The Construction of a Hyperbolic 4-Manifold with a Single Cusp, Following Kolpakov and Martelli

by

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ABSTRACT

Reprising the work of Kolpakov and Martelli, a manifold is constructed by face pairings of a four dimensional polytope, the 24-cell. The resulting geometry is a single cusped hyperbolic 4-manifold of finite volume. A short discussion of its geometry and underlying topology is included.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

The construction of hyperbolic finite-volume 4-manifolds appears to have been begun by Michael Davis in 1985, when he constructed such a manifold using the 120-cell as a starting point (Davis (1985)). The Davis manifold is compact, orientable, and of volume $\frac{104}{3}\pi^2$. Constructions continued, and at present the most complete classification of such manifolds is the one given by John Ratcliffe and Steven Tschantz (Ratcliffe and Tschantz (2000)). In it, they show the minimum volume for such manifolds is $\frac{4}{3}\pi^2$ and, using computer arithmetic, they classify 1171 minimum volume manifolds. Leone Slavitch has constructed a minimal volume manifold with two cusps, and a single cusped, non-orientable manifold of twice minimal volume (Slavich (2014)). Our goal is to construct a finite-volume, orientable, single-cusped manifold, and one of as small volume as possible.

One method for constructing manifolds involves the isometric gluing of polytopes along totally geodesic faces. In dimension two, for instance, we have the familiar construction of a flat torus by the orientation preserving gluing of opposite sides of a square. Such a construction is easliy visualized, and is thus convincing; we simply draw a square on a flat surface, indicate the direction each edge is to be glued to the opposite edge, and our intuition about how objects move and bend in three dimensional space convinces us that the resulting figure is a torus. If necessary we can physically cut such a figure from paper, glue the edges together, and hold it in our our hand. Such constructions *feel* natural.

In higher dimensions, however, such constructions become more arcane. Most of us posess some faculty in drawing three-dimensional figures, but not everyone can draw with the fluency needed to describe complicated manifolds. In dimension four and higher, sadly, such methods start to fail us entirely. What we need is a method, based on drawing, that works in higher dimensions. Our starting point will be a four dimensional polytope called a 24-cell; we will see it has octahedral 3-faces and can be given a hyperbolic structure with cusps at the vertices. Our gluing, then, occurs on these octahedral faces, but how can we easily describe the gluing?

Our solution is to draw a horosphere around each vertex, which then intersects its cusp along a horosection. We will further see such a section is a Euclidean cube, a familar and easily drawable object. Moreover each gluing of the faces of the cubical horosections uniquely extends to an orientation preserving isometric gluing of the octahedral 3-faces. Each 2-face of a horosection is the intersection of the horosection with a 3-face of the 24-cell, and thus gluing two 2-faces in section determines the n-faces to be glued between 3-faces of the 24-cell. We then rotate the octahedra so that the faces to be glued are aligned, and glue corresponding points. Each gluing between 2-faces of a horosection thus corresponds to a gluing between 3-faces of the 24-cell. This will motivate the rest of the paper.

