A Fun Way To Help Students Discover Discrete Mathematics

by

Stephanie Bell

A Thesis Presented in Partial Fulfillment
of the Requirement for the Degree
Master of Arts

Approved May 2014 by the
Graduate Supervisory Committee:

Susanna Fishel, Ph.D, Chair
Glenn Hurlbert, Ph.D
John Quigg, Ph.D

ARIZONA STATE UNIVERSITY

August 2014

ABSTRACT

This thesis focuses on sequencing questions in a way that provides students with manageable steps to understand some of the fundamental concepts in discrete mathematics. The questions are aimed at younger students (middle and high school aged) with the goal of helping young students, who have likely never seen discrete mathematics, to learn through guided discovery. Chapter 2 is the bulk of this thesis as it provides questions, hints, solutions, as well as a brief discussion of each question. In the discussions following the questions, I have attempted to illustrate some relationships between the current question and previous questions, explain the learning goals of that question, as well as point out possible flaws in students' thinking or point out ways to explore this topic further. Chapter 3 provides additional questions with hints and solutions, but no discussion. Many of the questions in Chapter 3 contain ideas similar to questions in Chapter 2, but also illustrate how versatile discrete mathematics topics are. Chapter 4 focuses on possible future directions. The overall framework for the questions is that a student is hosting a birthday party, and all of the questions are ones that might actually come up in party planning. The purpose of putting it in this setting is to make the questions seem more coherent and less arbitrary or forced.

ACKNOWLEDGEMENTS

I would like to thank Dr. Fishel for sticking with me for so long through this thesis process, Dr. Hurlbert for always being willing to entertain me, and Dr. Quigg for joining my committee at such a late hour! I want to point out that none of this would have been possible if it had not been for the wonderful Debbie Olson. She is the heart of the department.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

1.1 Why Guided Discovery?

Typical mathematics textbooks give students a problem with a full solution and then ask the students to follow the template provided by the author's solution to answer a very similar question. This turns "learning mathematics" into an exercise of following a template and mimicking techniques and ideas. There have been several papers written on the benefit of learning mathematics through guided discovery (see Mayer (2004) for several studies on these benefits). In my opinion, the main benefit is that the student is not handed the knowledge and simply asked to digest it, but rather she develops the ideas about whichever topic she just discovered and so she *owns* that knowledge. She spent the time to think through the idea and, along with a sense of ownership, she will also have a deeper understanding.

Some might argue that students learn best when they have no guidance and get to explore topics on their own, but many argue that having some guidance is more effective (Paul A. Kirschner and Clark, 2006). Learning through guided discovery can take advantage of a teacher's presence. The student does not need to beat their head against a wall when they get stuck because they can ask for more hints if they are making no progress. Without guidance, or a way to check accuracy in their thought process, students can practice incorrect ways of thinking, and potentially hinder their ability to learn. In Paul A. Kirschner and Clark (2006), the authors go as far as saying, "Not only is unguided instruction normally less effective; there is also evidence that it may have negative results when students acquire misconceptions or incomplete or disorganized knowledge."

1.2 Why Discrete Mathematics?

In this thesis, I chose to primarily focus on teaching middle school or high school students discrete mathematics (primarily combinatorics and basic set theory as opposed to some other branch of mathematics)through guided discovery for several reasons. The main reason for choosing discrete mathematics is that a student does not need to have many prerequisites to understand the ideas. With little more than basic algebra skills, and some hard thinking about the situation/problem, a student can discover many of the topics typically taught in an introductory discrete mathematics course. It gives the students a fresh start in mathematics (DeBellis and Rosenstein, 2008). They can approach the situations/ideas with fresh perspective since discrete mathematics topics are usually not taught in middle or high school, and they are not hampered if they did not understand the ideas of mechanics involved in studying calculus or geometry since many topics in discrete mathematics are not dependent on these courses. Furthermore, DeBellis and Rosenstein (2008) explains that discrete mathematics "facilitates a focus on problem-solving and reasoning at all grade levels." Some even argue that students should be taught discrete mathematics in elementary school to pique their interests sooner, and to give them more practice with problem solving. Friedler (1996) has gone as far as suggesting that it's possible that students who are exposed to fun problems sooner will be more excited and will maintain that enthusiasm.

Additionally, this thesis aims at exposing students to a new topic at the same time as a new way of learning. If they have been subjected to a more traditional approach of learning math (i.e. mimicking a template), a new technique for learning (i.e. guided discovery) will not be as drastic of a change if it is accompanied by a fresh, new topic (DeBellis and Rosenstein, 2008). That is, a traditional approach to learning incorporates new techniques with new topics (i.e. students learn the technique of using the quadratic formula when they need to solve quadratic equations), so some students might just think of guided discovery as a new technique for learning the new topic of discrete mathematics, and might more readily accept the style of learning since discrete mathematics will typically be new to them.

In Kenneth Bogart's "Combinatorics Through Guided Discovery" (Bogart, 2005) he introduces several of the main topics in enumerative combinatorics through a series of guiding questions. Bogart's text differs from many mathematics textbooks because he gives minimal background information and then forces the student to think of a novel technique to solve a problem. Rather than set a student loose in a problem that might initially be over their head, he typically guides them with baby steps. For example, in his book, Bogart's second question is:

> Now some number n of schools are going to send their baseball teams to a tournament, and each team must play each other team exactly once. Let us think of the teams as numbered 1 through n.
>
> (a) How many games does team 1 have to play in?
>
> (b) How many games, other than the one with team 1, does team two have to play in?
>
> (c) How many games, other than those with the first $i - 1$ teams, does team i have to play in?

(d) In terms of your answers to the previous parts of this problem, what is the total number of games that must be played?

Notice that he starts the students off with easier questions which guides them to develop the thought process needed to answer the more difficult questions. Frequently in mathematics, if a student gets frustrated by a problem, the frustration is from not knowing where to even start a problem. Small baby steps which are designed to keep a student on track and thinking about a problem correctly can keep the frustration to a minimum.

Although I focus on middle and high school students, I also think older students will benefit from guided discovery, but that is not the focus on this thesis. I want to focus on middle school and high school students learning through guided discovery for several reasons. The first reason is that young adults are still malleable. Even if they have been exposed to traditional lectures, they can adapt more easily and they can learn to think deeply about a topic without being spoon-fed the information, and can benefit for the rest of their mathematical career (Friedler, 1996; DeBellis and Rosenstein, 2008). Secondly, it is easier to take the time to learn with guided discovery if students have the same class every day for an entire year (as opposed to a few times a week for a single semester). Finally, many of these hints will need to be facilitated by a teacher, and it will typically be a judgment call whether or not a student could benefit from a particular hint. A self-taught look at this material is possible, but not all of the hints will be necessary and too many might to more confusing than helpful.

1.3 What I Suggest

What I am proposing in this thesis is to try to alter Bogart's approach slightly. Many students are still frustrated with this approach despite the baby steps he typically provides in his textbook. I certainly see the benefits of guided discovery, but too much frustration is not beneficial to students. If there can be a teacher nearby to continually check in with the student, then we can make some adjustments to Bogart's guided discovery.

