}_p$ is compact, $f : \mathbb{Z}_p \to R$ is uniformly continuous. Fix $s \in \mathbb{N}$. There exists $t \in \mathbb{N}$ such that

$$\|f(x) - f(y)\|_R < \|p\|_R^s$$

whenever $|x-y|_p \le p^{-t}$. In particular, for each $k \in \mathbb{N}$,

$$\|f(k+p^t) - f(k)\|_R < \|p\|_R^s.$$

Since f is continuous, f is bounded. By Lemma 47, scaling f by an element of R scales each $a_{\boldsymbol{n}}(f)$ by that same element. Replacing f with a scalar multiple of f, if necessary, we may assume $\|f\|_\infty \le 1$ since this will not change the convergence to zero of the $a_{\boldsymbol{n}}(f)$. Since the norm on R is non-archimedean, $\|a_n(f)\|_R \le 1$ for all $n \in \mathbb{N}$. Notice now that for each $k, n \in \mathbb{N}$,

$$\begin{aligned}\|\Delta^n f(p^t) - \Delta^n f(0)\|_R &= \left\| \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (f(p^t+k) - f(k)) \right\|_R \\ &\le \max_{0 \le k \le n} \left\{ \left\| \binom{n}{k} (-1)^{n-k} \right\|_R \|f(p^t+k) - f(k)\|_R \right\} \\ &< \|p\|_R^s.\end{aligned}$$

From Lemma 48, for each $n \in \mathbb{N}$,

$$\Delta^n f(p^t) = \sum_{j=0}^{p^t} \binom{p^t}{j} \Delta^{n+j} f(0).$$

This gives

$$\begin{aligned}a_{n+p^t}(f) = \Delta^{n+p^t} f(0) &= \Delta^n f(p^t) - \sum_{j=0}^{p^t-1} \binom{p^t}{j} \Delta^{n+j} f(0) \\ &= (\Delta^n f(p^t) - \Delta^n f(0)) - \sum_{j=1}^{p^t-1} \binom{p^t}{j} \Delta^{n+j} f(0) \\ &= (\Delta^n f(p^t) - \Delta^n f(0)) - \sum_{j=1}^{p^t-1} \binom{p^t}{j} a_{n+j}(f)\end{aligned}$$

But for all $1 \le j \le p^t - 1$,

$$\binom{p^t}{j} \equiv 0 \pmod{p},$$

by Lemma 51, so

$$\left\| \binom{p^t}{j} \right\|_R \leq \|p\|_R.$$

Therefore,

$$\begin{aligned}
\|a_{n+p^t}(f)\|_R &\leq \max \left\{ \left\| \binom{p^t}{j} a_{n+j}(f) \right\|_R, \|\Delta^n f(p^t) - \Delta^n f(0)\|_R : 1 \leq j \leq p^t - 1 \right\} \\
&\leq \max \left\{ \left\| \binom{p^t}{j} \right\|_R, \|p\|_R^s : 1 \leq j \leq p^t - 1 \right\} \\
&\leq \|p\|_R.
\end{aligned}$$

Since the above holds for all $n \in \mathbb{N}$, we conclude

$$\|a_n(f)\|_R \leq \|p\|_R$$

for all $n \geq p^t$. Using this, the same argument shows

$$\|a_n(f)\|_R \leq \|p\|_R^2$$

for all $n \geq 2p^t$, and an inductive argument gives

$$\|a_n(f)\|_R \leq \|p\|_R^s$$

for all $n \geq sp^t$. Since $\|p\|_R^s \to 0$, this gives $\|a_n(f)\|_R \to 0$.

From Lemma 46, the function

$$g(x) = \sum_{n \geq 0} a_n(f) \binom{x}{n}$$

is continuous $\mathbb{Z}_p^d \to R$. Note that $\mathbb{Z}_+ = S$ is a multiplicative subset of R. We work formally with power series in $(S^{-1}R)[[X]]$. Put

$$F(X) = \sum_{n \geq 0} f(n) \frac{X^n}{n!} \in (S^{-1}R)[[X]]$$

and

$$\exp(X) = \sum_{n\geq 0} \frac{X^n}{n!} \in (S^{-1}R)[[X]],$$

as usual. We have the following formal identity of power series in $(S^{-1}R)[[X]]$:

$$\begin{aligned}\exp(-X)F(X) &= \left(\sum_{n\geq 0}(-1)^n\frac{X^n}{n!}\right)\left(\sum_{n\geq 0} f(n)\frac{X^n}{n!}\right)\\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}f(k)\right)\frac{X^n}{n!}\\ &= \sum_{n\geq 0} a_n(f)\frac{X^n}{n!}.\end{aligned}$$

Now in $(S^{-1}R)[[X]]$ we have the formal identity

$$\exp(X)\exp(-X) = 1,$$

whence

$$\begin{aligned}F(X) &= \left(\sum_{n\geq 0} a_n(f)\frac{X^n}{n!}\right)\left(\sum_{n\geq 0}\frac{X^n}{n!}\right)\\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}a_k(f)\right)\frac{X^n}{n!}.\end{aligned}$$