Chapter 2

DEFINITIONS AND PRELIMINARIES

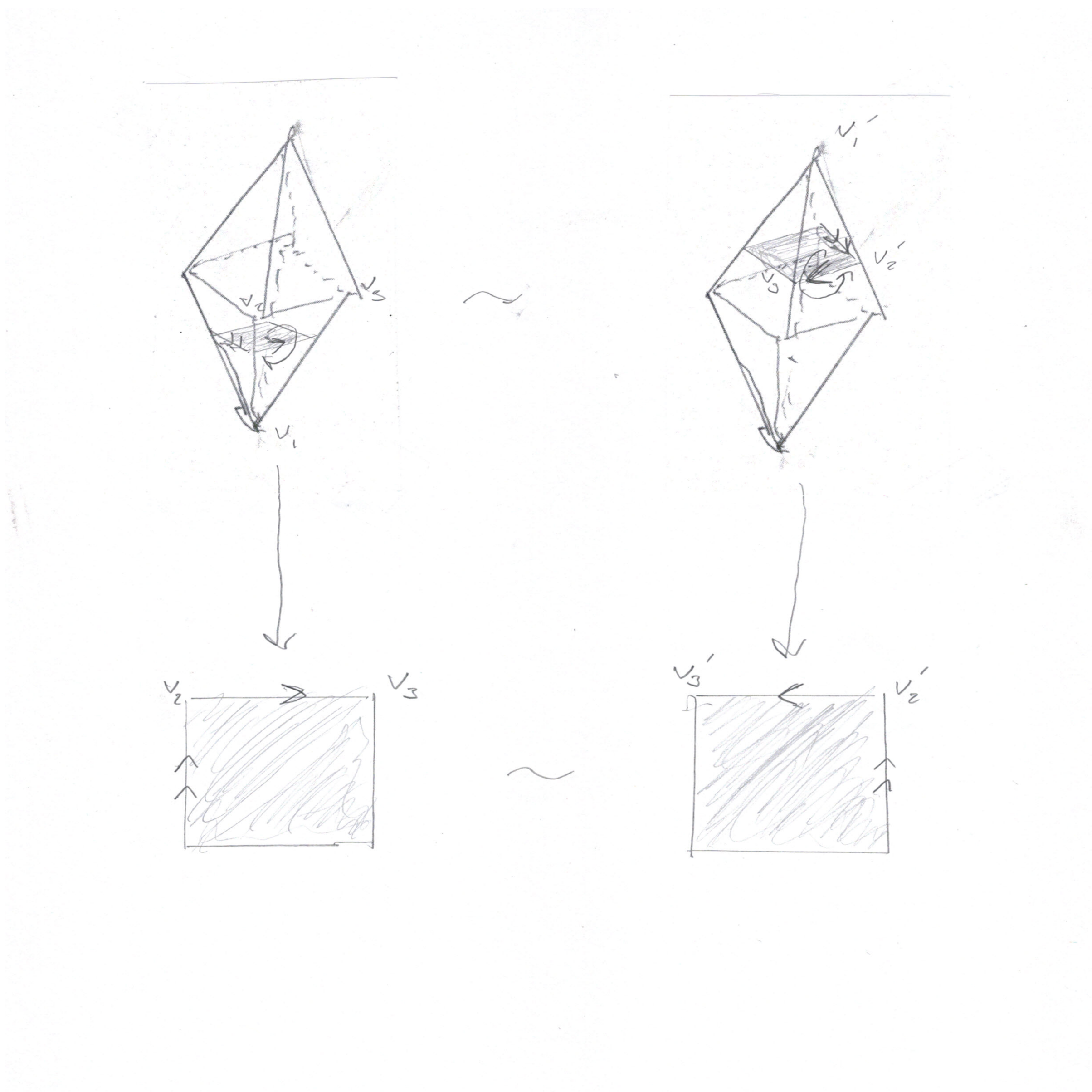

Figure 2.1: Sections of the 3-faces Determine How the 3-faces Must Be Glued

2.1 Definitions

Since we are interested in gluing along the boundary of hyperbolic polytopes, we begin by defining hyperbolic space.

Definition 2.1.1. *Hyperbolic space* $\mathbb{H}^n$ *is the necessarily unique simply-connected complete Riemannian manifold of constant sectional curvature -1.*

We will assume the reader is familiar with the canonical models for $\mathbb{H}^n$, specifically the ball, $\mathbb{B}^n$, and half-space models. We also define polytopes.

Definition 2.1.2. *A half-space is the closure of either component of* $\mathbb{H}^n \setminus S$ *for some totally geodesic hypersurface* S*.*

Definition 2.1.3. *A polytope is a non-empty compact subset of* $\mathbb{H}^n$ *that is the intersection of a finite number of half-spaces.*

Definition 2.1.4. *An ideal polytope is the non-empty intersection of a finite number of half-spaces of* $\mathbb{H}^n$ *that has no vertices and which intersects the boundary of the ball model at a finite number of points.*

2.2 Preliminaries

Our plan is to glue pairs of 3-faces of the 24-cell, and thus we need a method of determining if the quotient space of such a gluing is a hyperbolic manifold. We cite the following theorem:

Theorem 2.2.1. *Let* M *be a topological space obtained by gluing a finite collection* $P_1, \cdots, P_k$ *of hyperbolic* n*-dimensional polytopes or ideal polytopes via isometries between pairs of* $(n-1)$*-faces. Let* P *be the disjoint union of* $P_1, \cdots, P_k$*. If each point*

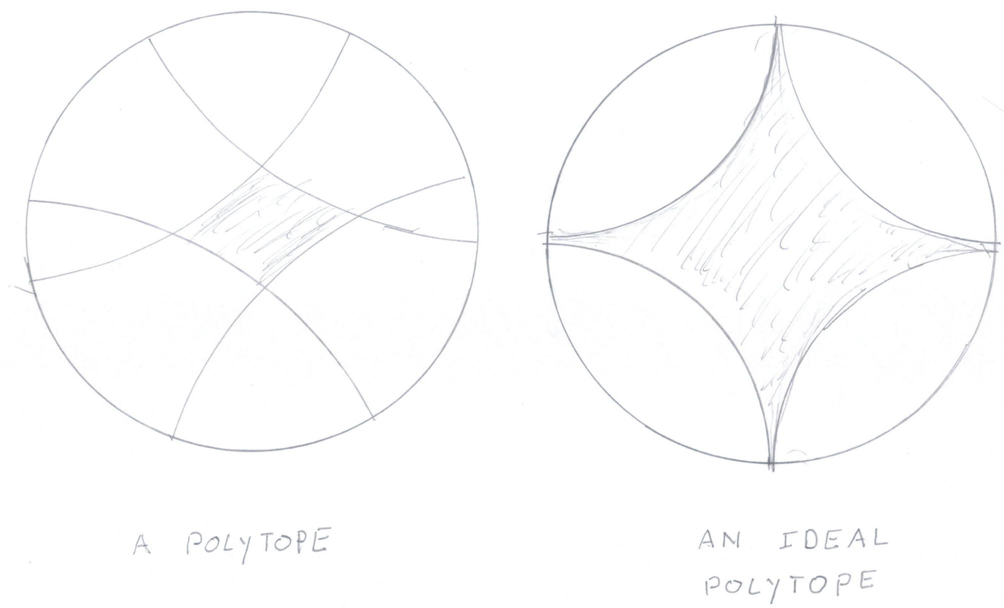

Figure 2.2: A (Nonideal) Polytope and an Ideal Polytope

$x \in M$ has a neighborhood N_x and a mapping $\phi_x \colon N_x \to B_{\epsilon(x)} \subset \mathbb{B}^n$ which is a homeomorphism onto its image, which sends x to 0, and which restricts to an isometry on each component of $N_x \cap q(P \setminus \partial P)$, then M inherits a hyperbolic structure.

