An Investigation of the Teaching and Learning of Function Inverse

by

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ABSTRACT

Based on poor student performance in past studies, the incoherence present in the teaching of inverse functions, and teachers' own accounts of their struggles to teach this topic, it is apparent that the idea of function inverse deserves a closer look and an improved pedagogical approach. This improvement must enhance students' opportunity to construct a meaning for a function's inverse and, out of that meaning, produce ways to define a function's inverse without memorizing some procedure. This paper presents a proposed instructional sequence that promotes reflective abstraction in order to help students develop a process conception of function and further understand the meaning of a function inverse. The instructional sequence was used in a teaching experiment with three subjects and the results are presented here. The evidence presented in this paper supports the claim that the proposed instructional sequence has the potential to help students construct meanings needed for understanding function inverse. The results of this study revealed shifts in the understandings of all three subjects. I conjecture that these shifts were achieved by posing questions that promoted reflective abstraction. The questions and subsequent interactions appeared to result in all three students moving toward a process conception of function.

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DEDICATION

This thesis is dedicated the people who have made my last two years possible. First of all, I dedicate this to Marilyn Carlson for all of her encouragement and for having read each of these pages at least 10 times. I dedicate this to Scott and Kristin for keeping me sane and happy. And, finally, I dedicate this to my parents for always supporting me in every adventure.

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CHAPTER 1

INTRODUCTION TO THE PROBLEM

Consider the common function inverse context:

> The temperature can be converted from degrees Celsius to degrees Fahrenheit by the following formula:

$$F = \tfrac{9}{5}C + 32$$

According to the typical approach to finding a function's inverse, students interchange the variables as a first step. In this example, students would write:

$$C = \tfrac{9}{5}F + 32$$

But, is this a true statement? This statement must be false if the original was true. Why do we swap the variables if it leads to a false statement? What meanings might students make of a strategy that produces false statements along the way, but leads to desired answers? I contend that looking past the incoherence of this procedure requires the student to ignore the quantities being related in the given problem, which does not have a beneficial influence on the development of quantitative reasoning.

The common approach for determining an inverse function that is presented in textbooks and used in instruction is to swap x and y in the function formula, and then solve the equation for y. This procedure is typically presented with little or no explanation, leaving students with a weak understanding of the idea of function inverse and no choice but to memorize the steps for defining an inverse. Oehrtman, Carlson, and Thompson (2008) claimed that when students are asked to memorize a procedure without understanding why the procedure works, the students are unable to recognize when to apply the procedure. To complicate matters, Van Dyke (1996) has also noted that, “the idea of undoing what a function did gets lost in the mechanics of switching the x and the y and then solving for the newly named y (p. 121).”

I contend that teaching students to apply this procedure contributes to their developing a view that mathematics is not always logical, but instead is about memorizing rules and procedures. Thompson, Thompson, Philipp, and Boyd (1994) wrote, “the most powerful approach to solving problems is to understand them deeply and proceed from the basis of understanding, and that a weak approach is to search one’s memory for the ‘right’ procedure (p. 11).”

Another commonly taught idea is that the graphs of a function and its inverse are reflected about the line $y = x$ when graphed on the same axes. In a conceptually oriented classroom where the focus is on meaning making (Thompson et. al., 1994), it is likely that numerous attempts are made to have students label the horizontal and vertical axes with the names of the quantities being represented when graphing a function. Then, what sense does it make to place a function and its inverse on the same axes? I contend that ignoring the quantities being related in the context and focusing solely on the geometric shapes of a function and its inverse’s graphs further motivates student beliefs that a graph is just some shape in the plane rather than a mapping between two covarying quantities. Moore, Weber, and Thompson (submitted) describe this tendency of students to consider the graph of a function as an object in the plane and to take cues from the visual properties of the graph as static shape thinking.

Esty (2005) argued that the typical textbook approach to function inverse focuses on an algorithm for finding the inverse of a simple function. In doing so, he claims that the common approaches highlight topics that will not reappear in future mathematics, and fail to focus on related topics that will reappear frequently. Specifically, he argues that the algorithm learned for simple inverses is typically never needed again as students are introduced to inverse trig functions, as well as the inverse relationship between exponential and logarithmic functions.

It is well documented that secondary and college students struggle with the concept of function and have particular difficulty understanding the idea of function inverse (e.g., Carlson, 1998; Harel & Dubinsky, 1992). In a study that Carlson, Oehrtman, and Engelke (2010) performed with over 2,000 precalculus students at the end of their semester, only 17% were able to correctly determine the inverse of a function for a specified value when given a table of the function's values. More recently, the task, "Given that the functions *h* and *k* are inverses of each other, and that $h(\sqrt{2}) = 0.13$ and $k(5) = \sqrt[3]{2\pi}$ determine the value of $k(h(2\pi))$," was given to a class of 30 precalculus students on their final exam at a large public university. Only 30% of the students applied the definition of function inverse and responded with the answer 2π.

Classroom Observations

Having seen the problems with this topic at the university level, I was interested in seeing how secondary teachers approached function inverse. I observed classes of four secondary teachers who were using a conceptually based Algebra II curriculum. Based on previous visits to these classrooms, I expected to see these teachers employing the problems in the curriculum's workbook to lead a discussion on function inverse, while also making slight modifications and varying pedagogical choices of how to utilize the problems. To my surprise, all four of the teachers decided to use materials they had created rather than use the conceptually focused activities that were available to them in their curriculum.

For each lesson I observed, missing were the contextual problems from the curriculum and employed were functions given without context. One teacher informed his students that they were celebrating "Freaky Friday" and that they were swapping all of the x and y variables, similar to the story of how a mother and daughter switched places. Another teacher attempted to begin with the notion of undoing the process of the function by reversing each operation, in the reverse order that they were applied to the input. However, when students appeared confused, he reverted to swapping the variables and solving for y. A third teacher also used the method of interchanging the variables, but she instructed her students to thenceforth, unfortunately, use the notation x' and y' to denote that these were no longer the same variables from the original function. And finally, the fourth teacher had her students "solve for the other variable", but at the end of that process, she had them swap the variables.