First, I would like to suggest having more hints readily available. Not everyone will need all of the hints, but the teacher should be able to gauge when to give out additional hints. Having too few hints can be frustrating, and having too many hints spoils the "discovery" and it could make the problem seem more like filling in an outline than thinking for themselves.

Secondly, I would like to suggest including or providing partial solutions for steps to the student. If a student is not on the right track, it can be frustrating for them to spend additional time only to get an answer that doesn't make sense or is missing a component/case. There can be partial solutions freely accessible to the student, just so they can check that they are on the right track, and/or there can be frequent progress checks by the teacher with full solutions checking to ensure the student did indeed reach the correct conclusion. Bogart does provide full solutions to his problems, but they are the full solution, and can spoil the problem. If the student is asked to count the number of ways to do something, the final answer can be checked with the teacher when the student thinks they have arrived at a solution. I am not suggesting that the student can see the correct answer. If the student has access to the correct count, they might just try to justify why that answer is correct, rather than truly believe their reasoning. That being said, I am in no way suggesting a solution manual be given to the students to accompany the questions, since this will be too much of a crutch for many students and again spoil the discovery, but rather, enough resources to make sure they are on the right track. If a series of problems are designed to teach a student a specific technique or idea, it is a detriment to the student if they did not actually get the ideas/techniques out of the problems that they were supposed to get (Paul A. Kirschner and Clark, 2006).

This thesis primarily focuses on sequencing questions in a way that provides students with manageable steps to understand some of the most fundamental concepts in discrete mathematics. Chapter 2 is the focus on this thesis as it provides questions, hints (for the students), solutions (for the instructor), as well as a brief discussion of each question. In the discussion following the questions, I have attempted to illustrate some relationships between the current question and previous questions, explain the learning goals of that question, as well as point out possible flaws in students' thinking or point out ways to explore this topic further. Chapter 3 provides bonus questions with hints and solutions, but no discussions. Many of the questions in Chapter 3 contain ideas similar to questions in Chapter 2, but also illustrate how versatile discrete mathematics is. The overall setting of the questions is that a student is hosting a birthday party, and each of the questions might actually come up in party planning. The purpose of this is to make the questions seem less arbitrary and forced (as opposed to "How many ways can Billy choose 4 fruit from a bag given that the bag contains 35 apples, 12 oranges, and 10 bananas?" [1]). Chapter 4 provides a couple possible future directions of this work.

[1] The answer is 395010 ways

1.4 Synopsis For the Student

You are planning a birthday party and there are a lot of aspects you need to consider. You and your friends are planning on making dinner and dessert, setting the table for dinner, playing games, and dancing. Several of your friends' younger siblings will also be attending, but you need to keep them entertained as well, so you will certainly need to think of a game that will keep their interest piqued for quite some time. While planning your party, you find yourself in the situations below. Can you figure out answers to your questions?

Chapter 2

ACTIVITIES WITH PROBLEMS, HINTS, SOLUTIONS AND DISCUSSIONS

2.1 Entertaining The Kids

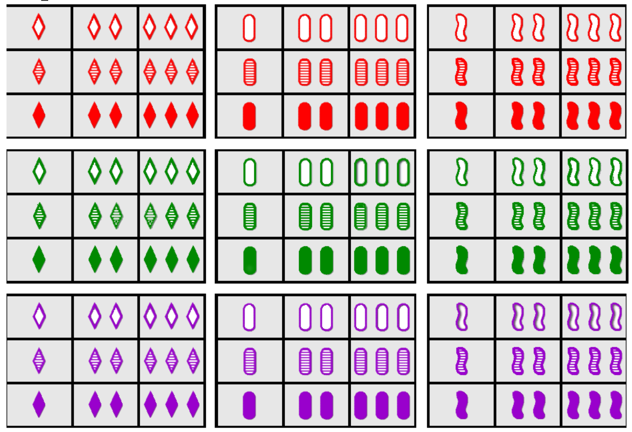
To the student: To keep the younger siblings occupied, you plan on teaching them to play the game of SET® and giving them some questions to consider. First they need to know how to play (see below for rules or http://www.setgame.com/set for official rules). Then after they begin to grow tired of this game, you can give them some problems to think about that relate to the game. The questions you thought of are listed below.

2.1.1 Prerequisite: The Rules of SET®

The cards in a deck of SET® are given below for reference. Note each card has four features: shape (diamond, oval, or squiggles), color (red, green, or purple), shading (solid, open, or shaded), and number (one, two, or three).

A *Set* (note the capital "S") consists of 3 cards in which each of the cards' features, looked at one by one, are the same on each card, or are different on each card. All of the features must independently satisfy this rule. This means, in order for a group of 3 cards to be a Set, shape must be either the same on all 3 cards, or different on each of the 3 cards; color must be either the same on all 3 cards, or different on each of the 3; shading must be either the same on all 3 cards, or different on each of the 3 cards; and number must be either the same on all 3 cards, or different on each of the 3 cards. A good way to determine if a collection of three cards is a Set or not is to ask yourself if you have exactly two of anything (two reds, two ovals, etc.).

Figure 2.1: The Complete Collection of Cards in SET®



If there are exactly two of anything, the collection of cards being considered is *not* a Set. The object of the game is to identify Sets in a collection of 12 cards placed face up on the table. For a complete list of rules, see SET® Enterprises' website at http://www.setgame.com/set. In hopes of avoiding confusion, we will use "SET®" to indicate the name of the game, "Set" when referring to a group of 3 cards having the desired properties in SET®, and "set" for all other uses of the word. So, a Set is a set of 3 cards having specific properties set by the makers of SET®.

2.1.2 Activity 1: Set Theory and Learning How to Count

What the Student Will Discover

To the teacher: In the following questions, not only will students become more familiar with the game of SET®, but students should begin to develop some counting techniques and ideas such the multiplicative and additive rules as well as permutations and possibly discovering $\binom{n}{k}$. They will be introduced to terms from set theory such as *complement*, *cardinality*, *union*, *intersection*, *symmetric difference*, and even *partition*. Students already have some intuition about what many of these terms mean, but here they can begin to use precise language and can start to give names to ideas they have (like “complement” or “union”). The students will be able to look at a physical deck of cards or at Figure 2.1 to help visualize the situations described in the following set of questions. No prior knowledge of such topics are prerequisites. In the future, there could be a series of questions that naturally follow that can take the small step from counting the number of outcomes to calculating the probability of a specific outcome or set of outcomes. These questions are in no way exhaustive of the types of questions you could ask or topics you could investigate that could relate to the game of SET®. See http://www.setgame.com/teachers-corner/set for further ways to explore the mathematics hidden in this great game. Future directions would include emphasizing Venn diagrams, more counting questions involving permutations and combinations, and then some basic probability.

Following most questions are hints for the students. Following each of the questions in this chapter are solutions for the teacher to divvy out as she sees fit, and a discussion on learning goals for that particular question. The discussion will also be used to point out how one could tie the current question to previous seemingly-unrelated questions, or how you can use that question as a jumping point for a new topic (such as graph theory or even summing up the first n positive integers).

Questions and Hints

1. (a) Can 2 cards form a Set?

   Hint: What is a Set?