Proof. See Lackenby (2000) □

This theorem has an obvious corollary:

Theorem 2.2.2. *Let M be obtained by gluing polytopes as in theorem 2.2.1. If M is a topological manifold, then M is a inherits a hyperbolic structure.*

Proof. See Lackenby (2000) □

What now remains is to define a hyperbolic 24-cell, and provide the specific gluings that produce a topological manifold on the quotient.

Chapter 3

THE 24-CELL

3.1 Constructing the Hyperbolic 24-Cell

We begin by constructing a hyperbolic 24-cell. A Euclidean 24-cell is defined to be the convex hull in $\mathbb{R}^4$ of the 24 vertices consisting of all permutations of the coordinates of $(\pm 1, \pm 1, 0, 0)$. As a polytope, it is regular, self-dual, and has:

24 3-faces

96 2-faces

96 1-faces

24 vertices

(Slavich (2014)).

As a regular polytope, we can embed the 24-cell with vertices removed into $\mathbb{B}^4$, the ball model for $\mathbb{H}^4$; we rescale each vertex until it has norm 1, and then place such vertices canonically on the boundary of $\mathbb{B}^4$. We then pull back the metric on $\mathbb{B}^4$ to the 24-cell giving us a hyperbolic manifold with boundary on which 3-faces are totally geodesic. Call such a space $\mathcal{C}$.

Lemma 3.1.1. *Each 3-face of the 24-cell is a regular octahedron. Furthermore each 3-face abuts 8 2-faces, 8 1-faces, and 6 vertices.*

Proof. Let x be the number of 2-faces each 3-face abuts. We count 2-faces by multiplying x by the number of 3-faces. However we have now overcounted by 2 as each 2-face touches 2 3-faces, one on each side. This gives us:

$$x \cdot 24 = 96 \cdot 2$$

and we easily see $x = 8$. By regularity, each 3-face must be a regular octahedron; there is only one polyhedron with 8 sides. We then observe the octahedron contains 8 2-faces, 8 1-faces, and 6 vertices. □

We color the 3-faces of the 24-cell in red, green or blue as follows:

1) Color the 3-face whose vertices are all permutations of (1, 1, 0, 0) green.

2) The 8 2-faces of the top face can be tiled in red and blue. Color each 3-face that abuts the top face along a red 2-face red; similarly for blue.

3) Color green the 3-face with vertices consisting of permutations of (-1, -1, 0, 0). Similarly to (2), color the vertices abutting the bottom face red and blue.

4) We have accounted for $1 + 8 + 8 + 1 = 18$ 3-faces. Color the remaining 6 green.

No 3-face of the 24-cell now abuts a similarly colored face (Kolpakov and Martelli (2013)).