My interest peaked after attending four classes, back-to-back, of teachers using the same curriculum, yet each had abandoned the conceptual approach supported by the curriculum for introducing the idea of function inverse and instead each used a different approach that was selected by each individual teacher. I only observed one of these four teachers explain to their students why they were swapping the variables and, unfortunately, the reasoning was that they were celebrating “Freaky Friday”. Therefore, I hypothesize that these students were left with no other reason than to follow their teacher’s lead and memorize the given procedure.

Each teacher not only provided their own problems in place of the contextual problems in the workbook, but also employed their own approach for finding the inverse of a function. Their conceptual Algebra II curriculum never mentions the action of interchanging any variables. Yet, why had these teachers chosen this approach? Up until this point in the workbook, each new idea was presented with a variety of contextual problems, where functions were defined as a relationship between two covarying quantities in a situation, and the variables used in those functions represented all possible values each quantity of interest could assume. In contrast, these teachers had resorted to simple algebraic formulas using the notation of $y = f(x)$ where the variables x and y held no contextual meaning. Why had these teachers discarded the contextual problems provided to them in the workbook for simple (meaningless) algebraic functions?

Further, the Algebra II curriculum used by each teacher included no mention of graphing a function and its inverse on the same axis to illustrate that they were reflections over the line $y = x$. However, each teacher spent a significant portion of their lesson discussing this fact and having their students graph numerous examples. Why was such importance placed on an idea that was absent from their curriculum?

Teacher Interview

Fortunately, some of these motives were revealed in a group interview of three teachers during a professional development workshop focused on their conceptually based curriculum. The group initially consisted of three teachers, from three different high schools, who had chosen to work together on a function inverse investigation. To set the stage, Alice was a veteran teacher of the curriculum who had worked with the program and used its materials for several years. Both Brad and Cheryl were beginning their second year of teaching with the curriculum's Algebra II materials. Upon being asked about their general approach to teaching function inverse, Alice responded by saying that she typically used the problems in the curriculum, but when teaching her students to find a function's inverse she taught her students "to simply solve for the other variable." She followed by saying that students created or were given a function with a particular input variable and a particular output variable and in solving for the other variable, the students were to express the original input variable in terms of the original output variable. In this way, she described the conversation her class had about how the domain and range of the function and those of its inverse were interchanged.

In contrast, Brad and Cheryl's responses were more aligned with the approaches used by teachers that I had observed in the classroom. They each shared that they had chosen to not use the problems provided in their curriculum when teaching about function inverse and instead used materials that had worked for them in the past. Both Brad and Cheryl said that they used the method of swapping x and y, and solving for y. As they were describing their methods, Alice chimed in by saying that her teaching practice placed an emphasis on contextual problems and that she required her students to clearly define their variables. She claimed that this traditional method of swapping the variables "does not make sense to the students when they have clearly defined their variables to represent specific quantities." Brad returned with, "I know that this is a confusing topic, but there are just so many conventions we have to worry about." When probed to discuss this further, he described that his approach to inverse functions was guided by his worry of what his students saw in other textbooks, as a convention used by their graphing calculators, on the SAT®, and the ACT®. He clarified that these exams are so important to his students and that they "always express y as a function in terms of x." As expected, Cheryl agreed, but Alice also agreed and admitted that she had the same worries about her teaching strategies regarding inverses.

Brad went on to say that the approach and the notation Alice used made more sense than his approach of swapping variables, yet by adopting this approach he said he would be "shoveling against the tide." To further clarify this expression he again brought up the conventions that students were expected to use as default when reading other books and taking standardized tests. Cheryl also commented that she would not change her teaching approach and would continue instructing her students to switch the variables and solve for y. It was clear here that factors beyond the intuition and meaningfulness of an approach to understanding function inverse were significantly influencing two of the three teachers' practices.

One thing that all three of the teachers agreed upon was that they would begin their discussion of function inverse using the contextual problems from the investigation. An example that they each highlighted was a problem involving conversions between dollars and euros. Cheryl, who had previously avoided using contextual problems for this topic, remarked that "this type of context gives the students a concrete understanding of the undo idea involved in finding a function's inverse." Brad ended the session by exclaiming, "I just realized that I have been working in a world where functions have no context- x was just *the* input and y was just *the* output."

Based on poor student performance in past studies, the incoherence present in the teaching of inverse functions, and teachers' own accounts of their struggles to teach this topic, it is apparent that the notion of a function's inverse deserves a closer look and an improved pedagogical approach. This improvement must enhance students' opportunity to construct a meaning for a function's inverse and, out of that meaning, produce ways to define a function's inverse without memorizing some procedure.

CHAPTER 2

LITERATURE REVIEW & THEORETICAL PERSPECTIVE

As indicated by its name, the notion of reflective abstraction involves the actions of reflection and abstraction. Thus, any author's attempt to define reflective abstraction will, at least implicitly, provide insights into their interpretations of what it means to reflect and what it means to abstract.

In von Glasersfeld's (1991) case, he takes an approach similar to Locke in providing his interpretation of the act of reflection. Both contrasted the notion of reflection with the notion of sensation, where sensation relies on external objects while reflection is internal in nature. Von Glasersfeld included the actions of re-presenting and comparing as acts of reflection. For example, to compare the taste of two apples, he wrote that one would have to experience the sensations of eating each of the apples, but would then have to re-present those sensations to himself to then be able to compare them. In this way, he makes the clear distinction between having an experience (i.e. eating an apple) and reflecting on that experience. He further describes reflection to be the action of ceasing our collection of perceptual information, trying to recreate what we previously experienced, and examining that creation as if it were a direct experience, all the while being conscious of the fact that it is not, in fact, a direct experience.