   Solution (for teachers): No! By definition, a Set needs exactly 3 cards.

Discussion: This question is to ensure that all players know what a Set is before trying to answer the rest of these questions. It also builds confidences, as it is a very easy problem if one knows the rules of the game.

(b) Given any two cards, how many cards are in the deck that form a Set with them?

Hint: If you are having trouble seeing the answer, try picking two cards at random, and try to figure out what features a third card would need in order to complete the Set. You may also use Figure 2.1 if you do not have a physical deck with you.

Solution (for teachers): There is exactly 1 card that goes with any pair of cards to form a Set. This is since, for each of the 4 characteristics, if the first two cards have the same type (i.e. both red, or both ovals) then the third card must be the same, and if the first two cards have different types (i.e. one red and one purple, or one oval and one squiggle) then the third card must be different from both. So you only have one choice for what card you must look for (and there is only one of each card in the deck).

Discussion: This question aims to get students familiar with the deck of cards so that they can visualize Sets later on. This is also a very easy question, to help build up some confidence and practice thinking about what a Set is.

2. Intuitive counting questions and set theory:

   (a) Call the collection of cards in the deck of a game of SET®, U. How many cards are in U given that there are no duplicate cards, and each card has exactly 1 out of 3 of each of the 4 features?

   Notation: The set (notice the lowercase "s") U is called the *universe* of the problem, and the size of U (i.e. the number of cards in U) is called the *sample size* of the problem and denoted by $|U|$. This is also called the *cardinality* of U.

   Hint: To check your answer, you can count the number of cards shown in Figure 2.1. Did you come up with a better way to find the size of the deck than just counting one by one? If you did not, try grouping cards together in a methodical way. How many groups are there? How many cards are in each group? Note that there are many natural ways to group these cards together, and one such way is illustrated in Figure 2.1.

   Solution (for teachers): If your student is using multiplicative reasoning, there are $3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81$ cards in the deck. This is because there are 3 choices for each of the 4 properties. If a student gives a solution of $9 \cdot 9$, ask them to describe the 9 groups of 9.

   If your student is using additive reasoning, they might also isolate one property (say "color") and say there are $9 \cdot 3$ red cards, plus $9 \cdot 3$ green cards, plus $9 \cdot 3$ purple cards, which gives them $9 \cdot 3 + 9 \cdot 3 + 9 \cdot 3 = 3(9 \cdot 3) = 81$ cards.

Take a moment to point out that they counted $3(9 \cdot 3)$ because there are 3 colors, each with $9 \cdot 3$ cards. If they are using additive reasoning, try to see if they can break down their thought process to explain how you can count only the red cards. Just as they did in isolating the feature "color," they should move on to picking another property (say "shape"). They might say there are 9 diamonds, plus 9 ovals, plus 9 squiggles. Take a moment to point out that this is also $9 \cdot 3$ because there are 3 shapes, each with 9 cards. Next ask them to break down how they found there are 9 diamonds. They will likely pick another property (say "number") and say there are 3 ones, 3 twos, and 3 threes. Point out that this is $3 \cdot 3$ because there are 3 numbers, each with 3 cards (one for each of the shadings). Finally, if they don't reach the conclusion, point out that that they essentially just showed there are (3 colors) $\cdot$ (3 shapes) $\cdot$ (3 numbers) $\cdot$ (3 shadings) $= 81$ cards.

Discussion: This is the first time the student is asked to use multiplicative reasoning in this series of problems. If they are using additive reasoning here, the teacher is asked to help the student also understand how to think multiplicatively too. The cards allow the students to actually visualize each of the cards, so they can group the cards naturally. In Figure 2.1, they can see an orderly way to organize the deck. The point of looking at it organized in such a way is so that the student can multiply the number of groups they see, by the number in each group.

(b) Call the set of cards with diamonds on them D. How many cards have diamonds on them? In other words, find $|D|$.

Hint: To check your answer, you can count the cards one by one, but make sure you can think of another way to find $|D|$ besides counting one by one.

Solution (for teachers): Since there are 81 cards total, and exactly 1/3 have diamonds on them, there are $81/3 = 27$ cards with diamonds. Alternatively, there are $3^3 = 27$ cards with diamonds on them since if we fix the property diamonds, then there are still 3 choices for each of the other 3 properties (number, shape, and shading).

Discussion: This questions is to reinforce the concepts of cardinality, multiplicative reasoning, and getting more familiar with the deck of cards.

(c) Call the set of cards with no diamonds on them D^C. How many cards do not have diamonds on them? In other words, find $|D^C|$. Note: D^C is common notation to indicate the *complement* of a set D. See question 3 for more information.

Solution (for teachers): Since there are 81 cards, and 27 with diamonds on them, there are $81 - 27 = 54$ cards with no diamonds on them. Alternatively, there are the same number of cards with diamonds as there are of either cards with ovals on them, or cards with squiggles on them, thus there are 27 cards of each. This means there are $27 + 27 = 54$ cards that do not have diamonds. Alternatively, since there are 3 shapes, and 2 of those shapes are not diamonds, 2/3rds of the deck do not have diamonds, so there are $81 \cdot \frac{2}{3} = 54$ cards without diamonds on them.

Discussion: This is the first time the students are exposed to *complements*. This question is to give students an opportunity to use deletion thinking. If they do not catch this way of thinking yet, the next few questions are designed to give the students several more opportunities to try. In this problem, if they did not use deletion thinking, they likely will reinforce their multiplicative reasoning.

(d) How are the sets D and D^C related to U?

Notation: When two sets (such as D and D^C) have the same relationship you just observed with the universe U, we call those two sets *complements* and denote the complement of a set with a superscript C in the set, like we did with D^C to denote the complement of D.

Solution (for teachers): Make sure the students get the correct definition of a complement. They should have observed that the D^C was exactly all of the elements in U that are not in D. If they say that U is just the sum of all the elements in D^C and D, make sure to point out that D and D^C are have no elements in common.

Discussion: Everyone has been exposed to set complements before (i.e. the set of odd integers and the set of even integers), but they may not have called these sets complements. This question is designed to be a gentle introduction to some common notation and terms.

(e) What is another was to describe the set of cards with no diamonds on them? If you are having trouble on this question, see the next question first, and come back to this one.

Hint: If there are no diamonds on the card, what do we know has to be on the card?

Solution (for teachers): The set of cards with either a squiggle or an oval on it.

Discussion: This question was designed to help students see that we can typically describe sets several ways, and that if there answer does not match their friend's answer, then neither student is necessarily wrong. Many students at this age might assume that there is exactly one correct answer to each problem, and all other ideas are wrong.

(f) Call the set of cards with squiggles or ovals on them A. How many cards have squiggles or ovals on them? That is, find $|A|$.

Solution (for teachers): The set of cards with squiggles or ovals on them are exactly the set of cards without diamonds on them. We have already counted this: 81 total cards, minus 27 diamonds leaves $81 - 27 = 54$. Or, if the student has not yet realized that the cards with ovals or squiggles on them are exactly the cards without diamonds on them, they can count 27 cards with ovals, and 27 cards with squiggles, (and 0 cards with both) for a total of 54 cards with either an oval or a squiggle.