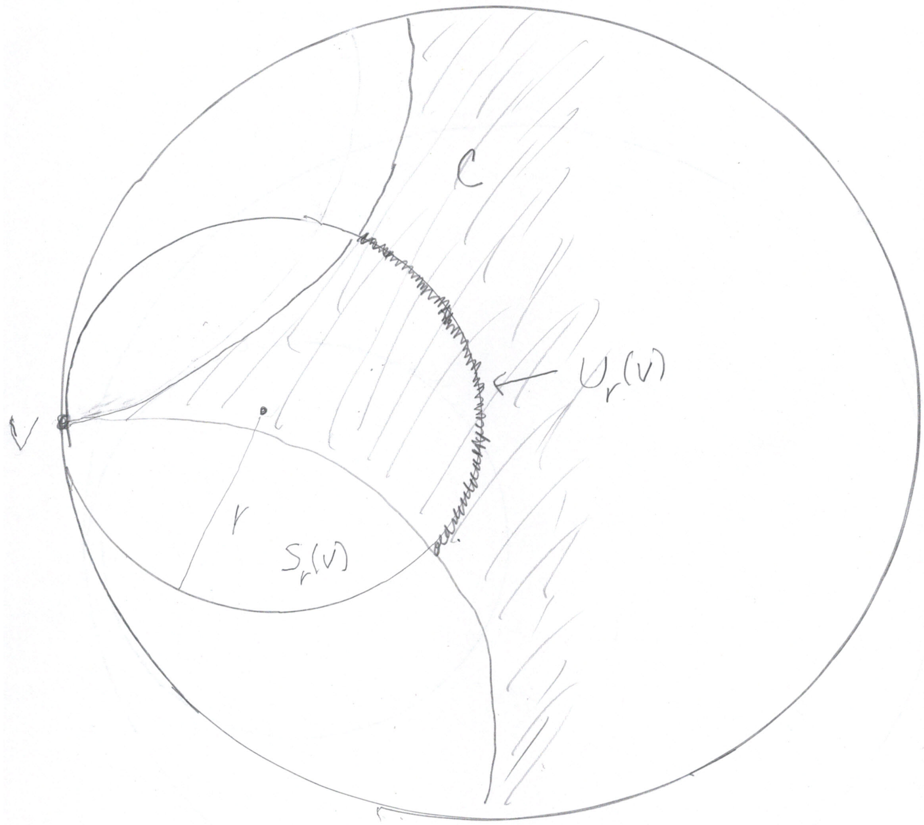

Figure 3.1: A Horosection of a Regular Polytope

3.2 The Horosections

Choose a vertex v of $\mathcal{C}$. Thurston defines horospheres to be a hypersurface orthogonal to all geodesics into v, and notes that they take the form of Euclidean spheres tangent to the boundary of $\mathbb{B}^n$ at v (Thurston (2002)). We make similar definitions.

Definition 3.2.1. *Let $S_r(v)$ be a sphere of Euclidean radius r tangent to $\mathbb{B}^4$ at v. We call $S_r(v)$ the horosphere of radius r at v. If $r > 0$ is sufficiently small to intersect $\mathcal{C}$ as a cusp section, we define the horosection $U_r(v)$ to be this intersection.*

Lemma 3.2.2. *Each horosection $U_r(v)$ is orthogonal to every geodesic into v.*

Proof. See Thurston (2002). □

The next lemma shows $U_r(v)$ is a cube, and the intersection of $U_r(v)$ with a 3-face is a square.

Lemma 3.2.3. *Each horosection U_r is a Euclidean cube, i.e. a cube with flat sides and dihedral angle between sides equal to $\frac{\pi}{2}$.*

Proof. First consider the intersection of a horosection with a 3-face of $\mathcal{C}$. We have seen each 3-face is a regular hyperbolic octahedron with vertices corresponding to cusps of $\mathcal{C}$. We may model this polyhedron in the half-space model for $\mathbb{H}^3$ as follows. Call the vertex to which the horosection is tangent v, and place it at ∞. Call the opposite vertex $-v$ and the four other vertices $v_1 \cdots v_4$, these are cusps and so must be on the boundary. By regularity, the opposite vertex to v, $-v$, must be placed at the origin, and the vertices v_i will be symmetrically arranged in a square around $-v$. Each edge connecting v_i to v is a geodesic and hence a vertical line over v_i. Our horosphere is now a plane parallel the the boundary - it is a cusp section orthogonal to geodesics into v. This sections the octahedron as a square with dihedral angle $\frac{\pi}{2}$.