While Dubinsky (1991) did not explicitly discuss the meaning of reflection, aspects of his interpretation of this activity are apparent in his differentiation between reflective abstraction and other forms of abstraction introduced by Piaget. In making this distinction, Dubinsky wrote that empirical abstraction concerns objects, whereas reflective abstraction concerns actions on those objects. Further, reflective abstraction differs from pseudo-empirical abstraction in that the latter is concerned with actions, while the former is more concerned with relationships between those actions. Dubinsky clarified that reflective abstraction is “completely internal”, which seems to imply that he would agree with von Glasersfeld and Locke in saying that reflection is an internal activity.

In discussing the notion of abstraction, von Glasersfeld again refers to Locke’s description and seems to, for the most part, agree. I interpret von Glasersfeld’s meaning of abstraction to be the action(s) by which ideas from particular objects (whether they are physical or mental objects) become more general and apply to all other objects that fit the criteria of the original objects in consideration. Glasersfeld argues that once a concept has been abstracted, one can then use it to recognize and categorize particular objects that they perceive. For example, I have abstracted the concept of function and can now, regardless of their nuances, recognize and categorize individual objects as functions or non-functions.

In describing Piaget’s different types of abstraction, Dubinsky uses the language of “extracting” or “teasing out” of properties and describes how one “comes to know” the properties of objects and actions. It appears as though he views abstraction as the act by which one is able to recognize, and possibly accumulate and coordinate, certain properties of a particular object or action. Thus, it seems that Dubinsky and von Glasersfeld both agree that the subjects of abstraction can be physical or mental objects or actions.

Dubinsky and von Glasersfeld both described the different forms of abstraction that Piaget distinguished, however, their interpretations are not identical. On one hand, von Glasersfeld claimed that Piaget distinguished between two main types of abstraction: empirical and reflective, where the latter type was divided into three distinct forms. He wrote that empirical abstractions are the abstractions of properties of sensory-motor experiences. Thus, the sensory material, or observable material (from the perspective of the individual), must be available to the individual in order for empirical abstractions to be made. Glasersfeld distinguishes three different forms of reflective abstraction and claims that the differences in the names Piaget attributed to them were lost when the three were translated into English, altogether, as reflective abstractions. First, there are the types of reflective abstractions where an individual takes the actions performed at one level of thought, and projects and organizes them on a higher level of thought. The second type, von Glasersfeld writes, is similar to the first but is differentiated by the fact that the subject is now aware of what is being abstracted. And finally, the third type of reflective abstraction, according to von Glasersfeld, is called pseudo-empirical abstraction. Like empirical abstraction, he describes pseudo-empirical abstraction to involve material objects, however, in this form of abstraction, the perceived properties of the object(s) were produced by the actions of the individual. For example, it may seem that when learning to add that the child simply reads the results directly from the material object of the abacus, but what allows the child to do so is not just the existence of the physical objects, but the child's action of counting the beads on the abacus.

Rather than four, Dubinsky claims that Piaget defined three distinct types of abstraction. Like von Glasersfeld, he defines one of these to be empirical abstraction, in which the individual extracts knowledge from the properties of objects that they perceive. On the other hand, Dubinsky defines a second form of abstraction as reflective abstraction in which coordinations of actions are made. Unlike his description of empirical abstractions, Dubinsky claims that the source for reflective abstractions is the individual making these abstractions, rather than perceived objects, and that, therefore, this form of abstraction is entirely internal. Dubinsky describes the third form of abstraction, pseudo-empirical abstraction, to be an intermediate between empirical and reflective abstraction. He defines pseudo-empirical abstraction to be the act of extracting properties from the meanings we attribute to objects through our actions on them (for example, the counting of beads on the abacus).

In terms of defining the individual forms of abstraction, I find von Glasersfeld and Dubinsky to be mostly in agreement, but note that they seem to have formed different interpretations on Piaget's overarching organization of these forms. From both descriptions, I conclude that reflective abstraction involves the interiorization of actions and then coordination of those actions to create new mental actions and objects. In particular, reflective abstraction allows an individual to construct mental objects and then to act on those objects.

Tightly tied to understanding the teaching and learning of function inverse, is the concept of function. My hypothesis is that the students (and sometimes teachers) who struggle to have a productive understanding of function inverse have not completed the necessary abstraction of the concept of function to enable them to internalize a function's process, let alone construct a function as a mental object. From the observations I have made and the data I have collected, it appears that students are (sometimes) capable of finding a function's inverse, but their actions appear to be extracted from imitation, or a memorized procedure, rather than meaningful actions on the mental object of a function. This claim is further supported by the inability of students, upon correctly calculating a function's inverse, to describe the meaning of what they have just calculated. Dubinsky (1991) claimed that an individual could construct a process to reverse an original process, only when that original process existed internally for the individual. I fear that many of the students I have seen struggle with reversing the process of a particular function lacked the ability to internalize the given process and act on it, but, rather, saw the function as simply markings and symbols on the paper in front of them. In the words of von Glasersfeld, it seems that these students have not made the necessary abstractions in order to work with the notion of reversing the process of a function on the re-presentational level, and, therefore, are left needing the sensory-motor experience of manipulating the symbols on their page.

Piaget's notion of reflective abstraction developed from his research on the learning of children. The mental constructions of APOS Theory (Action, Process, Object, Schema) were introduced as an attempt by researchers to extend the notion of reflective abstraction to undergraduate mathematics education (Dubinsky & McDonald, 2001). Dubinsky and McDonald describe that the main idea behind APOS Theory is that the learner constructs actions, processes, and objects, and organizes these constructions into schemas in an attempt to make sense of situations that they view as mathematical.