Comparing coefficients of the two representations of $F(X)$ above, we find that

$$f(n) = \sum_{k=0}^{n}\binom{n}{k}a_k(f) = g(n)$$

for all $n \in \mathbb{N}$. But now $g(x)$ and $f(x)$ are two continuous functions $\mathbb{Z}_p \to R$ which agree on all of $\mathbb{N}$. By the density of $\mathbb{N}$ in $\mathbb{Z}_p$, we in fact must have

$$f(x) = g(x) = \sum_{k=0}^{\infty} a_k(f)\binom{x}{k},$$

as desired. □

By repeated application of Mahler's Theorem, we obtain the following corollary.

Corollary 32 (Multivariate Mahler Theorem)**.** *Let $f : \mathbb{Z}_p^d \to R$ be a continuous function. For $\boldsymbol{n} \in \mathbb{N}^d$, put*

$$a_{\boldsymbol{n}}(f) = \sum_{\boldsymbol{k} \preccurlyeq_d \boldsymbol{n}} \binom{\boldsymbol{n}}{\boldsymbol{k}} (-\boldsymbol{1})^{\boldsymbol{n}-\boldsymbol{k}} f(\boldsymbol{k}) \in R.$$

Then $a_{\boldsymbol{n}}(f) \to 0$ and f can be written uniquely in the form

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} a_{\boldsymbol{n}}(f) \binom{\boldsymbol{x}}{\boldsymbol{n}}.$$

Proof. We will show by induction that the result holds for each $d \geq 1$. The result holds for $d = 1$ by Mahler's theorem. Suppose the result holds for some $d \geq 1$, and let $f : \mathbb{Z}_p^{d+1} \to R$ be continuous. For each fixed $y \in \mathbb{Z}_p$, define $f_y : \mathbb{Z}_p^d \to R$ by $f_y(\boldsymbol{x}) = f(x_1, \ldots, x_d, y)$. Then f_y is a continuous function $\mathbb{Z}_p^d \to R$. Moreover, the function $g : \mathbb{Z}_p \to C(\mathbb{Z}_p^d, R)$ given by $g(y) = f_y$ is continuous. By Mahler's theorem (for the ring $C(\mathbb{Z}_p^d, R)$), we have that

$$g(y) = \sum_{n=0}^{\infty} a_n(g) \binom{y}{n},$$

where

$$a_n(g) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} g(k) \in C(\mathbb{Z}_p^d, R),$$

and $\|a_n(g)\|_\infty \to 0$.

For a $d+1$-tuple $\boldsymbol{m} = (m_1, \ldots, m_d, m_{d+1})$, we denote by $\overleftarrow{\boldsymbol{m}}$ the d-tuple $\overleftarrow{\boldsymbol{m}} =$

$(m_1, \ldots, m_d)$. Then for $\boldsymbol{n} \in \mathbb{N}^d$

$$\begin{aligned}
a_{\boldsymbol{n}}(f) &= \sum_{\boldsymbol{k} \preccurlyeq_{d+1} \boldsymbol{n}} (-\boldsymbol{1})^{\boldsymbol{n}-\boldsymbol{k}} \binom{\boldsymbol{n}}{\boldsymbol{k}} f(\boldsymbol{k}) \\
&= \sum_{\boldsymbol{k} \preccurlyeq_{d+1} \boldsymbol{n}} (-\boldsymbol{1})^{\boldsymbol{n}-\boldsymbol{k}} \binom{\boldsymbol{n}}{\boldsymbol{k}} f_{k_{d+1}}(\overleftarrow{\boldsymbol{k}}) \\
&= \sum_{\overleftarrow{\boldsymbol{k}} \preccurlyeq_d \overleftarrow{\boldsymbol{n}}} (\overleftarrow{-\boldsymbol{1}})^{\overleftarrow{\boldsymbol{n}}-\overleftarrow{\boldsymbol{k}}} \binom{\overleftarrow{\boldsymbol{n}}}{\overleftarrow{\boldsymbol{k}}} \left[\sum_{k_{d+1}=0}^{n_{d+1}} (-1)^{n_{d+1}-k_{d+1}} \binom{n_{d+1}}{k_{d+1}} g(k_{d+1}) \right] (\overleftarrow{\boldsymbol{k}}) \\
&= \sum_{\overleftarrow{\boldsymbol{k}} \preccurlyeq_d \overleftarrow{\boldsymbol{n}}} (\overleftarrow{-\boldsymbol{1}})^{\overleftarrow{\boldsymbol{n}}-\overleftarrow{\boldsymbol{k}}} \binom{\overleftarrow{\boldsymbol{n}}}{\overleftarrow{\boldsymbol{k}}} [a_{n_{d+1}}(g)](\overleftarrow{\boldsymbol{k}}).
\end{aligned}$$

This gives

$$\|a_{\boldsymbol{n}}(f)\|_R \leq \|a_{n_{d+1}}(g)\|_\infty.$$

This holds for all $\boldsymbol{n} \in \mathbb{N}^d$, so that $a_{\boldsymbol{n}}(f) \to 0$ in R. Consequently, the series

$$\sum_{\boldsymbol{n} \in \mathbb{N}^{d+1}} a_{\boldsymbol{n}}(f) \binom{\boldsymbol{x}}{\boldsymbol{n}}$$

converges in R for each $\boldsymbol{x} \in \mathbb{Z}_p^{d+1}$.