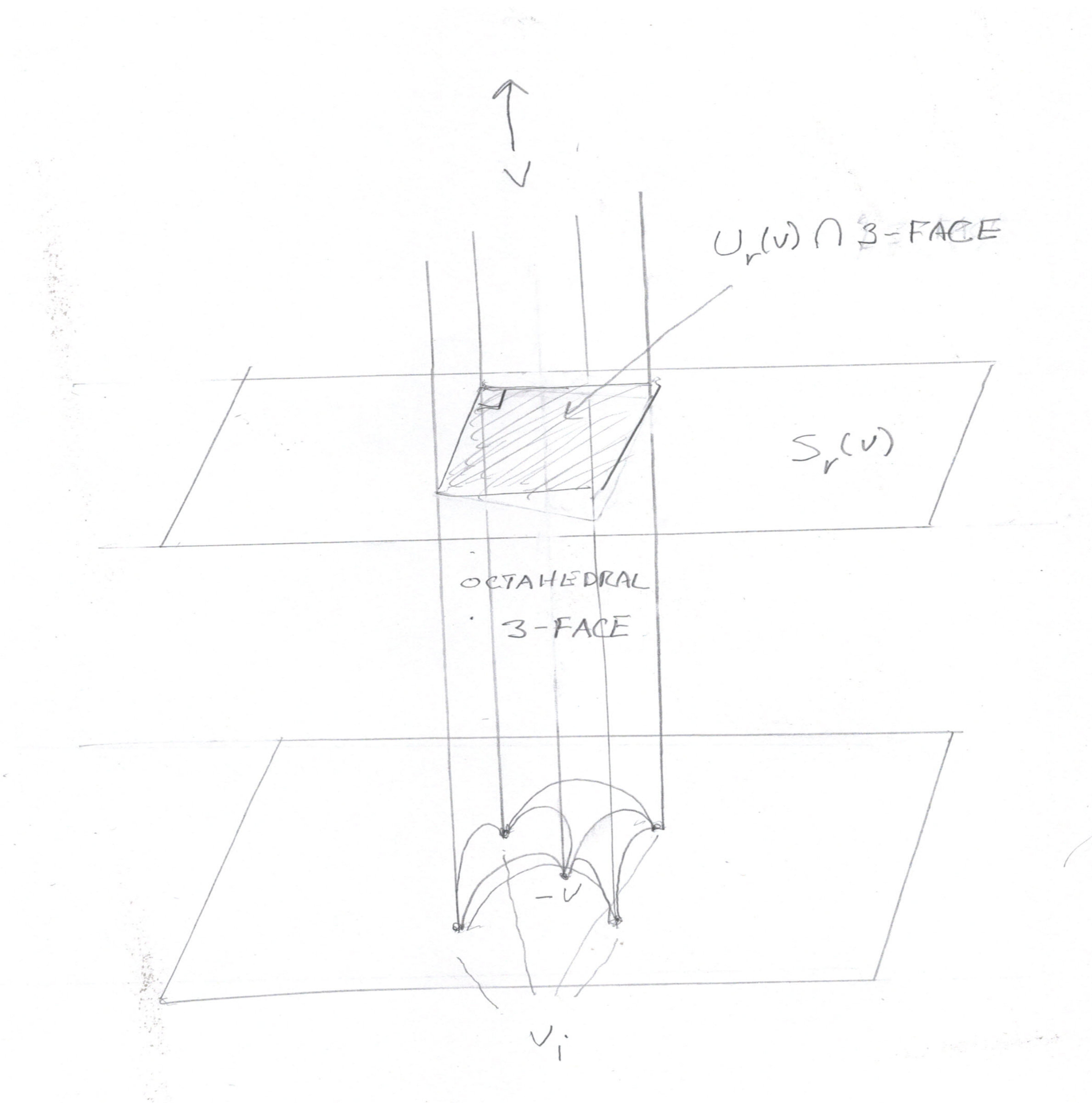

Figure 3.2: A Horosphere Sectioning a Regular Octahedron

Recall each 3-face of $\mathcal{C}$ abuts 6 vertices. By self-duality, each vertex must abut 6 3-faces, and so our horosection U_r must be a regular 6-sided polyhedron with each face the square resulting from the intersection of U_r with a facet. $\square$

Remember that we have colored each 3-face of $\mathcal{C}$ one of three colors. Each 2-face of U_r is contained in some 3-face of $\mathcal{C}$ and thus inherits a coloring. But no coloring of a 3-face of $\mathcal{C}$ abuts another similarly colored 3-face, hence no 2-face of the square U_r abuts the same color. We conclude U_r has opposite sides of the same color.

Chapter 4

THE BUILDING BLOCK

Let $\mathcal{C}$ be the hyperbolic 24-cell with colored 3-faces as constructed above. We make four copies of $\mathcal{C}$, say $\mathcal{C}_{11}$, $\mathcal{C}_{12}$, $\mathcal{C}_{21}$, and $\mathcal{C}_{22}$ preserving the colorings of each. We then glue isometrically and in a orientation preserving fashion:

1) each red 3-face of $\mathcal{C}_{11}$ to the corresponding red 3-face of $\mathcal{C}_{12}$ and each red 3-face of $\mathcal{C}_{21}$ to the corresponding red 3-face of $\mathcal{C}_{22}$.

2) each blue 3-face of $\mathcal{C}_{11}$ to the corresponding blue 3-face of $\mathcal{C}_{21}$ and each blue 3-face of $\mathcal{C}_{12}$ to the corresponding 3-face of $\mathcal{C}_{22}$

and call this topological space $\mathcal{B}$.

Lemma 4.0.4. *$\mathcal{B}\backslash\{$green 3-faces of $\mathcal{B}\}$ is a manifold.*

Proof. We need show that $\mathcal{B}$ is locally Euclidean. Let q be the quotient map from the four unglued copies of $\mathcal{C}$ to $\mathcal{B}$. Choose $x \in \mathcal{B}$

1) If $x = q(p)$ for some p in the interior of some $\mathcal{C}_{ij}$, then p is left unglued by q. Hence q is a local homeomorphism from a ball around p to a ball around x.

2) If $x = q(p)$ for some p in the interior of some red or blue 3-face, then the fiber over x consists of two points, one on each similarly colored face of different $\mathcal{C}_{ij}$s. Recall