To be specific, an action is any manipulation or transformation of objects in a repeatable manner that is either physical or mental (Harel & Dubinsky, 1992; Breidenbach et. al., 1992). In the context of function inverse, I argue that a student with an action conception would be able to solve for an input value of a function given a particular output value, provided that the student was given the algebraic rule for the function. I would also argue that students with an action conception of function are capable of memorizing and applying meaningless procedures. That is, I hypothesize that students with an action conception of function are highly capable of applying the typical procedure of swapping x and y, and solving for y, but I would argue that students who only possess an action conception of function would be unable to explain why this procedure makes sense or why their resulting answer is correct.

When it is possible for manipulations or transformations to be carried out in the mind, without having to consider each step, the action has been interiorized to become a process. Harel and Dubinsky (1992) call a process a "dynamic transformation of quantities" in such a way that given some initial quantity, the process will always transform that quantity to produce the same new quantity, and that this transformation can be considered as a complete activity rather than the learner having to consider the transformation step-by-step. Breidenbach, et al. (1992) claimed that a process conception is necessary for a student to be able to envision the reversal of a function process. When the student has completed the necessary mental actions to see the new formula as a generalized rule for determining input values of the original function when given specific output values from the original function, I contend that the student has a process conception of the inverse function. The student with this conception could also see that the output quantity of the original function is now the input quantity of the inverse function and the input quantity of the original function is now the output quantity of the inverse function. Additionally, I contend that any understanding of logarithmic functions as the inverse of exponential functions requires a process conception of function, as with a logarithmic function the student is unable to consider the process in a step-by-step manner. This argument can be extended to inverse trigonometric functions for the same reasoning.

When the learner can consider a process as something that can be acted upon, then APOS theorists claim that the process has been encapsulated and is now considered an object. An example of an application of an object conception could include creating a set of functions.

Then, as these actions, processes, and objects are constructed, they begin to be organized into a linked collection that can serve the learner when confronted with the topic in the future. This linked collection can serve the student well in applied problems where the student might have to consider various relationships between quantities to determine whether a particular function or its inverse is needed, whether an inverse function may need to be composed with another function, etc.

These mental constructions of actions, processes, and objects are ordered in such a way that each must be constructed before the next is possible (i.e. a process must be constructed before it can be encapsulated into an object) (Dubinsky & McDonald, 2001). However, these are ways of thinking about a mathematical concept and not stages of development. Therefore, these ways of thinking are not necessarily mutually exclusive and can co-exist. A student with multiple conceptions of function might spontaneously apply, for example, an action conception in one context and a process conception in another context. Further, a learner may also be in a state of transition from one conception to another (Breidenbach et al, 1992).

Researchers in undergraduate mathematics education have applied APOS Theory to the notion of functions to further examine how students construct their meanings for function (e.g. Breidenbach et al., 1992; Harel & Dubinsky, 1992; Dubinsky & McDonald, 2001: Asiala et al. 2004). Asiala et al. (2004) described that with an action conception of function, the student can consider a step-by-step process applied to some quantity to produce a new quantity. In doing so, the student with an action conception of function requires some precise definition of what steps to take to be able to consider the function. In contrast, Harel and Dubinsky (1992) describe a student with a process conception of function as having the ability to imagine the activity as a complete activity involving some input quantity, some transformation, and an output quantity. As the details are not required when a student possesses a process conception of function, students with this conception are able to envision a function process without being given the algebraic rule of assignment for the function.

Harel and Dubinsky (1992) claim that a process conception of function is necessary for a student to be able to envision reversing the given process, or, in other words, envisioning the inverse of the given function. In agreement, Asiala et al. (2004) claimed that the struggle many students face when function inverses are introduced can be linked to the fact that these students have not been able to construct beyond an action conception of function.

CHAPTER 3

CONCEPTUAL ANALYSIS

An intended understanding of function that is propitious to understanding function inverse is an understanding of a function as a process that maps a continuum of input values to output values (Dubinsky & Harel 1992, Carlson et al 2005). Dubinsky and Harel (1992) claim that a process conception of function allows the student to reverse the process of a function (i.e. consider the inverse of a function.) Consider, for example, the function *g* that accepts temperature measured in degrees Celsius as its input, and produces the corresponding temperature measured in degrees Fahrenheit as output. According to a process conception, the student can imagine starting with some temperature (or a continuum of temperatures) measured in degrees Celsius, perform some process as defined by the function's rule, and end with the corresponding temperature(s) in degrees Fahrenheit. With an understanding of this process from inputs to outputs, the student has the potential to understand reversing that process. As in the reversal of everyday tasks like making your way back home from a destination, you begin with the final product (or output) and work your way backwards to the initial state (or input). Then the reversal of the function *g* will begin with a temperature measured in degrees Fahrenheit and end with the corresponding temperature in degrees Celsius (see *Figure 1*).

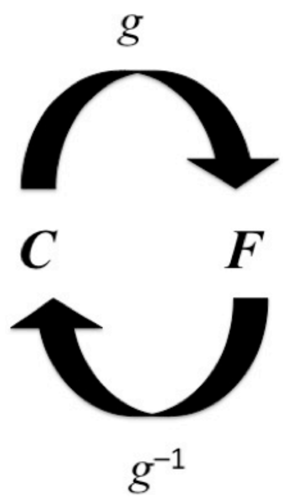


Figure 1

Once the student is able to understand that the function's inverse is the reversal of the function's process and that the input of the original function is the output of the function's inverse and vice versa, then many important properties of the function's inverse can become available for discussion.

For example, the rule for the inverse function can be considered as a representation of the same relationship between the two quantities being considered, but in such a way that the direction of the relationship of the function and its inverse are opposite. For example, the function g is defined as $F = g(C) = 9/5C+32$, which can be written as $F = 9/5C+32$. As this function defines F in terms of C, the inverse function will define C in terms of F. To find a rule that takes values of F as its input and produces values of C as its output, the student simply needs to solve the original formula for C. This solving approach effectively reverses the directional relationship.

The understanding of the reversal of inputs and outputs also provides a bridge into a discussion of the graphs of function inverses. If the coordinate point (a, b) is on the graph of the function, then the point (b, a) will be on the graph of the inverse function as the input and output quantities are related in the reverse order. This way of thinking also extends to the construction and interpretation of tables of values for the function and its inverse.