By the induction hypothesis, for each $y \in \mathbb{Z}_p$, the d-net $(a_{\boldsymbol{n}}(f_y))$ converges to 0, and

$$f_y(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} a_{\boldsymbol{n}}(f_y) \binom{\boldsymbol{x}}{\boldsymbol{n}}.$$

Then for $\boldsymbol{x} \in \mathbb{Z}_p^{d+1}$, we have

$$\begin{aligned}
f(\boldsymbol{x}) &= f_{x_{d+1}}(\overleftarrow{\boldsymbol{x}}) \\
&= \sum_{\boldsymbol{n} \in \mathbb{N}^d} a_{\boldsymbol{n}}(f_{x_{d+1}}) \binom{\overleftarrow{\boldsymbol{x}}}{\boldsymbol{n}} \\
&= \sum_{\boldsymbol{n} \in \mathbb{N}^d} \left(\sum_{\boldsymbol{k} \preccurlyeq_d \boldsymbol{n}} \binom{\boldsymbol{n}}{\boldsymbol{k}} (-\mathbf{1})^{\boldsymbol{n}-\boldsymbol{k}} f_{x_{d+1}}(\boldsymbol{k}) \right) \binom{\overleftarrow{\boldsymbol{x}}}{\boldsymbol{n}} \\
&= \sum_{\boldsymbol{n} \in \mathbb{N}^d} \left(\sum_{\boldsymbol{k} \preccurlyeq_d \boldsymbol{n}} \binom{\boldsymbol{n}}{\boldsymbol{k}} (-\mathbf{1})^{\boldsymbol{n}-\boldsymbol{k}} g(x_{d+1})(\boldsymbol{k}) \right) \binom{\overleftarrow{\boldsymbol{x}}}{\boldsymbol{n}} \\
&= \sum_{\boldsymbol{n} \in \mathbb{N}^d} \left(\sum_{\boldsymbol{k} \preccurlyeq_d \boldsymbol{n}} \binom{\boldsymbol{n}}{\boldsymbol{k}} (-\mathbf{1})^{\boldsymbol{n}-\boldsymbol{k}} \left(\sum_{n_{d+1} \geq 0} a_{n_{d+1}}(g)(\boldsymbol{k}) \binom{x_{d+1}}{n_{d+1}} \right) \right) \binom{\overleftarrow{\boldsymbol{x}}}{\boldsymbol{n}} \\
&= \sum_{\boldsymbol{n} \in \mathbb{N}^d} \sum_{n_{d+1} \geq 0} \left(\sum_{\boldsymbol{k} \preccurlyeq_d \boldsymbol{n}} \binom{\boldsymbol{n}}{\boldsymbol{k}} (-\mathbf{1})^{\boldsymbol{n}-\boldsymbol{k}} a_{n_{d+1}}(g)(\boldsymbol{k}) \right) \binom{\overleftarrow{\boldsymbol{x}}}{\boldsymbol{n}} \binom{x_{d+1}}{n_{d+1}} \\
&= \sum_{\boldsymbol{n} \in \mathbb{N}^d} \sum_{n_{d+1} \geq 0} a_{\boldsymbol{n}, n_{d+1}}(f) \binom{\overleftarrow{\boldsymbol{x}}}{\boldsymbol{n}} \binom{x_{d+1}}{n_{d+1}} \\
&= \sum_{\boldsymbol{n} \in \mathbb{N}^{d+1}} a_{\boldsymbol{n}}(f) \binom{\boldsymbol{x}}{\boldsymbol{n}}.
\end{aligned}$$

□

Definition 38. *Let $f \in C(\mathbb{Z}_p^d, R)$ and write*

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} a_{\boldsymbol{n}}(f) \binom{\boldsymbol{x}}{\boldsymbol{n}},$$

as in Mahler's theorem. The element $a_{\boldsymbol{n}}(f) \in R$ is called the $\boldsymbol{n}$th Mahler coefficient of f.