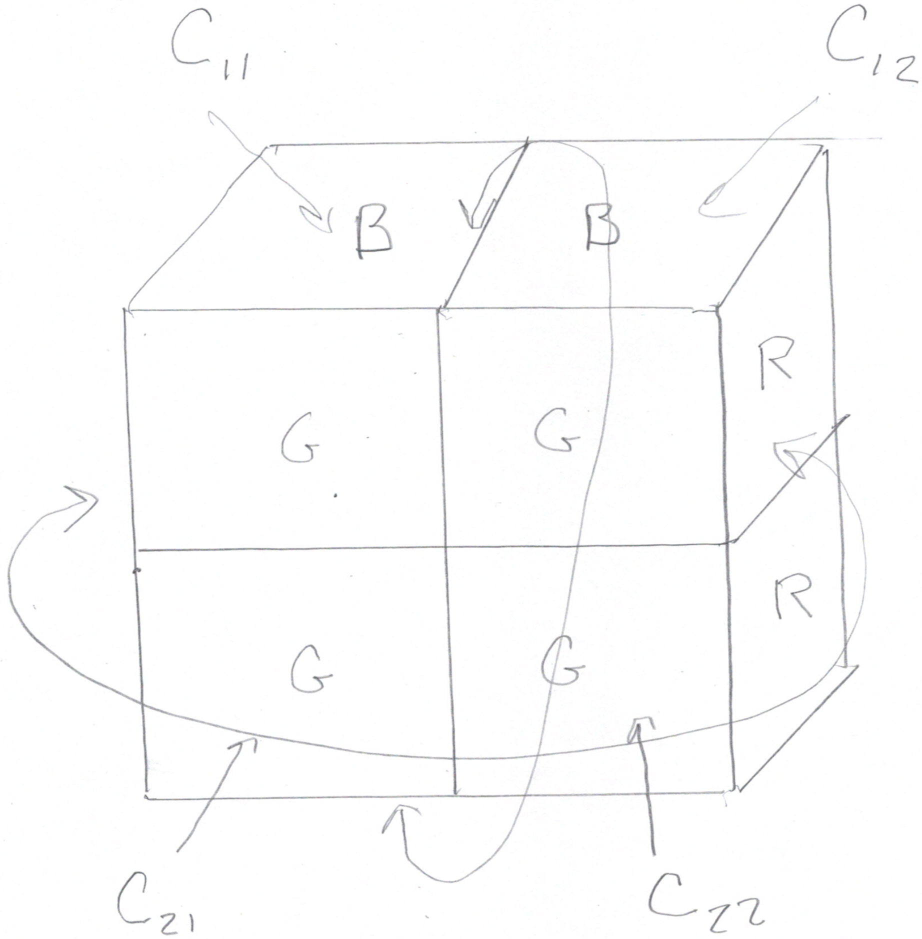

Figure 4.1: The Construction of $\mathcal{B}$ by Gluing Faces, in Cusp Section

each $\mathcal{C}_{ij}$ is a manifold with boundary consisting of its 3-faces. We may then choose a neighborhood of each point in the fiber identical to the half-ball model for manifolds with boundary. Then q glues each half-ball to a space containing a ball around x.

3) If $x = q(p)$ for some p on a 2-face or a 1-face and x is contained in a non-maximal horosection U_r of radius r containing x, we pass to horosections. Recall that the horosections look like a flat cube with oppositely colored sides. But gluing red to red and blue to blue glues horosections into a larger cube with x in its interior. This interior is plainly a manifold; we may therefore choose a 3-ball B around x inside U_r. Now thicken r by an ϵ-neighborhood to produce an open neighborhood of x homeomorphic to $B \times (r - \epsilon, r + \epsilon)$. This contains a ball around x.

So now $\mathcal{B}$ is locally Euclidean at x - provided x is close enough to a cusp to be contained in a horosection. But our gluing along faces is homogeneous, if our quotient space is locally Euclidean at any point glued along faces, it must have the same topological property at every point glued along the same faces. We conclude $\mathcal{B}$ is locally Euclidean at every point along a gluing of 2-faces or 1-faces.

4) There are no vertices. □

Theorem 4.0.5. *$\mathcal{B}$ is a hyperbolic manifold with totally geodesic boundary consisting of the 32 green 3-faces of the $\mathcal{C}_{ij}$s*

Proof. We now know $\mathcal{B}\backslash\{\text{green 3-faces of } \mathcal{B}\}$ is a manifold. It is hyperbolic as it is realized by isometrically gluing totally geodesic boundary components of hyperbolic manifolds with boundary. $\mathcal{B}$ is therefore a manifold with boundary consisting of the 32 unglued green 3-faces of the $\mathcal{C}_{ij}$s which were previously seen to be totally geodesic. □

We now count the distinct cusps of $\mathcal{B}$. Each cusp section intersects 8 green 3-faces,