The reversal of input and output quantities also allows for a discussion about the relationship between the domain and range of the original function and its inverse. When the student is able to reason about either the algebraic definition of the inverse or the graph of the inverse, then there can be an understanding about whether or not the inverse is a function itself, or, in other words, whether the inverse maps each input to exactly one output. In exploring when a function's inverse will be a function or not, there arises an intellectual need for the classification of one-to-one functions.

In understanding that the function inverse is a reversal of the process of the original function, the student has the potential to comprehend why the composite function created by the composition of a function and its inverse (when also a function) produces the identity function. While the student may not be able to articulate this understanding in this way, they should be able to understand why $f^{-1}(f(a)) = a$ for all values a in the domain of f and $f(f^{-1}(b)) = b$ for all values b in the domain of f^{-1}.

An object conception of function will also be propitious for students' understanding of function inverse. With this conception, students have the potential to understand that processes and their inverses are objects in and of themselves, rather than transformations that act on other objects. This understanding of processes as objects can lead to the understanding that processes and their inverses, then, can be acted upon (e.g. taking the inverse of a function, composing two functions, etc.). Without an understanding that, for example, g^{-1} is an object that can be acted upon, then the notation $(g^{-1})^{-1}$ will likely appear as an opportunity to apply a procedure for a student, rather than an expression involving the transformation of a relationship.

CHAPTER 4

METHODS

Three students were selected from my pre-calculus course based on their ability to vocalize their thinking. Two of the students, Randy and Madison, are categorized as A-level students. Randy has been out of high school for 13 years and took College Algebra the semester prior to this experiment. Madison is a freshman in her first year out of high school, and stopped taking secondary mathematics courses before she reached the pre-calculus level. The third student, Tess, is a sophomore in her second year out of high school, where she previously took pre-calculus. Tess is categorized as a C-level student who stated, “I do not like math in any way, shape, or form…” The mix of educational levels and backgrounds was intended to provide a potential for gaining insights into the usefulness of the instructional sequence for a variety of students.

Each student participated in two one-on-one teaching sessions. The first session lasted about 90 minutes. This session included a pre-test that allowed me to document the meanings that the student had previously constructed. In order to construct a model of the students’ thinking rather than trying to influence that thinking, I treated the pre-test as a clinical interview (Hunting, 1997). Following the pre-test, I began a teaching experiment (Steffe & Thompson, 2000) in which I utilized the tasks outlined in the instructional sequence. The first session ended once the student had worked through each of the nine tasks in the instructional sequence.

Randy and Madison met with me two days after the first session to complete the second session, while Tess did so four days later. The second session lasted, on average, about 25 minutes in which I administered a post-test to document the meanings the student had constructed throughout the instructional sequence and in the following days. In order to document these meanings without continuing to influence the student, I treated the second session as a clinical interview. The details of the pre-test, instructional sequence, and the post-test will be discussed in more detail in the following section titled Hypothetical Learning Trajectory.

I video recorded each of the sessions in order to analyze the student's work and statements following the interview. Before beginning the second session, I completed the analysis of the first session in order to form hypotheses about the students' meanings and prepare questions to ask during the post-test clinical interview.

Hypothetical Learning Trajectory

As a starting point, the Pre-test (see *Figure 2*) will allow the researcher to document the meanings and ways of thinking the student associates with inverse function. That is, I designed this Pre-test with the intent to learn about the student's current concept image (Tall and Vinner, 1981), or the total cognitive structure including properties and processes the student associates with the concept of function inverse.

Pre-Test

A local apple orchard uses the function *f* to determine the cost of a customer's load of apples, *c,* given the number of pounds of apples that the customer picked, *n*, where $c = f(n)$ and *f* is defined by $f(n) = .45n + 7$.

i. Determine $f^{-1}(25)$ and describe its meaning in the given context.

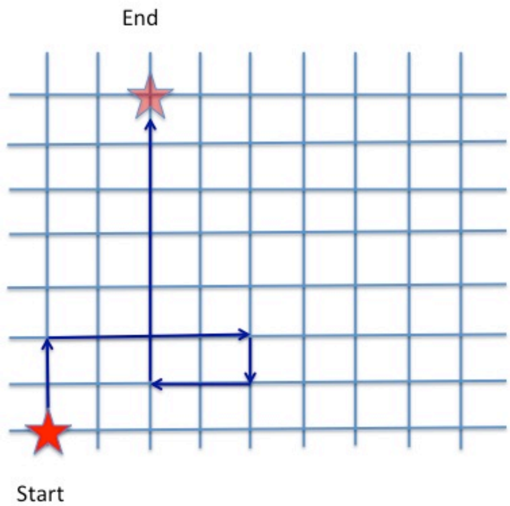
ii. Define f^{-1}.

iii. Sketch graphs for the functions *f* and f^{-1}.

Figure 2

Based on my experience teaching pre-calculus students and on the general consensus in APOS literature, I conjecture that the student will begin the instructional sequence with an action conception of function. That is, it is likely that the student can only consider a function step-by-step when given all the details about said function. The first four tasks of the instructional sequence are designed to be approachable for a student with an action conception of function, while helping him construct a process conception of function and, further, imagine the reversal of a function as the reversal of a process.

The first task provides the student with a process (see *Figure 3*) for moving from one point on the grid to another. The student is then prompted to construct the reverse of this process. This is not a presentation of a function in any sense that a student would likely be familiar with, but the purpose of this task is to have the student reflect on the salient features of a process and its reversal. For instance, the student can reflect on the relationship between the "Start" and "End" of the process and its inverse, as well as what happens when a process and its inverse are applied in succession. By having the student name the process and name the reverse process, it is hoped that the focus is removed from the details of each step of the process, towards the entire process as a whole. Overall, this task provides a potential building block for students to reflect on their actions and the products of those actions in such a way that promotes reflective abstraction.