Discussion: If the students did not realize that the set of cards without diamonds on them are exactly the set of cards with ovals or squiggles, then this question is supposed to help them reach that conclusion.

(g) How are the sets A related to D^C?

Solution (for teachers): They are the same! The set of cards with ovals or squiggles is exactly the same set of cards without diamonds. Make sure the student doesn't just say that the two sets have the same number of cards. They can be more specific and say the two sets contain the exact same cards.

Discussion: This question is designed to cement the idea that it is possible to describe a set in multiple ways. Furthermore, students practice either deletion reasoning (either with or without realizing they are using the complement) or additive reasoning.

3. For the following parts, use the following definitions and notation (many of these definitions you learned in the previous question, so use these as a review):

- A *set* is a collection of objects called *elements*. For example, in this game, we will consider sets of cards where the elements in the set will be individual cards.

- The *size* (or *cardinality*) of a set A is the number of elements in the set, and is denoted by $|A|$. For example, in this game, this is just the number of cards in a set of cards.

- The *universe* (or *universal set*) is the entire collection of elements. For example, in this game, the universe is simply the deck of cards.

- The *union* of two sets A and B is the set of all objects in either A or B, and this is denoted by $A \cup B$. For example, in this game, the union of the set of cards with ovals and the set of cards which are purple would be the set of cards that either have ovals on them or are colored purple.

- The *intersection* of two sets A and B is the set of all objects in both A and B, and this is denoted by $A \cap B$. For example, in this game, the intersection of the set of cards with ovals and the set of cards which are purple would be the set of cards that have both ovals on them and at the same time are colored purple.

- The *complement* of a set A is the set of elements not in A, and is denoted by A^C. For example, in this game, the complement of the set of cards with purple ovals is the set of cards which are neither purple, nor have ovals.

- The *set difference* between two sets A and B is the set of elements which are in A, but not in B, and is denoted by $A \setminus B$. For example, in this problem, the set difference of purple solid cards and squiggle cards would be the set of cards that are both purple, solid, and do not have squiggles.

- The set B is a *subset* of the set A, if all elements in B are also in A, and this relationship is denoted by $B \subseteq A$. For example, in this game, the set of cards with purple squiggles is a subset of the set of cards with squiggles.

(a) Consider the set of cards which both have ovals and are colored purple. Call this set A.

   i. Are there any cards in A? If yes, how many are there?

   Solution (for teachers): Yes. See Figure 2.1 to observe that there are 9.

   Discussion: This part of the question is to get the students to start thinking about how two sets can have a nonempty intersection (as opposed to question 2 parts e and f).

   ii. Would these cards be in the union of the set of purple cards and the set of cards with ovals?

   Solution (for teachers): Yes. They are in the union, but they do not make up the entire union.

   Discussion: This part is designed to get the students more familiar with common terminology.

   iii. Does the previous part mean that A is a subset of the union of the set of cards which are purple and the set of cards with ovals?

Hint: See the definition of *subset* at the beginning of question 3.

Solution (for teachers): Yes! The cards contained in A are cards which have both ovals and are colored purple, so these cards are contained in the set that has all cards with ovals or which are purple.

Discussion: This question lets the student practice with definitions of what it means to be a subset and with the idea of the union of two sets.

iv. Would these cards be the union of the set of purple cards and the set of cards with ovals?

Hint: Notice that this part is different from the previous part in that here, we are asking if the set under consideration is, by definition, exactly the union of the set of purple cards and the set of cards with ovals, whereas the previous part wanted to know if the set under consideration can just be found within the union of the set of purple cards and the set of cards with ovals (and not necessarily contain all elements in the union).

Solution (for teachers): No. The union of these two sets would be the cards which have either purple or ovals; the cards do not need to have both properties to be in the union.

Discussion: This part is designed to reinforce terminology and attempt to distinguish union from intersection and what it means to be a set from what it means to be a subset.

v. Would these cards be in the intersection of the set of purple cards and the set of cards with ovals?

Solution (for teachers): Yes. By definition, the cards under consideration are exactly the cards in the intersection.

Discussion: This part is aimed to give the students practice with understanding the terminology. Additionally, this parts gives the students an opportunity to realize that the set under consideration in this part is exactly the intersection of purple cards and the set of cards with ovals.

vi. Would these cards be the intersection of the set of purple cards and the set of cards with ovals?

Hint: Notice that this part is different from the previous part in that here, we are asking if the set under consideration is, by definition, exactly the intersection of the set of purple cards and the set of cards with ovals, whereas the previous part wanted to know if the set under consideration is merely contained in the intersection of the set of purple cards and the set of cards with ovals (and not necessarily contain all elements in the intersection).

Solution (for teachers): Yes. Not only is this set contained in the intersection, but it happens to contain every element in the intersection, so it is, by definition, the intersection.

Discussion: This part is to contrast the situation where the set of cards under consideration was contained in the union, but not equal to the entire union. The students should now have an example they can go back to when referring to the difference between union and intersection, and what it means to be contained in a set vs. what it means to be exactly equal to a set.

vii. Would you say that the intersection of purple cards and the set of cards with ovals is a subset of the union of purple cards and the set of cards with ovals?

Hint: See previous parts of this question to try to label the union and the intersection of purple cards and the set of cards with ovals, and then refer back to the definition of a subset to decide if the intersection is a subset of the union.

Solution: Yes. All elements of the intersection are in both the set of purple cards and the set of cards with ovals, so they are each certainly in the union.

Discussion: This part is designed to make sure the students did not miss the fact that the intersection was contained in the union, but are not necessarily the same thing.

(b) For the following parts, let the set of cards with squiggles on them be denoted by A, and the set of cards with light shading on them be denoted by B.

i. What is $|A|$? What is $|B|$?

Hint: See problem 2, part d.

Solution (for teachers): The answer is 27 for both questions.

Discussion: This part is to gently remind students that there are 27 cards of each type (that is, 27 solid cards, 27 ovals, 27 purple, 27 red, etc.) and to get them more comfortable with cardinality notation.

ii. How can you describe the set of cards denoted by $A \cap B$? That is, which cards would in in $A \cap B$?

Solution (for teachers): These are the cards that are in the intersection of sets A and B, so that means these are the cards that have squiggles and are lightly shaded (at the same time).

Discussion: This question is designed to get them familiar with intersection notation and being able to describe physically what is in the intersection of two sets.

iii. What is $|A \cap B|$?

Solution (for teachers): 9. The student can count these one by one, or they can see that two properties are fixed, but the other two properties (number and color) are free, and so there are $3 \cdot 3$ combinations of cards with both light shading and squiggles.

Discussion: This is to help set the student up for discovering the fact that $|A \cup B| = |A| + |B| - |A \cap B|$.

iv. How can you describe the set of cards denoted by $A \cup B$?

Solution (for teachers): These are the cards that are in the union of sets A and B, so that means these are the cards that either have squiggles or are lightly shaded (does not have to be at the same time).

Discussion: This question is designed to get them familiar with union notation and being able to describe physically what is in the union of two sets.

v. What is $|A \cup B|$?

Solution (for teachers): $|A \cup B| = |A| + |B| - |A \cap B|$ so $|A \cup B| = 27 + 27 - 9 = 45$.