Corollary 33. *For any $f \in C(\mathbb{Z}_p^d, R)$,*

$$\|f\|_\infty = \sup_{\boldsymbol{n} \in \mathbb{N}^d} \{\|a_{\boldsymbol{n}}(f)\|_R\} = \max_{\boldsymbol{n} \in \mathbb{N}^d} \{\|a_{\boldsymbol{n}}(f)\|_R\}.$$

Proof. For each $\boldsymbol{n} \in \mathbb{N}$, we have $\|a_{\boldsymbol{n}}(f)\|_R \leq \|f\|_\infty$ by the explicit expression for $a_{\boldsymbol{n}}(f)$. On the other hand, for $\boldsymbol{x} \in \mathbb{Z}_p^d$, we have

$$\begin{aligned}\|f(\boldsymbol{x})\|_R &= \left\| \sum_{\boldsymbol{n}\in\mathbb{N}^d} a_{\boldsymbol{n}}(f)\binom{\boldsymbol{x}}{\boldsymbol{n}} \right\|_R \\ &\leq \sup_{\boldsymbol{n}\in\mathbb{N}^d} \left\{ \left\| a_{\boldsymbol{n}}(f)\binom{\boldsymbol{x}}{\boldsymbol{n}} \right\|_R \right\} \\ &\leq \sup_{\boldsymbol{n}\in\mathbb{N}^d} \left\{ \|a_{\boldsymbol{n}}(f)\|_R \right\}.\end{aligned}$$

□

The uniqueness of Mahler coefficients implies the following two results.

Corollary 34. *Let $f, g \in C(\mathbb{Z}_p^d, R)$ and $c \in R$. For each $\boldsymbol{n} \in \mathbb{N}^d$,*

$$\begin{aligned}a_{\boldsymbol{n}}(f+g) &= a_{\boldsymbol{n}}(f) + a_{\boldsymbol{n}}(g) \\ a_{\boldsymbol{n}}(cf) &= c a_{\boldsymbol{n}}(f).\end{aligned}$$

Proof. For each $\boldsymbol{x} \in \mathbb{Z}_p^d$, we have

$$\begin{aligned}f(\boldsymbol{x}) + g(\boldsymbol{x}) &= \sum_{\boldsymbol{n}\in\mathbb{N}^d} a_{\boldsymbol{n}}(f)\binom{\boldsymbol{x}}{\boldsymbol{n}} + \sum_{\boldsymbol{n}\in\mathbb{N}^d} a_{\boldsymbol{n}}(g)\binom{\boldsymbol{x}}{\boldsymbol{n}} \\ &= \sum_{\boldsymbol{n}\in\mathbb{N}^d} (a_{\boldsymbol{n}}(f) + a_{\boldsymbol{n}}(g))\binom{\boldsymbol{x}}{\boldsymbol{n}}.\end{aligned}$$

Since $(a_{\boldsymbol{n}}(f) + a_{\boldsymbol{n}}(g)) \to 0$, the result follows from the uniqueness statement in Theorem 32. Similarly,

$$\begin{aligned}cf(\boldsymbol{x}) &= c\sum_{\boldsymbol{n}\in\mathbb{N}^d} a_{\boldsymbol{n}}(f)\binom{\boldsymbol{x}}{\boldsymbol{n}} \\ &= \sum_{\boldsymbol{n}\in\mathbb{N}^d} (c a_{\boldsymbol{n}})\binom{\boldsymbol{x}}{\boldsymbol{n}},\end{aligned}$$

and $(c a_{\boldsymbol{n}}(f)) \to 0$, so the result follows from the uniqueness statement in Theorem 32.

□

Corollary 35. *Let R be as above, and let $f : \mathbb{Z}_p^d \to R$ be a continuous function. Suppose for $i = 1, \ldots, d$ there are continuous functions $f_i : \mathbb{Z}_p \to R$ such that*

$$f(\boldsymbol{x}) = \prod_{i=1}^{d} f_i(x_i)$$

for all $\boldsymbol{x} \in \mathbb{Z}_p^d$. For each $\boldsymbol{n} \in \mathbb{N}^d$,

$$a_{\boldsymbol{n}}(f) = \prod_{i=1}^{d} a_{n_i}(f_i).$$

Conversely, if for $i = 1, \ldots, d$ there are continuous functions $f_i : \mathbb{Z}_p \to R$ such that for all $\boldsymbol{n} \in \mathbb{N}^d$,

$$a_{\boldsymbol{n}}(f) = \prod_{i=1}^{d} a_{n_i}(f_i),$$

then

$$f(\boldsymbol{x}) = \prod_{i=1}^{d} f_i(x_i)$$

for all $\boldsymbol{x} \in \mathbb{Z}_p^d$.

Proof. Let $\boldsymbol{n} \in \mathbb{N}^d$. By definition, we have

$$\begin{aligned} a_{\boldsymbol{n}}(f) &= \sum_{\boldsymbol{k} \preccurlyeq_d \boldsymbol{n}} (-\boldsymbol{1})^{\boldsymbol{n}-\boldsymbol{k}} \binom{\boldsymbol{n}}{\boldsymbol{k}} f(\boldsymbol{k}) \\ &= \sum_{\boldsymbol{k} \preccurlyeq_d \boldsymbol{n}} \prod_{i=1}^{d} (-1)^{n_i - k_i} \binom{n_i}{k_i} f_i(k_i) \\ &= \prod_{i=1}^{d} \sum_{k_i=0}^{n_i} (-1)^{n_i - k_i} \binom{n_i}{k_i} f(k_i) \\ &= \prod_{i=1}^{d} a_{n_i}(f_i). \end{aligned}$$

The converse follows from the above computation and the uniqueness of the Mahler expansion. ∎

APPENDIX B

POWER SERIES

We collect here some important results on rings of formal power series. For proofs of these results, consult any of Chapter III of Bourbaki (1972), Chapter 1 of Frolich (1968), or Appendix A of Hazewinkel (1978).