Process:

1. Shift 2 units up
2. Shift 4 units right
3. Shift 1 unit down
4. Shift 2 units left
5. Shift 6 units up

Figure 3

The student is provided repeated opportunities to envision the reversal of specific processes in the first three tasks since this reasoning ability has been shown to be challenging for students to learn. In having the student engage in the action of reversing a process and prompting them to reflect by asking them to compare the tasks, my intention was to provide an opportunity for the student to engage in reflective abstraction that would allow them to generalize those actions. I designed Task IV (see *Figure 4*) to provide the student an opportunity to generalize their thinking about the actions involved in reversing a process. As there are no step-by-step directions for the student to apply to an input value to determine an output value, I contend that a student who was only capable of utilizing an action conception of function would not be able to successfully answer the question. In addition, Breidenbach, et al. (1992) argued that a student is only capable of envisioning the reversal of a process once that student has developed a process conception of function.

Task IV
Consider the process called *h* that has three steps:
Step 1: ? Step 2: ? Step 3: ?
Describe (with as much detail as possible) the process needed to undo *h*.

Figure 4

Task V (see *Figure 5*) provides the student with a real-world motivation for constructing an inverse function. The task gives the student the function used to convert a temperature measured in degrees Celsius to the corresponding temperature measured in degrees Fahrenheit. In the first two subtasks, the student is asked to calculate the temperature in degrees Celsius when given the temperature in degrees Fahrenheit (i.e. solve for the input of the function, when the output is known.) The first two subtasks are possible to perform with an action conception of function and the third task requires the student to envision calculating a temperature in degrees Celsius given any temperature in degrees Fahrenheit. In order to do so, the student must envision and determine how to reverse the process of the given function.

Task V: Part I

Since many countries measure the temperature in Celsius units instead of Fahrenheit, we need a way to convert between these two units. We use the following function: $F = \frac{9}{5}C + 32$, where F represents the temperature measured in degrees Fahrenheit and C represents the temperature measured in degrees Celsius to convert from degrees Celsius to degrees Fahrenheit.

1. Think about the question: "What is the value of C when F = 96?" How is this question like the "find my number" game? (Then find your number.)
2. Find the value of C when F = 25?
3. How can you find the value of C for any value of F? Can you write this as a function?

Figure 5

This task is designed to begin a discussion about the usefulness of a function's inverse, how to determine the rule of a function's inverse, and what quantities are being related by the function's inverse. As students are likely to see inverse functions again when they uncover logarithms and exponentials and inverse trigonometric functions, this initial introduction is likely the last time the student will actually determine the rule for a function's inverse. Thus, I found it important to introduce the procedure used to determine the inverse of a function in a conceptual way that mimics the action of solving for a function's input, in terms of that function's output. In addition, this task provides an opportunity to discuss function inverse notation.

When discussing the relationship between the two quantities (temperature in degrees Celsius and temperature in degrees Fahrenheit) and how they are related by the original function and its inverse, the researcher can develop the opportunity to have the student envision the outcome of applying the two processes in succession. That is, this is an opportunity to help the student discover the outcome of composing a function and its inverse. If it does not arise spontaneously, the interviewer can ask the student to compare this notion of composing a function and its inverse, to applying the original and reverse processes in Task I on the grid. Having the student compare these two tasks might help him develop more meaning for what the composed function represents, but also requires the student to reflect on the products of their actions in these two different tasks. This continued focus on having the student reflect on their actions and the products of those actions was the result of designing an instructional sequence with the goal of promoting reflective abstraction

In addition to having the student determine the rule for the inverse function in Part I, Task V: Part II requires the student to graph both the original function and the inverse function. In combination, these two parts are intended to confront the incoherence with the common approaches of determining the inverse by swapping the two variables and flipping the graph of the function across the line $y = x$. Whether these approaches are used spontaneously by the student or introduced by the instructor, these tasks are designed to initiate a conversation about the problems that arise with each of these procedures in the given context of temperature conversion. If the student has either of these procedures in his scheme for function inverse, this conversation is intended to perturb the student in ways that lead to him abandoning his reliance on these procedures in lieu of a conceptual understanding of inverse. If the student does not rely on these procedures at the time of this interview, explaining the problems that arise by the use of these procedures may cause him to reflect on the understandings that he has in hopes of strengthening his meanings. This might also help to prepare the student to confront the incoherencies about function inverse in future textbooks he may use.

The sixth and seventh tasks are designed to engage students in using function inverse in the contexts of algebraic rules, tables, and graphs. The tasks will provide the student an opportunity to reflect on the relationship between the input and output quantities of the original function compared to the input and output quantities of the inverse function. For example, Task VI: Part II provides the student a graph of the function *g* that takes the number of years since Bill made an investment and outputs the value of that investment. The student is then given the questions in *Figure 6.* This focus on considering the relationship between the input and output quantities of the function and its inverse, when the algebraic rule is not known, allows the student to build meanings for function inverse that will be beneficial when he is introduced to logarithms and inverse trig functions, in which there is no simple algebraic rule for step-by-step consideration.

Task VII: Part II	
i.	Determine $g^{-1}(400)$ and explain its meaning in the given context.
ii.	Determine the value of a so that $g^{-1}(a)= 8$. Explain the meaning of this in the given context.
iii.	Determine the value of b so that $g(b)= 500$. What similarities and differences are there between the statements $g(b)= 500$ and $g^{-1}(a)= 8$?
iv.	What are the domain and range of g? What are the domain and range of g^{-1}?
v.	Sketch the graph of g^{-1}?