Discussion: This is to help set the student up for discovering the fact that the $|A \cup B| = |A| + |B| - |A \cap B|$.

vi. How are $|A \cup B|$, $|A \cap B|$, $|A|$, and $|B|$ related? Can you come up with a formula for this relation? Does this always work in the situation where sets A and B come from the deck of cards in a game of SET®? How do you know?

Solution (for teachers): $|A \cup B| = |A| + |B| - |A \cap B|$. As the teacher, you may want to introduce Venn diagrams at this point (not done in this worksheet). This always works in these finite situations because the number of cards in either A or B would be the set of cards in A, plus the cards in B, minus the cards you counted twice.

Discussion: This is a fundamental application of the Inclusion-Exclusion Principle.

(c) Let A be the set of lightly shaded cards, B be the set of purple cards, C be the set of cards with one shape on it, D be the set of cards with ovals, and E be the set of cards with diamonds.

i. Find $|A \cap B|$.

Solution (for teachers): 9. There are 9 cards that are purple and lightly shaded. See the 8th row of Figure 2.1 for the specific 9.

Discussion: This question is to gives more practice with the concept of the intersection of two (non-disjoint) sets. Additionally, it give more practice with cardinality.

ii. Find $|C \cap D|$.

Solution (for teachers): There are 9 cards with 1 oval on them. See the 4th column of Figure 2.1 for the specific 9.

Discussion: This question is aimed to let the students feel confident that they have a firm grasp of the cards in the deck, and can understand the idea of the intersection of two (non-disjoint) sets. Additionally, it gives more practice with cardinality.

iii. Find $|D \cap E|$.

Notation: Sets of the size you just observed are *empty sets* and are denoted by $\emptyset$.

Solution (for teachers): 0.

Discussion: This part is designed to just make sure the students notice that not all sets have elements in their intersection. This will also help with the next part.

iv. Find $|D \cup E|$.

Solution (for teachers):$|D \cup E| = 54$ since $|D| = 27$, $|E| = 27$, and $|D \cap E| = 0$.

Discussion: This question is aimed to give students more practice with Inclusion-Exclusion. Additionally, it give more practice on cardinality.

v. Notice that $A \cap B$ is a set and $C \cap D$ is a set, so we can find the intersection of them. Describe $(A \cap B) \cap (C \cap D)$ and find $|(A \cap B) \cap (C \cap D)|$.

Solution (for teachers): The set $(A \cap B) \cap (C \cap D)$ is the set containing all cards that are in each A, B, C, and D at the same time. That is, the set of cards that are lightly shaded, purple, have one shape that is an oval. Since $|(A \cap B) \cap (C \cap D)|$ is the number of lightly shaded purple single ovals, it is clear that $|(A \cap B) \cap (C \cap D)| = 1$.

Discussion: This question is designed to give students practice with the intersection of several sets. Although $(A \cap B) \cap (C \cap D) = A \cap B \cap C \cap D$, I chose to ease the students into recognizing that they can find the intersection of four sets, by first considering considering $A \cap B$ and $C \cap D$, and then (after emphasizing that these themselves are sets) finding the intersection of these new sets.

vi. Describe the cards in $A \setminus C$ and find $|A \setminus C|$.

Hint: See the definition of the *set difference* at the beginning of question 3.

Solution (for teachers):

One solution: The set $A \setminus C$ is the set of cards that are in A, but not in C. That is, the set of cards that are lightly shaded, but not the cards with one shape on them. You can see this set as the set of cards that are in rows 2, 5, and 8, but not in columns 1, 4, or 7 in Figure 2.1. Since $|A \setminus C|$ is the number of cards we are considering, $|A \setminus C| = 3 \cdot 9 - 3 \cdot 3 = 27 - 9 = 18$.

Another solution: Alternatively, we can also describe this as the set of cards that are lightly shaded and have either two or three shapes on them, and can be any of the 3 colors and any of the 3 shapes. Hence $|A \setminus C| = 1 \cdot 2 \cdot 3 \cdot 3 = 18$.

Discussion: This problem is designed to give students an idea of what the difference of two sets is and to practice with the *set difference* notation.

vii. Describe the cards in A^C and find $|A^C|$.

Solution (for teachers):

One solution: The set A^C is the set of cards that are *not* in A. So it is the set of cards that are not lightly shaded. We have counted similar sets before (see problem 2 part c above) and have seen that there are $81 - 27 = 54$ cards that are not lightly shaded, so $|A^C| = 54$.

Another solution: Alternatively, we can also describe this as the set of cards solid shading, or no shading, so $|A^C| = 27 + 27 = 54$.

Discussion: This question is designed to give students more practice with complements and cardinality. They have seen before (as in problem 2 part c) that there are 27 cards of each characteristic, and now they get to recall and reapply that observation.

4. Five kids are going to have a SET® tournament. In this tournament, two kids will play each other at a time, and each kid plays each other kid exactly once. How many games will have been played after the tournament is over?

Hint 1: Here is one possible way to start: Say the 5 kids playing are Alan, Billy, Corey, Danielle, and Elizabeth.

(a) How many games will Alan be in? Keep in mind that Alan plays each of the other 4 kids exactly once.

(b) How many games will Billy be in?

(c) How many games total were Alan or Billy in? (If you said 8, you made a really common mistake, and you should list out the pairings for each of the games. Do you see what you overlooked?)

(d) How many games were Alan, Billy, or Corey in?

(e) How many games were Alan, Billy, Corey, or Danielle in?

(f) Are there any games that you have not counted yet? If yes, which two people are playing?

Hint 2: Try drawing a diagram of some kind that can illustrate the situation of this tournament.

Follow up hint: If you are having trouble thinking of a way to draw a diagram for this, try the following: Draw 5 dots, with each dot representing one kid. Now draw lines between two dots to represent a game played between those two kids. Draw all lines that would represent all games played in the tournament. How many lines did you draw?

Solution (for teachers):

Following hint 1:

(a) Alan will be in 4 games since he has to play each other kid exactly once.

(b) Billy will also be in 4 games since he too must play each kid exactly once.

(c) Alan and Billy together will be in 7 games total since Alan plays each of the 4 kids (including Billy) and Billy must play all 4 children too, but we've already counted the game he is playing against Alan, so we only need to count the other 3 games Billy is in. Thus the total Alan and Billy play in is $4 + 3 = 7$.

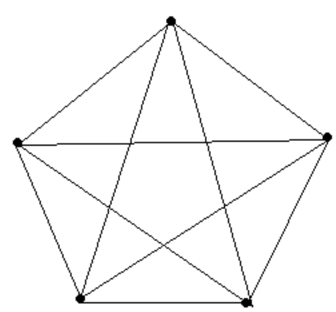
(d) Alan, Billy and Corey are in $4 + 3 + 2 = 9$ games since in addition to the $4 + 3 = 7$ games Alan and Billy are in (including the two against Corey), Corey is in another two games that we haven't counted yet (against Danielle and Elizabeth).

(e) Following the pattern, Alan, Billy, Corey and Danielle are in a total of $4+3+2+1 = 10$ games since in addition to the $4+3+2 = 9$ games Alan, Billy and Corey are in (including the 3 against Danielle), we still need to count the game that Danielle plays against Elizabeth.