Let R be a commutative ring with identity $1 \neq 0$. Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a d-tuple of pairwise independent commuting indeterminates. The ring of d-variable formal powers series over R is denoted $R[[\boldsymbol{X}]] = R[[X_1, \ldots, X_d]]$. $R[[\boldsymbol{X}]]$ is the collection of all power series

$$F(\boldsymbol{X}) = \sum_{\boldsymbol{m} \in \mathbb{N}^d} a_{\boldsymbol{m}} \boldsymbol{X}^{\boldsymbol{m}}$$

with coefficients $a_{\boldsymbol{m}} \in R$ under "termwise addition" and "Cauchy multiplication":

$$\left(\sum_{\boldsymbol{m} \in \mathbb{N}^d} a_{\boldsymbol{m}} \boldsymbol{X}^{\boldsymbol{m}} \right) + \left(\sum_{\boldsymbol{m} \in \mathbb{N}^d} b_{\boldsymbol{m}} \boldsymbol{X}^{\boldsymbol{m}} \right) = \sum_{\boldsymbol{m} \in \mathbb{N}^d} (a_{\boldsymbol{m}} + b_{\boldsymbol{m}}) \boldsymbol{X}^{\boldsymbol{m}}$$

$$\left(\sum_{\boldsymbol{m} \in \mathbb{N}^d} a_{\boldsymbol{m}} \boldsymbol{X}^{\boldsymbol{m}} \right) \left(\sum_{\boldsymbol{m} \in \mathbb{N}^d} b_{\boldsymbol{m}} \boldsymbol{X}^{\boldsymbol{m}} \right) = \sum_{\boldsymbol{m} \in \mathbb{N}^d} \left(\sum_{\boldsymbol{k} \preccurlyeq_d \boldsymbol{m}} (a_{\boldsymbol{k}} b_{\boldsymbol{m}-\boldsymbol{k}}) \right) \boldsymbol{X}^{\boldsymbol{m}}$$

With these operations, $R[[\boldsymbol{X}]]$ is a commutative ring. The power series

$$I(\boldsymbol{X}) = \sum_{\boldsymbol{m} \in \mathbb{N}^d} a_{\boldsymbol{m}} \boldsymbol{X}^{\boldsymbol{m}}$$

with $a_{\boldsymbol{0}} = 1$ and $a_{\boldsymbol{m}} = 0$ for $\boldsymbol{m} \neq \boldsymbol{0}$ is the identity element of $R[[\boldsymbol{X}]]$. The ring R can be naturally identified with a subring of $R[[\boldsymbol{X}]]$ by

$$r \mapsto rI(\boldsymbol{X}) = \sum_{\boldsymbol{m} \in \mathbb{N}^d} r a_{\boldsymbol{m}} \boldsymbol{X}^{\boldsymbol{m}}$$

with $a_{\boldsymbol{0}} = 1$ and $a_{\boldsymbol{m}} = 0$ for $\boldsymbol{m} \neq \boldsymbol{0}$. In fact, for $r \in R$, the action

$$r \left(\sum_{\boldsymbol{m} \in \mathbb{N}^d} a_{\boldsymbol{m}} \boldsymbol{X}^{\boldsymbol{m}} \right) = \sum_{\boldsymbol{m} \in \mathbb{N}^d} r a_{\boldsymbol{m}} \boldsymbol{X}^{\boldsymbol{m}}$$

endows $R[[\boldsymbol{X}]]$ with the structure of an R-algebra.

The ring of d-variable polynomials over R is denoted $R[\boldsymbol{X}] = R[X_1, \dots, X_d]$, and is naturally identified with a subring of $R[[\boldsymbol{X}]]$. A power series

$$F(\boldsymbol{X}) = \sum_{\boldsymbol{m} \in \mathbb{N}^d} a_{\boldsymbol{m}} \boldsymbol{X}^{\boldsymbol{m}} \in R[[\boldsymbol{X}]]$$

belongs to $R[\boldsymbol{X}]$ if and only if there is $\boldsymbol{n} \in \mathbb{N}^d$ for which $a_{\boldsymbol{m}} = 0$ for all $\boldsymbol{n} \preccurlyeq_d \boldsymbol{m}$.