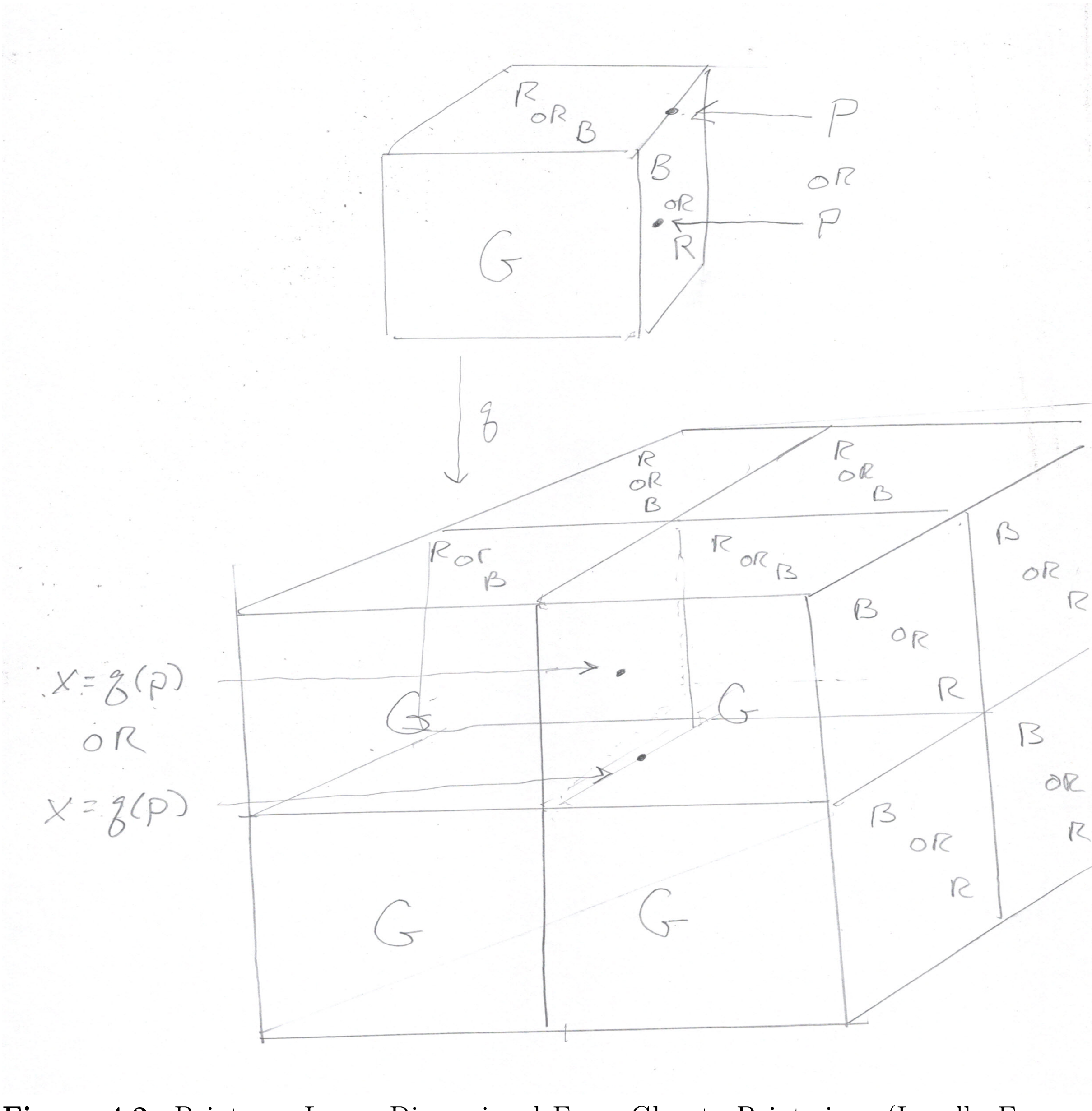

Figure 4.2: Points on Lower Dimensional Faces Glue to Points in a (Locally Euclidean) Packing of Cubes

now glued into two groups of four as in figure 3.1. $\mathcal{B}$ has 32 such faces, and therefore 4 cusps. We also determine the topology of the cusps.

Theorem 4.0.6. *Each cusp of $\mathcal{B}$ is homeomorphic to $T^2 \times [0, 1] \times [0, \infty)$*

Proof. As pictured, each horosection decomposes into a cube with one pair of opposite sides colored red and glued in an orientation preserving manner, and one pair colored blue and glued similarly. The green sides are unglued. We conclude the horosection is homeomorphic to $S^1 \times S^1 \times [0, 1]$. The cusp itself is therefore homeomorphic to $T^2 \times [0, 1] \times [0, \infty)$ □

Chapter 5

THE CONSTRUCTION

5.1 A single cusped 4-manifold

Our plan is now to construct a manifold without boundary by isometrically gluing the green 3-faces of $\mathcal{B}$. In each horosection, we have seen that there are two groups of four green faces which have been glued along edges into a larger square and any further gluing must respect this identification. Kolpakov and Martelli produce an entire menagerie of manifolds by gluing in this manner (Kolpakov and Martelli (2013)); we can, for instance, glue the front four green faces of a horosection to the back four and repeat similarly for each section, we can glue the front of the first section to the back of the second, and continue in a cycle, there are orientation reversing gluings, and so on. Moreover, we can make n copies of $\mathcal{B}$ and glue the $32n$ resulting green faces in some similar manner. This paper will attempt to construct only one manifold, one with a single cusp.

Begin by orienting the four cubes forming a horosection with green faces front and back, blue up and down, and red left and right. Repeat this for all four horosections. Now isometrically glue the front four green faces of the first section to the back four faces of the second section, glue the front of the second to the back of the third, and repeat cyclically; the natural packing of cubes as in figure 5.1 preserves orientation. This gives a gluing in horosections, now extend this gluing to the coresponding isometric gluing of 3-faces discussed is the introduction. Call the resulting topological space $\mathcal{M}$

We immediately see that such a section is a topological torus - it is a cube with opposite sides glued in an orientation preserving manner - and hence a manifold. Each point, therefore, has an open 3-neighborhood B within the section, and, by a thickening argument similar to that in lemma 4.0.2, $\mathcal{M}$ is locally Euclidean. We have isometrically glued all remaining 3-faces, hence $\mathcal{M}$ is a hyperbolic manifold.

Lastly, we note that we have glued cusp sections for each of the four cusps into a single connected manifold. Since section gluing determines vertex (i.e. cusp) gluing, we have glued all four cusps into one. $\mathcal{M}$ is a single cusped hyperbolic manifold.