(f) We have counted all possible games! If there are only 5 players, the fifth player (Elizabeth) can only play against the other 4 players and we've already counted the games that the other players are in.

Thus, the total number of games played in a tournament with 5 players in which each player must play every other player exactly once is 10.

Figure 2.2: Diagram Representing a Tournament with 5 Players



Following hint 2: If we draw this diagram, we get something like illustrated in Figure 2.2, where each dot (i.e. each node or vertex) represents a child, and each line (i.e. edge) connecting the dots represents 1 game played between the two corresponding children. Notice there are 10 total lines (i.e. edges), thus there would be 10 games played.

Discussion: This question can be taken at least two ways, depending on which hint the student follows. If they follow hint 1, this question can easily lead up to summing the first k integers. If they follow hint 2, then they get exposed to basic graph theory. If they are encouraged to follow both hints, then they can see the beauty between the number of edges on a complete graph with $k+1$ vertices (that is, a graph in which each vertex is adjacent to each other vertex) and the sum from 1 to k. See Bondy and Murty (2008) for more about graph theory.

See additional discussion below.

5. Answer question 4 with n kids instead of just 5. Note the answer should be in terms of n.

   Hint: If you didn't discover a pattern in question 4, either try it again following hint 1, and/or try it again with 6 kids. If you still can't see a pattern, try a few more small examples.

   Solution (for teachers): There are

$$(n-1)+(n-2)+(n-3)+\ldots+3+2+1$$

   games played (note that this sum equals $\frac{(n-1)n}{2}$, and it could be a future questions to have students figure out this closed form). This follows the pattern in hint 1 from question 4, since the first kid plays each of the other kids so she is in $n-1$ games, then the second kid is in an additional $n-2$ games, etc.. Do not worry if students do not get the closed form answer for this question.

Discussion: See the discussion for the above problem for motivation to include this problem.

This question would be a good introduction to combinations. To count the number of edges on a complete graph with $n-1$ vertices, you can note that since each vertex is adjacent to every other vertex, this is the number of ways to choose 2 of the vertices and draw an edge between them. Since there are $n-1$ vertices, this means there are $\binom{n-1}{2} = \dfrac{(n-1)n}{2}$ edges. See Bondy and Murty (2008); Bogart (2005) or any other resource on combinatorics for more information on notation and uses.

6. (a) If you were to go through the deck of cards in SET® (see Figure 2.1 for the complete deck of cards) and systematically lay out groups of 3 cards that form Sets (see prerequisites above for the rules of SET® and what conditions you need to satisfy to ensure your 3 cards form a Set), is it possible to lay out all of the cards? In other words, can you arrange all of the cards in the deck into Sets, with no card left out? If yes, is there exactly one way to arrange the cards into groups, or more than one way? If no, why not?

   Hint: Try it! Use a physical deck of cards or see Figure 2.1 to help.

Solution (for teachers): Yes. There are several ways to lay out all of the cards in a deck of SET® so that the cards are groups into Sets. One way is to go across the rows in Figure 2.1 and take the first 3 cards, then the next 3, and continue this way across all rows to group the deck into Sets. Another way would be to do the same process, but go down along the columns instead. There are several other ways, but those two are most obvious from the way the cards are laid out in Figure 2.1.

Discussion: This problem is to introduce the idea of a *partition*.

(b) If you complete the task of grouping all cards into Sets, how many Sets will there be?

Solution (for teachers): There will be $81/3 = 27$ Sets since each of the 81 cards must be used and each Set contains 3 cards.

Discussion: Even if the student could not figure out a nice way to partition the deck of SET® cards, they can still make the observation that a partition of a deck needs to include all cards in the deck, and if each part is of size 3, then there will be 27 parts.

7. (a) If you are in the middle of a game, and you are about to claim a Set that you see, how many ways can you pick up those 3 cards (assuming you can only pick up one at a time)?

Hint 1: You can always label your three cards as 1, 2, and 3, and then list out all possible orders (i.e. 312 could mean you picked up card 3, then card 1, and then card 2).

Hint 2: How many choices do you have for your first card? After you choose your first card, how many choices do you have for your second card? Third card?

Solution (for teachers): Following hint 1, the possible orders are 123, 132, 213, 231, 312, and 321, so there are 6 total ways. Following hint 2, you have 3 choices for the first cards, then 2 choices for your second card, then only 1 choice for the last card, so overall you have $3 \cdot 2 \cdot 1 = 6$ choices.

Discussion: This is the first question that deals with permutations.

(b) How many ways can you pick up 5 cards (assuming you can only pick up one at a time)?

Hint: Use the hints from part (a). Notice hint 1 could get tedious, so try to follow hint 2 here. If you are having trouble understanding how hint 1 and hint 2 will give you the same answer here, go hack and redo a with both hints to compare, and then come back and try hint 2 on this question.

Solution (for teachers): Following hint 2, there are $5 \cdot 4 \cdot 3 \cdot 2 = 5! = 120$ ways to do this.

Discussion: Here is a second chance at permutations. Even if the student did not understand how to count the number of permutations in part (a), hopefully they went back and followed hint 2 to be exposed to counting permutations efficiently (i.e. using multiplication instead of just listing out all possible permutations).

Chapter 3

ADDITIONAL ACTIVITIES WITH PROBLEMS, HINTS AND SOLUTIONS

3.1 Entertaining Your Friends

To the student: The kids are off playing SET®, but you have to entertain your guests. You plan on dancing and playing games. While planning activities for your friends, you run into the following questions.

3.1.1 Activity 2: More Permutations, Combinations, and Basic Graph Theory

What the Student Will Learn

To the teacher: In this section, the student will be exposed to more counting problems, including the classic "handshake problem." Additionally, they will develop some graph theory concepts. See Hart (2008); Bondy and Murty (2008) for further reasons to include more graph theory and some sources for what you could include. Following most questions are hints for the students, and solutions for the teacher to divvy out as she sees fit. These questions are in no way exhaustive of the types of questions you could ask or topics you could investigate that could relate to other games. See Chapter 4 for some more ideas of what to include in these questions.

Questions and Hints

1. At a party 6 boys and 6 girls dance together. In how many ways can they make couples for dancing, provided each couple is made up of one boy and one girl?

Hint: Name the girls A, B, C, D, E and F. If girl A gets to pick a boy first, how many choices does she have? How many choices are left for B to pick from? What about for the rest of the girls?

Solution (for teachers): The first girl has 6 choices, the second girls has 5 boys left to choose from, the third girl has 4 choices, and if you continue this way, the sixth girl will only have 1 boy to choose from, so there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6! = 720$ ways to form couples.

2. You plan on playing team games, and naturally there will be cheering and high fiving when teams win different games.

   (a) If there are 5 people on a team, how many high fives will there be assuming there can only be two people involved in a high five and each person high fives each other person exactly one time?

   Hint: Draw five dots spread out around a circle where each dot represents one person. Call each dot a *vertex* (plural would be *vertices*). Draw a line connecting vertices if they high five. Call a line between two vertices an *edge*. How many edges did you draw? See question 4 to check for similarities.