For any $d \geq 1$, we have

$$R[[X_1, \dots, X_{d-1}]][[X_d]] \cong R[[X_1, \dots, X_d]].$$

Let I be the ideal of $R[[\boldsymbol{X}]]$ generated by the set $\{X_1, \dots, X_d\}$. We endow $R[[\boldsymbol{X}]]$ with the I-adic topology. With this topology, $R[[\boldsymbol{X}]]$ is a complete Hausdorff topological ring, and $R[\boldsymbol{X}]$ is a dense subring of $R[[\boldsymbol{X}]]$.

Definition 39. *A term in $R[[\boldsymbol{X}]]$ is an element of the form $a\boldsymbol{X}^{\boldsymbol{n}}$ for some $a \in R$ and $\boldsymbol{n} \in \mathbb{N}^d$. A term $a\boldsymbol{X}^{\boldsymbol{n}}$ is nonzero if $a \neq 0$ and is called a monomial if $a = 1$. The degree of a term $a\boldsymbol{X}^{\boldsymbol{n}}$ is*

$$|\boldsymbol{n}| = \sum_{i=1}^{d} n_i,$$

or equivalently is the smallest $k \geq 0$ with $a\boldsymbol{X}^{\boldsymbol{n}} \in I^k$. A constant term is a term of degree zero. An element $F(\boldsymbol{X}) \in R[[\boldsymbol{X}]]$ is homogeneous of degree k if all of the nonzero terms of F have degree k.

Lemma 49. *Let*

$$F(\boldsymbol{X}) = \sum_{\boldsymbol{m} \in \mathbb{N}^d} a_{\boldsymbol{m}} \boldsymbol{X}^{\boldsymbol{m}}.$$

Then $F \in R[[\boldsymbol{X}]]^\times$ if and only if $a_{\boldsymbol{0}} \in R^\times$.

Now let $G_1, \dots, G_d \in R[[X]]$ (so the G_i are all one-variable power series). If $F \in R[\boldsymbol{X}]$, then the composition

$$F(G_1(X_1), \dots G_d(X_d))$$

is a well-defined element of $R[[\boldsymbol{X}]]$ (the composition involves only finitely many operations in the ring $R[[\boldsymbol{X}]]$). On the other hand, if $G_i \in XR[[X]]$ for all $1 \leq i \leq d$, and $F \in R[[\boldsymbol{X}]]$ is arbitrary, then the composition

$$F(G_1(X_1), \ldots G_d(X_d))$$

is again a well-defined element of $R[[\boldsymbol{X}]]$ (this fact uses the completeness of $R[[\boldsymbol{X}]]$ in a fundamental way).

Definition 40. *An R-derivation of $R[[\boldsymbol{X}]]$ is an R-linear map $D : R[[\boldsymbol{X}]] \to R[[\boldsymbol{X}]]$ satisfying the "product rule":*

$$D(fg) = fDg + gDf.$$

Example 27. *For each i, let $\dfrac{\partial}{\partial X_i} : R[[\boldsymbol{X}]] \to R[[\boldsymbol{X}]]$ denote the termwise partial derivative operator with respect to X_i:*

$$\frac{\partial}{\partial X_i}\left(\sum_{\boldsymbol{n}\in\mathbb{N}^d} a_{\boldsymbol{n}} \boldsymbol{X}^{\boldsymbol{n}}\right) = \sum_{\boldsymbol{n}\in \boldsymbol{e}_i+\mathbb{N}^d} n_i a_{\boldsymbol{n}} \boldsymbol{X}^{\boldsymbol{n}-\boldsymbol{e}_i}.$$

Then $\dfrac{\partial}{\partial X_i}$ is an R-derivation of $R[[\boldsymbol{X}]]$. ◊

A routine induction argument shows that if D is an R-derivation of $R[[\boldsymbol{X}]]$, then

$$DX_i^n = nX_i^{n-1}DX_i = DX_i\frac{\partial}{\partial X_i}(X_i^n)$$

for all i. Repeated application of the above fact and the product rule gives that for each $\boldsymbol{n} \in \mathbb{N}^d$,

$$D\boldsymbol{X}^{\boldsymbol{n}} = \sum_{i=1}^{n} DX_i\frac{\partial}{\partial X_i}(\boldsymbol{X}^{\boldsymbol{n}}).$$

The product rule now implies that if $f \in I^n$, then $Df \in I^{n-1}$. When coupled with R-linearity, this last fact gives that D is continuous with respect to the I-adic topology on $R[[\boldsymbol{X}]]$. Therefore, for any $f \in R[[X]]$, we have

$$Df = \sum_{i=1}^{n} DX_i \frac{\partial}{\partial X_i}(f).$$

Definition 41. *Let $\mathfrak{D}_d(R)$ denote the set of R-derivations of $R[[\boldsymbol{X}]]$. Then $\mathfrak{D}_d(R)$ is a $R[[\boldsymbol{X}]]$-module under the operations*

$$(D_1 + D_2)f = D_1 f + D_2 f$$

$$(aD_1)f = a(D_1 f)$$

for $D_1, D_2 \in \mathfrak{D}_d(R)$, $a \in R[[\boldsymbol{X}]]$.