5.2 Notes on the geometry of $\mathcal{M}$

We conclude by making a few remarks about the geometry and topology of $\mathcal{M}$. We have seen that $\mathcal{M}$ is a single cusped manifold which is thus homeomorphic to a closed manifold with a single puncture. It follows that $\mathcal{M}$ is complete. Its cusp section, as we have seen, is homeomorphic to T^3. Considering this section was constructed via isometric gluings of flat cubes, this cusp section is flat. The volume of $\mathcal{M}$ must be four times the volume of $\mathcal{C}$ - we have constructed $\mathcal{M}$ from four copies of $\mathcal{C}$ by identifying 3-faces which have zero volume.

The volume of $\mathcal{C}$ can be computed as $\frac{4\pi^2}{3}$ (Kolpakov (2012)), making $\mathrm{vol}(\mathcal{M}) = \frac{16\pi^2}{3}$. Furthermore, one consequence of Chern-Gauss-Bonnet is that every orientable, finite-volume, complete, hyperbolic manifold M has its volume and Euler characteristic linked via the formula

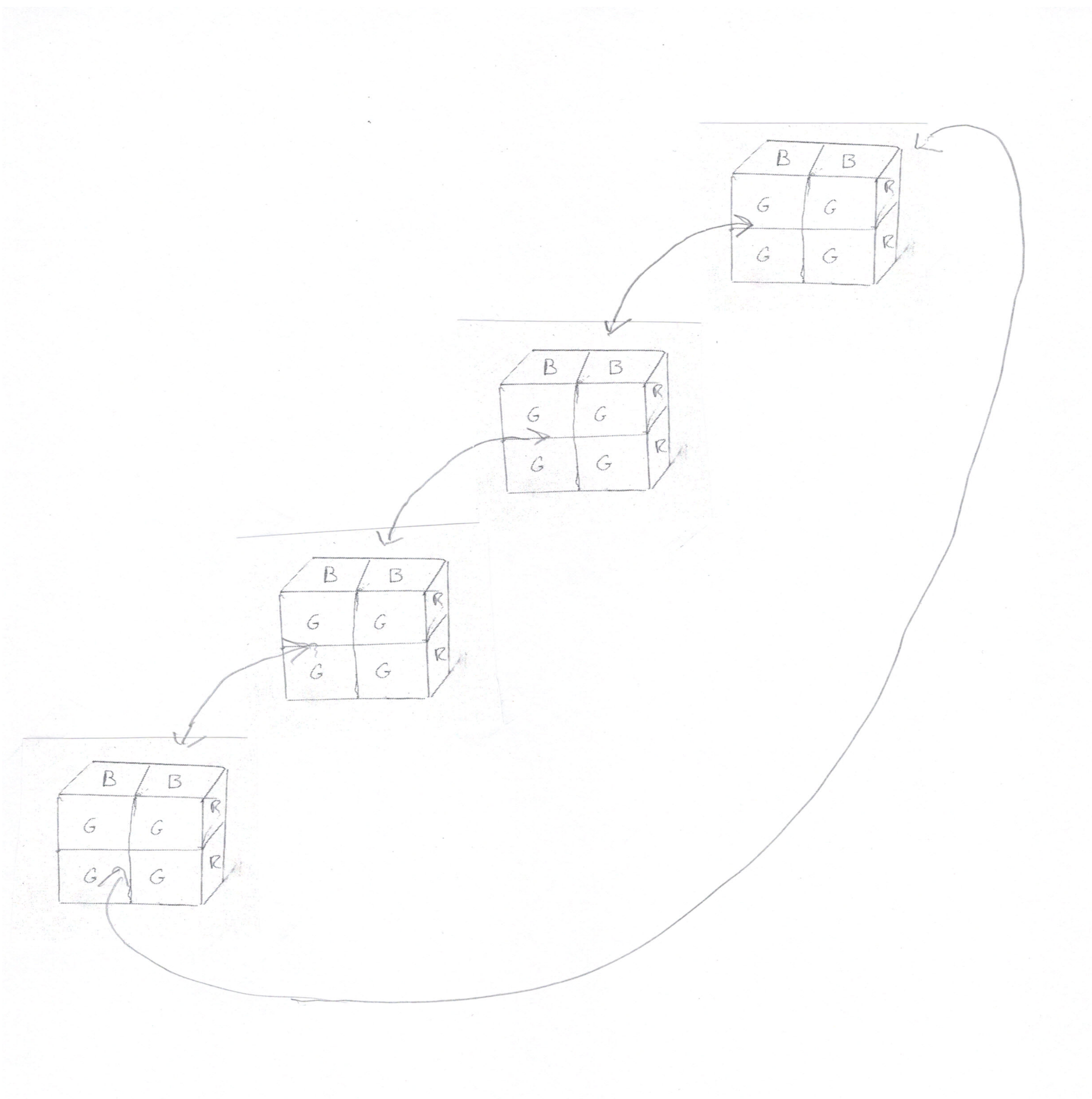

Figure 5.1: The Construction of $\mathcal{M}$ by Front-to-Back Gluing of Green Faces in Cusp Section

$$\text{vol}(M) = \frac{4\pi^2}{3}\chi(M)$$

Considering our volume calculation, this gives us $\chi(\mathcal{M}) = 4$. None of this completely classifies $\mathcal{M}$ up to homeomorphism, but it does give us some useful insights.

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