   Follow up hint: Did you draw all of your edges in a methodical way? One methodical way would be to first start at one vertex and connect it to each other vertex. Then move on to another vertex and draw all of the other edges that would come from it that you didn't draw already. Then keep doing this until you can't add any more edges. If you did not draw your edges like this, try to do so right now. How many edges did you draw when you looked at your first vertex? How many more did you draw at your second vertex? How many more did you draw at your third? Can you see a pattern?

Solution (for teachers): See the graph in Figure 2.2 to see a diagram of this situation. There are $4 + 3 + 2 + 1 = 10$ high fives exchanged.

(b) If there are n people on a team, how many high fives will there be assuming there can only be two people involved in a high five and each person high fives each other person exactly one time? (Note: Your answer should be dependent on n.)

Hint: see the "follow up hint" from part (a). Or, see the next section, question 2. In particular, look at part f.

Solution (for teachers): $(n-1) + (n-2) + \cdots + 1 = \frac{(n-1)n}{2}$. Do not worry if the student can't get the closed form yet. This could be a good time to explore the mathematics folklore about how young Gauss added the first 100 positive integers so quickly. Or, that can be a future question.

3.2 Dinner

To the student: You still need to feed your party guests. While trying to figure out where everyone will be sitting and what each guest will be eating, you find yourself trying to answer the following questions.

3.2.1 Activity 3: More Permutations and Combinations

What the Student Will Learn

To the teacher: In this section, students will be exposed to more counting problems. Here, they will use the techniques and knowledge they developed in the previous section to aid in solving these more complicated questions. Following most questions are hints for the students and solutions for the teacher to divvy out as she sees fit. These questions are in no way exhaustive of the types of questions you could ask or topics you could investigate that could relate to ordering around tables, and counting combinations.

Questions and Hints

1. You must figure out how to order and seat your guests.

    (a) If you have 14 guests, how many ways are there for your guests to line up for food?

    Solution (for teachers): There are $14 \cdot 13 \cdot 12 \cdot 11 \cdot \ldots \cdot 3 \cdot 2 = 14!$ ways for your guests to line up, since there are 14 choices for first in line, 13 choices remaining for who can be second in line, 12 choices for third in line, and so forth, until finally someone must be last in line.

(b) If you have 14 guests, how many ways are there for them to sit down at a big round table? Assume that you can't tell the difference in the seating arrangement if the table is rotated.

Hint: For any arrangement, try to focus on one person and then count how many ways she can see people arranged if she always looks around the table, say to her left.

Solution (for teacher): Following the hint, we see that it doesn't matter where the first person sits, but then there are 13 choices for who can sit directly to her left, then 12 choices for two people down on her left, then 11 choices, and so forth, until finally there is only one person not sitting and they must sit by her (immediately to her right). This means there are $13 \cdot 12 \cdot 11 \cdot \ldots \cdot 3 \cdot 2 = 13!$ ways to arrange your guests around a circular table.

2. You have some picky guests. Rather than give them all the same food to eat, you generously offered choices. For dinner, your guests have 3 choices for protein, 4 choices for vegetables, and 6 options for drinks. Note: Parts of this question could be difficult if the student has not yet seen combinations. If the student is not familiar with this yet, see Bogart (2005), Bondy and Murty (2008) or Rosen (2012) before attempting.

   (a) How many possible dinner combinations do your guests each have to choose from if they must have exactly 1 protein, 1 vegetable, and 1 drink?

   Solution (for teachers): There are $3 \cdot 4 \cdot 6 = 72$ possible dinner orders.

(b) How many possible dinner combinations do your guests each have to choose from if they must have exactly 1 protein, 2 vegetables, and 1 drink?

Solution (for teachers): There are $3 \cdot \binom{4}{2} \cdot 6 = 3 \cdot \frac{(4-2)!}{2!} \cdot 3 = 3 \cdot 6 \cdot 6 = 108$ possible dinner combinations.

(c) How many possible dinner combinations do your guests each have to choose from if they must have the choice of 1 or 0 proteins, 1 or 0 vegetables, and 1 drink?

Hint: Break up the possible dinners into groups based on how many proteins, vegetables, and drinks each has, then count the number of combinations in each group. What do you do with all of your counts for the groups to find the total dinner combinations?

Solution (for teachers): There are $3 \cdot 4 \cdot 6 = 72$ ways to have 1 protein, 1 vegetable, and 1 drink; $3 \cdot 1 \cdot 6 = 18$ ways to have 1 protein, 0 vegetables, and 1 drink; $1 \cdot 4 \cdot 6 = 24$ ways to have 0 proteins, 1 vegetable, and 1 drink; and $1 \cdot 1 \cdot 6 = 6$ ways to have 0 proteins, 0 vegetables, and 1 drink. Thus, in total, there are $72+18+24+6 = 120$ possible dinner combinations.

3. You have some After Dinner Mints, for after dinner, and you want to ask each of your guests how many they would like so you can go get enough from the kitchen for each person.

(a) If each of your n guests wants 2 mints, how many do you have to bring back?

Solution (for teachers): You need to get $2n$ mints.

(b) If you have an even number of guests and exactly half want 1 mint and the other half want 3 mints, how many do you have to bring back?

Solution (for teachers): You need $\frac{n}{2} \cdot 1 + \frac{n}{2} \cdot 3 = \frac{n}{2} \cdot 4 = 2n$ mints.

(c) Is there anything interesting about how parts (a) and (b) relate? Can you explain why this works?

Hint: In part (b), imagine pairing off your guests so that each pair has exactly one person that wanted 1 mint, and one person that wanted 3 mints. How many mints do you have to bring back for each pair? Now in part (a), think about pairing off people too. How many mints do you have to bring back for each pair?

Solution (for teachers): They are the same! Since in part (b), half your guests want 1 mint, and the other half wants 3 mints, pair everyone that wants 1 mint with someone that wants 3 mints, so each pair wants 4 mints. Thus, this is the same problem as part (a), since, on average, everyone wants 2 mints. So go back and get $2n$ mints.

(d) Imagine you have 10 guests, and each guest wants a different number of mints than anyone else wants, and each person wants at least 1 mint and at most 10 mints.

i. Will anyone get exactly 6 mints? Will anyone get exactly 10 mints?

Hint: How many different numbers are there between 1 and 10 including 1 and 10? List out these numbers. How many guests do you have? Call them A through J, and list them below your numbers 1 through 10. Imagine drawing a line between A and 2 if guest A gets 2 mints. Is there any way you can draw a line from each letter to a different number of mints, so that no letter has a line to 6? If no, can you argue why this is impossible?

Solution (for teachers): Following the hint, since there are 10 numbers between 1 and 10 (including both 1 and 10), and each of your 10 guests requires a different number of mints from 1 through 10, you must use both 6 mints and 10 mints. If you did not use some number of mints, you are trying to draw 10 edges from 10 vertices to 9 other vertices, and so at least two edges must go to the same vertex. This is not allowed since it means two guests will have the same number of mints.

ii. How many mints will you have to bring back?

Hint: Instead of adding them up one at a time, try to pair them up conveniently first. Try grouping 1 with 10, 2 with 9, 3 with 8, etc.