Figure 6

Once the student had responded to questions utilizing tables, graphs, and function notation to represent the inverse of a function, Task VIII was designed to provide an opportunity for him to extend his reasoning to consider the reversal of a combination of processes. During the interviews each student was told that if we start at x and apply some process A and then apply some process B, we end up at y. The student was then asked to undo this process to start at y and end up at x. By stripping away the details of the processes A and B, the student was no longer able to rely on an action conception of function, but must consider the complete process and its reversal. Additionally, Harel and Dubinsky (1992) claim that a student must have a process conception of function in order to envision not only the reversal of a function, but also the composition of functions.

Task VIII was designed to elicit the understanding needed to invert a composed function. For example, the understandings needed to answer this question mirrors the knowledge needed to understand that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. However, by utilizing “process A and process B” rather than “function f and function g”, we can avoid any pseudo conceptual knowledge about function notation that the student may have memorized without understanding (Oehrtman, Carlson, & Thompson, 2008).

The final task in the instructional sequence, Task IX (see *Figure 7*), requires the student to consider when it is appropriate to utilize a function’s inverse, and when it is not. It is important that the student understand the meaning of an inverse function and how to determine a function’s inverse, but these skills alone cannot properly serve the student if he or she does not understand when to utilize function inverse when solving problems.

Task IX

The function f accepts the number of molecules in a sample of the chemical Phosphorus as input and determines the mass (in kilograms) of the sample. The function g takes the volume (in milliliters) of the sample, and determines the mass (in kilograms) of the sample.

a. If possible, use function notation to express the number of molecules in the sample, given the volume in mL of the sample.

b. If possible, express the volume of the sample in mL in terms of the number of molecules present in the sample.

Figure 7

In completing this instructional sequence, the student has been given the opportunity to build a conceptual understanding of a function’s inverse as the process that undoes the original function. The student has also been given multiple opportunities to engage in repeated reasoning about how to determine the rule of a function’s inverse, how to graph, and how to produce tables of values for a function's inverse. Throughout the tasks, the student is required to reflect on the relationship between the input and output quantities of the original function and its inverse. The post-test was designed to assess the meanings that the student constructed through his participation in the instructional sequence.

The post-test starts by having the student examine his responses on the Pre-test and having him reflect on anything he might now approach or understand differently. Thus, the Pre-test not only serves as a benchmark for the researcher, but also a tool to help the student reflect on his own ways of thinking.

The post-test includes three items that mirror tasks from the instructional sequence. The first item provides the student with the graph of a function and requires the student to reason about the relationship between the input and output of f and f^{-1}, inverse function notation, and the composition of a function and its inverse. The second item of the Post-test mirrors Task IX of the instructional sequence (see *Figure 7*) and requires the student to reason about the quantities being related to decide when the original function is required and when the inverse function is required. And finally, the third item of the Post-test asks the student to determine the meaning of $(f^{-1})^{-1}$, given that f is a function. This novel task is designed to see whether the student can envision an inverse function as something that can be acted upon.

CHAPTER 5

RESULTS

Pre-test

In the pre-test, the students were given the definition of a function *f,* in the context of buying a certain number of pounds of apples, and were asked to define f^{-1}. (Refer to *Figure 1*.) None of the three students determined the correct answer of $f^{-1}(c) = \dfrac{c-7}{0.45}$. In their responses, it was apparent that all three students were attempting to recall a learned procedure rather than appealing to any meaning they may have had for the inverse of a function.

An excerpt of Randy's written work for his attempt to determine the rule for f^{-1} is included in *Figure 8*. As I had anticipated, Randy applied the typical approach of swapping the variables, but was unable to explain why his procedure made sense. Although Randy arrived at what many math teachers would consider the "correct answer", he did not appear to be viewing an inverse function as the reversal of the function process. von Glasersfeld's words seem appropriate here, in that Randy had not previously engaged in the abstractions necessary to envision reversing the given process on the re-presentational level. Thus, the sensory-motor activity of applying a memorized procedure to the markings on his paper was his only resort to getting the correct answer. Randy and I had the following conversation:

1.	I:	The only question I have…for now…is umm… I see how you got here [pointing to (y-7)/0.45 = n]. You solved for n…and you got this rule y minus 7 divided by .45 and then you changed it to n minus 7 divided by…I'm just curious why you…why you made that change. I'm not saying its right or wrong, I'm just curious why you did that.

2.	R:	Um…this is just the way I was taught last semester…how to do it… and I know that you said…um…(*long pause*)
3.	I:	You don't have to worry about what I said. I'm just curious why you…So someone taught you to do that?
4.	R:	Yes. Um I just remembered like setting it to *y* and then putting everything on that side and then once you've got everything over to this side its known as the inverse.
5.	I:	Ok.
6.	R:	And then you just swap it back.
7.	I:	Ok. So just some one taught you that?
8.	R:	Yes.
9.	I:	Ok. Do you have any meaning for it besides that….just beside that someone taught you that?
10.	R:	No. [laughs]

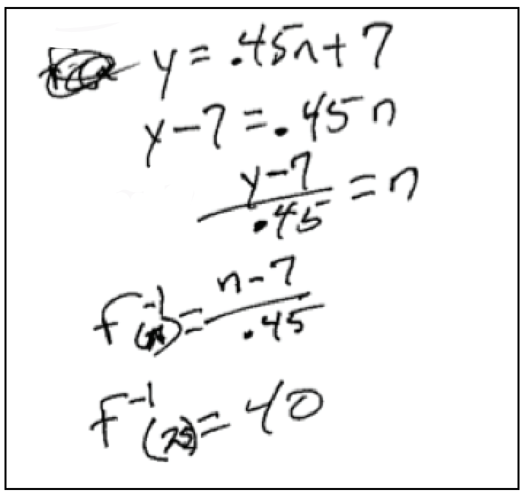


Figure 8

Neither Tess nor Madison utilized the typical procedure of swapping the variables, but both seemed to struggle to remember some procedure they had learned or, rather, memorized before. Tess, on one hand, did not appear to differentiate between the functions f and f^{-1}. When attempting to determine $f^{-1}(25)$, Tess simply plugged 25 into the rule for function f and did not seem perturbed when I paraphrased what she had done. In defining the rule for f^{-1}, Tess began by setting f^{-1} equal to the rule of *f*. Once she had written $f^{-1} = .45n + 7$ she continued by solving for *n*, to get that $n = \frac{f^{-1} - 7}{.45}$. When I asked her why she had done this, she claimed that she was trying to remember what she had done before, and she claimed, "it's just been a really long time since I've worked with this stuff."