We have seen that the $\dfrac{\partial}{\partial X_i}$ span $\mathfrak{D}_d(R)$ as an $R[[\boldsymbol{X}]]$-module. They are also $R[[\boldsymbol{X}]]$-linearly independent. Indeed, if $a_i \in R[[\boldsymbol{X}]]$ are such that

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial X_i}(f) = 0$$

for all $f \in R[[\boldsymbol{X}]]$, then evaluating at X_i gives

$$0 = \sum_{i=1}^{n} a_i \frac{\partial}{\partial X_i}(X_i) = a_i.$$

For this reason, an R-derivation D on $R[[\boldsymbol{X}]]$ is uniquely determined by DX_i for $i = 1, \ldots, d$.

p-ADIC POWER SERIES

Let R be a ring as in Section 2.5 which is a discrete valuation ring. As in Chapter 2, let ord_R denote the valuation on R and π a fixed uniformizer in R, and let Λ_d denote the d-variable power series ring $R[[\boldsymbol{T} - \mathbf{1}]]$. We will write elements of Λ_d as $F(\boldsymbol{T})$ rather than the more cumbersome $F(\boldsymbol{T} - \mathbf{1})$, with the hope that no confusion results.

In this setting, $\Lambda_d = R[[\boldsymbol{T}-\boldsymbol{1}]]$ is a local ring with maximal ideal $\langle \pi, T_1 - 1, \ldots, T_d - 1\rangle$, which we denote by $\mathfrak{m}$. Then Λ_d is complete with respect to the $\mathfrak{m}$-adic topology. In this case, if $G_1, \ldots, G_d \in \langle \pi, T-1\rangle \subseteq R[[T-1]]$, and $F \in \Lambda_d$, then the composition

$$F(G_1(T_1), \ldots, G_d(T_d))$$

gives a well-defined element of Λ_d (each of the $G_i(T_i)$ are topologically nilpotent elements in Λ_d).

An important collection of power series include the following objects. For $\boldsymbol{x} \in \mathbb{Z}_p^d$, we have the series

$$\boldsymbol{T}^{\boldsymbol{x}} = \sum_{\boldsymbol{m}\in\mathbb{N}^d} \binom{\boldsymbol{x}}{\boldsymbol{m}} (\boldsymbol{T}-\boldsymbol{1})^{\boldsymbol{m}} \in \mathbb{Z}_p[[\boldsymbol{T}-\boldsymbol{1}]] \subseteq \Lambda_d.$$

These series satisfy $\boldsymbol{T}^{\boldsymbol{x}}\boldsymbol{T}^{\boldsymbol{y}} = \boldsymbol{T}^{\boldsymbol{x}+\boldsymbol{y}}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}_p^d$ (see Appendix C). In particular, the set

$$E = \{\boldsymbol{T}^{\boldsymbol{x}} : \boldsymbol{x} \in \mathbb{Z}_p^d\}$$

is a (topologically) closed multipicative subgroup of Λ_d, isomorphic to $(\mathbb{Z}_p^d, +)$ (as topological groups) via $\boldsymbol{T}^{\boldsymbol{x}} \mapsto \boldsymbol{x}$. Moreover, E is generated topologically by the power series $T_1, \ldots, T_d$, i.e.

$$E = \overline{\langle T_1, \ldots, T_d\rangle}$$

as a subgroup of Λ_d under multiplication.

As in Section 2.2, for $\boldsymbol{m} \in \mathbb{N}^d$, let $I_{\boldsymbol{m}}$ be the ideal of Λ_d generated by $T_1^{p^{m_1}} - 1, \ldots, T_d^{p^{m_d}} - 1$. A consequence of the isomorphism $(\mathbb{Z}_p^d, +) \to E$ given by $\boldsymbol{x} \mapsto \boldsymbol{T}^{\boldsymbol{x}}$ is the following useful fact:

Lemma 50. *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}_p^d$. Then $\boldsymbol{T}^{\boldsymbol{x}} \equiv \boldsymbol{T}^{\boldsymbol{y}} \pmod{I_{\boldsymbol{m}}}$ if and only if $\boldsymbol{x} \equiv \boldsymbol{y} \pmod{\boldsymbol{p}^{\wedge\boldsymbol{m}}\mathbb{Z}_p^d}$.*

Consequently, the set

$$\{\boldsymbol{T}^{\boldsymbol{a}} : \boldsymbol{a} \preccurlyeq_d \boldsymbol{p}^{\wedge \boldsymbol{m}} - \boldsymbol{1}\}$$

is a complete set of distinct representatives of the quotient ring $\Lambda_d/I_{\boldsymbol{m}}$, and these elements are all R-linearly independent.

APPENDIX C

COMBINATORIAL IDENTITIES

We collect several important combinatorial identities which will be used in Chapter 2.