What do you notice about the sum of each pair? How many pairs do you have? So how many mints do you need?

Solution (for teachers): You need to bring back $1+2+3+4+5+6+7+8+9+10 = (1+10)+(2+9)+(3+8)+(4+7)+(5+6) = 11+11+11+11+11 = 110 \cdot 5 = 55$ mints.

(e) Imagine you have 11 guests, and each guest wants a different number of mints than anyone else wants, and each person wants at least 1 mint and at most 11 mints. How many mints will you have to bring back?

Hint: Instead of adding them up one at a time, try to pair them up conveniently first. Try grouping 1 with 11, 2 with 10, 3 with 9, etc. Does each number have a pair, or is someone left off? What do you notice about the sum of each group (where each group is either a pair or a single number)? How many groups do you have? So how many mints do you need?

Solution (for teachers):You need to bring back $1+2+3+4+5+6+7+8+9+10+11 = (1+11)+(2+10)+(3+9)+(4+8)+(5+7)+6 = 12+12+12+12+12+\frac{1}{2}\cdot 12 = \frac{11}{2}\cdot 12 = 66$ mints.

(f) Imagine you have n guests, and each guest wants a different number of mints than anyone else wants, and each person wants at least 1 mint and at most n mints. what can you conclude about this? How many mints do you have to bring back?

Hint: See hints for the previous two parts. Break it up into cases where n is odd and then do when n is even. Do you get the same number?

Solution (for teachers): If n is even, there will be $\frac{n}{2}$ pairs of people that want $n+1$ mints. So a total of $\frac{n(n+1)}{2}$ mints. If n is odd, there will be $\frac{n-1}{2}$ pairs of people that want $n+1$ mints, but one person that doesn't get a pair, and this single person will want $\frac{n+1}{2}$ mints. This totals to be $\frac{(n+1)(n-1)}{2} + \frac{n+1}{2} = \frac{(n+1)(n-1+1)}{2} = \frac{n(n+1)}{2}$. Note that this is the same, regardless of n being even or odd.

(g) What does the previous part have to do with section 3.1 question 2b?

Solution (for teachers): These are the same, except section 3.1 question 2b is the sum of 1 through $n-1$, and here we have the sum of 1 through n.

Chapter 4

FUTURE DIRECTIONS

Since these questions are aimed at middle school or high school aged students, it is possible that they will need additional small examples. Any problem that asks for a solution in terms of n, could be lead up to by more small cases, such as $n = 3$ or $n = 4$. In the future, I would like to add these smaller cases.

In addition to slowing down on each problem, we could delve deeper in many topics. There are many ways to take the framework of these questions, and these ideas, and explore further. Several questions in Chapter 2 are related to some other very fun problems, and those are mentioned in the discussions. There are many different topics to explore that easily fit in the framework of "the student is throwing a birthday party." It would be nice to discuss more games. You could play with various easy counting problems and talk about interesting topics like counting necklaces with various beads available which could also lead to discussions of equivalence classes and more about partitions. Once students have sufficient practice at counting sets, questions about probability could easily follow. Most of the questions in this thesis are set theory questions, but there are many topics in graph theory that could be introduced to young students without needing to teach them much background first. There are many problems that can be added that give the student the opportunity to draw a graph or diagram the situation in some way. There are many sources for questions in which diagrams are helpful (Bogart, 2005; Bondy and Murty, 2008; Rosen, 2012).

Some ways to fit these topics into the party planning framework would be to add sections like "Order of Activities" where the scenario for the student could be: "You have every part of your party planned, except the order in which you are going to enjoy each part! While trying to create a schedule for your party, you run into the following problems and possible ways to organize the events." Then, you could include decision trees and flow diagrams (Rosen, 2012); more counting problems from counting possible orders of events; scheduling problems; and even optimization problems (like the traveling salesman problem) and modeling other real world problems with graph theory. Some sample questions/hints or topics that could be covered include the following:

1. You plan on making dinner, making dessert, setting the table for dinner, playing games, and dancing.

   Hints for the parts below: If you get stuck on any of the following, try doing a smaller example if you need to. Try drawing a diagram to represent the decisions you need to make. How would your diagram look different in the situation of restrictions as opposed to no restrictions?

   (a) How many options do you have for what order are you going to do all of your planned activities in (assuming you can only do one at a time)?

   (b) How many options do you have for what order are you going to do all of your planned activities in if dessert must come after dinner and you must set the table before you eat?

(c) How many options do you have for what order are you going to do all of your planned activities in if you have 3 games that you can play, and you can play them in any order and at any time in the evening?

Hint: list out all of the activities you now need to schedule, calling the 3 games game 1, game 2, and game 3. How is this problem different from the first problem?

(d) How many options do you have for what order are you going to do all of your planned activities in if you have 3 games that you can play, but you have to play all of the games (in any order) before moving on to your next non-game activity?

(e) Can you think of any other restriction might you need to impose? Can you count how many options do you have for what order are you going to do all of your planned activities in with your new restriction?

Besides just sequencing more questions, in the future I would like to provide a way for students to have guidance and the ability to check their answers if a teacher is not available. If there is not a way for a teacher to be present with the student almost continually, I would still like to alleviate as much potential frustration as possible. In the future, I propose using WeBWorK (http://webwork.maa.org/), or some other online answer checker to help the student ensure they are indeed on the right track. In particular, for problems with multiple parts in which each part depends on either the answer or reasoning behind the answer to the previous part, it can be tremendously frustrating to get all subsequent parts incorrect. In fact, if a student has incorrect reasoning for the first part, and continues with the subsequent parts using the same logic, then it is likely that the student will cement the incorrect line of reasoning in their mind (Paul A. Kirschner and Clark, 2006). It would be much better to correct any misconceptions before they solidify. Especially if there is not a teacher around, WeBWorK could be an excellent resource while discovering combinatorics. So many discrete math questions have a correct answer and the answer is a simple number. For problems in which a student is asked to count something, it would be very beneficial if the student were able to check their count before moving on to the next part of a question. One caveat of using WeBWorK would be that there is no way to gauge which hints a student might need. When using WeBWorK, I would attempt to include enough hints in the questions to help the students, but not enough as to spoil the discovery.

REFERENCES

Bogart, K. P., *Enumerative Combinatorics Through Guided Discovery* (Free Software Foundation, 2005).

Bondy, J. A. and U. S. R. Murty, *Graph theory*, vol. 244 of *Graduate Texts in Mathematics* (Springer, New York, 2008).

DeBellis, V. and J. Rosenstein, "Discrete mathematics in the schools: Experiences from the U.S.A.", Mathematics in School (2008).

Friedler, L. M., "Problem solving with deiscrete mathematics", Teaching Children Mathematics (1996).

Hart, E. W., "Vertex-edge graphs: An essential topic in high school geometry", The Mathematics Teacher (2008).

Mayer, R. E., "Should there be a three-strikes rule against pure discovery learning?", American Psychologist (2004).

Paul A. Kirschner, J. S. and R. E. Clark, "Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching", (2006).

Rosen, K. H., *Discrete Mathematics and Its Applications* (McGraw-Hill, 2012), 7 edn.