Let $\mathcal{A}$ be a $\mathbb{Q}$-algebra. For $n \in \mathbb{N}$, consider the polynomial

$$\binom{X}{n} = \begin{cases} \frac{X(X-1)\cdots(X-n+1)}{n!} & : n > 0 \\ 1 & : n = 0 \end{cases} \in \mathcal{A}[X].$$

When $x \in \mathbb{N}$, then $\binom{X}{n}$ evaluated at $X = x$ gives the usual binomial coefficient. For this reason, we will refer to this polynomial as the nth binomial coefficient polynomial in $\mathcal{A}[X]$.

For $x, y \in \mathbb{N}$ and $n \in \mathbb{N}$, we have the following identities, all of which may be found in Chapter 1 of Stanley (2012).

$$x\binom{x}{n} = n\binom{x}{n} + (n+1)\binom{x}{n+1}$$

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}$$

$$n!\binom{x}{n} = \sum_{k=0}^{n} s(n,k)x^k$$

$$x^n = \sum_{k=0}^{n} k!S(n,k)\binom{x}{k}$$

The integers $s(n, k)$ and $S(n, k)$ appearing above are the Stirling numbers of the first and second kind, respectively (see Stanley (2012)). Since a polynomial in $\mathcal{A}[X]$ may have only finitely many zeros in $\mathcal{A}$, the identities above yield the following polynomial identities in $\mathcal{A}[X, Y]$:

$$X\binom{X}{n} = n\binom{X}{n} + (n+1)\binom{X}{n+1}$$

$$\binom{X+Y}{n} = \sum_{k=0}^{n} \binom{X}{k}\binom{Y}{n-k}$$

$$n!\binom{X}{n} = \sum_{k=0}^{n} s(n,k)X^k$$

$$X^n = \sum_{k=0}^{n} k!S(n,k)\binom{X}{k}$$

In particular, the above formulas hold in $\mathbb{Q}_p[X,Y]$. However, whenever $x \in \mathbb{Z}_p$, also $x^n \in \mathbb{Z}_p$ and $\binom{x}{n} \in \mathbb{Z}_p$ for all $n \in \mathbb{N}$. Thus, when substituting $X = x$ and $Y = y$ for $x, y \in \mathbb{Z}_p$, each of the above identities hold in $\mathbb{Z}_p$.

Finally, recall that for $\boldsymbol{x} \in \mathbb{Z}_p^d$ and $\boldsymbol{n} \in \mathbb{N}^d$, we defined

$$\binom{\boldsymbol{x}}{\boldsymbol{n}} = \prod_{i=1}^{d} \binom{x_i}{n_i}.$$

The above identities can then be applied to these objects. For example, we have the useful identity

$$\binom{\boldsymbol{x}+\boldsymbol{y}}{\boldsymbol{n}} = \sum_{\boldsymbol{m} \preccurlyeq_d \boldsymbol{n}} \binom{\boldsymbol{x}}{\boldsymbol{m}}\binom{\boldsymbol{y}}{\boldsymbol{n}-\boldsymbol{m}}$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}_p^d$ and $\boldsymbol{n} \in \mathbb{N}^d$.

We also need the following identity, which appears in Section 1.9 of Stanley (2012):

$$\sum_{j=0}^{m} \binom{m}{j}\binom{j}{i}(-1)^j = \begin{cases} (-1)^m & : i = m \\ 0 & : i < m \end{cases}.$$

Finally, we will use the following result from Section XXI of Lucas (1878) several times in Chapter 2. (see also Dickson (1966) and exercise 14 in Chapter 1 of Stanley (2012)).

Lemma 51. *Let $m, n \in \mathbb{N}$ and let*

$$m = \sum_{i=0}^{\infty} m_i p^i \quad \textit{and} \quad n = \sum_{i=0}^{\infty} n_i p^i$$

be the base p expansions of m and n, respectively. Then

$$\binom{m}{n} \equiv \prod_{i=0}^{\infty} \binom{m_i}{n_i} \pmod{p}.$$

For $x \in \mathbb{Z}_p$, let $x^{(k)}$ denote the kth partial sum in the p-adic expansion of x. Fix $n \in \mathbb{N}$. Then by Lemma 51, the sequence

$$\left(\binom{x^{(k)}}{n} \pmod{p} \right)_k$$

is eventually constant. By the continuity of the function $x \mapsto \binom{x}{n}$, we have

$$\binom{x}{n} \equiv \prod_{i=0}^{\infty} \binom{x_i}{n_i} \pmod{p},$$

where x_i, n_i are the ith digits in the p-adic expansions of x and n, respectively.

BIOGRAPHICAL SKETCH

Scott Zinzer was born December 2, 1987 in Aurora, Illinois. He holds a Bachelor of Science in Mathematics from Aurora University and a Master of Arts in Mathematics from Arizona State University. He served as a Teaching Assistant in the School of Mathematical and Statistical Sciences at Arizona State University beginning in 2009. His Ph.D. thesis was written under the guidance of Nancy Childress.