Tess exhibited an action conception of function as she was able to step-by-step calculate an output when she plugged 25 into the rule for *f*. However, her use of notation (i.e. using f^{1} without expressing an input variable) could suggest a pre-function conception, or a lack of function concept (Breidenbach, et. al., 1992). In not expressing an input variable for f^{-1} and manipulating it as if it were a variable, Tess' work suggests that she imagines f^{-1} as a variable that represents some value or values. Tess did not appear to be viewing the name of the function as the name of the process that maps input values to output values. This impoverished understanding of function notation resulted in Tess mindlessly manipulating mathematical symbols when determining her answer.

Much like Tess, Madison seemed to reach for a memorized algorithm to determine the rule for f^{-1}. Madison and I had the following conversation:

11.	I:	Ok…so it looks like when you were finding *f* inverse of 25…can you tell me what you did?
12.	M:	Well…I don't even know if I did that right, but I thought to find the inverse it would be -1 times the…
13.	I:	Ok. That's just what you remember? Or do you have some meaning for that- why you multiply by -1?
14.	M:	That's just what I remember.
15.	I:	Ok.
16.	M:	So yeah, that was just kind of a guess.

Again, rather than describing the meaning of an inverse or discussing the reversal of the given process, Madison resorted to applying a procedure that she admitted held no meaning for her. Many of my pre-calculus students describe a function's inverse is "the opposite". It is possible that multiplying by negative one was consistent with Madison's interpretation of "the opposite" function.

Tess utilized the notation of f^{-1} without including an input variable, Randy claimed that you just "swap it back " (line 6) and then ended up with n as the input to f^{-1}, and Madison included n as the input to f^{-1} only after I asked her about a possible input. Thus, not one of the three students was able to correctly identify the input quantity to the inverse function. In applying their individual procedures for determining the rule for the inverse, none of the students mentioned the input or output quantities. And, although each of the three students graphed f and f^{-1} on separate axes and did not utilize the procedure of flipping the graph over $y = x$, none of the students labeled their axes with the quantities that the functions were relating. It is noteworthy that none of the three students were referring to the input and output quantities for the given function and its inverse. Their responses suggest that rather than imagine the reversal of a directional relationship between two quantities, the three subjects were attempting to apply memorized algorithms to determine the inverse function.

Since I was treating the pre-test as a clinical interview to observe the students' ways of thinking about function inverse, I did not try to influence their thinking at this stage of the data collection. After finishing the clinical interview, I began the teaching experiment with new tasks in my hypothesized instructional sequence. During the second session of the data collection, I also gave students a chance to return to the work they provided in the pre-test. This provided me information about how my subjects' thinking about function inverse was being impacted by the teaching experiment.

Teaching Experiment

The first three tasks of the teaching session prompted students to carry out the mental actions involved in reversing a set of steps. When attempting to carry out the actions involved in reversing a process all three of my subjects were successful. They were able to conclude that the inverse of each operation or, the inverse of each "action", as noted by Tess, needed to be applied in the reverse order. The first three tasks allowed the student to come to this conclusion three times.

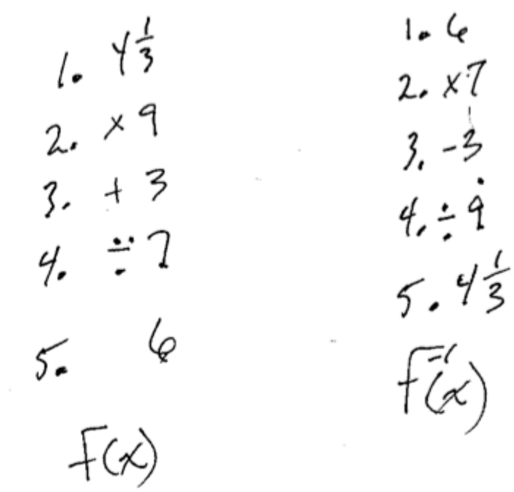
By prompting the student to describe his thought process and compare his conclusions to any previous tasks, I was intending to provide an opportunity for him to reflect on his actions, or to participate in reflective abstraction, in the hopes of allowing him the ability to generalize those conclusions. An example of the intended type of reflection is evident in the following conversation I had with Randy after he constructed the inverse process for a given set of actions in Task II:

17.	I:	Do you see anything similar with this task and this one [Task I] that you just did?
18.	R:	Yes.
19.	I:	Or different?
20.	R:	The same thing.
21.	I:	The same thing? How so? It doesn't look like the same to me.
22.	R:	No, but its just going in the same routine, as in you're asking me to go one way [runs pen down given process in Task II] and then back the other way [runs pen down inverse process he wrote for TII], if I were to do these two processes [points to the two processes in Task I]
23.	I:	Ok. Sounds good. Anything else?

24.	R:	Mm, no. I mean, I can, its asking me the same thing. You were asking me the same thing about starting and ending point, where I can get from the starting point [points to 5 in given process of TII] to the end [points to 14 in given process], and then I can also get from the end back [points to 14] to the [points to 5 in given process]…

In the first four tasks of the teaching session, the students, first, coordinated the input and output with a given process. In Tasks II-IV, Randy repeatedly used the phrases "starting point", "apply the process" and "end up with" (e.g. line 6 and *Figure 9*). He kept returning back to his work on Task I with the grid and set of directions to compare his conclusion on later tasks, so I conjecture that this imagery of the input as the "starting point" and the output as the "end" may have developed from his participation in solving Task I.

Randy's ability to coordinate the inputs and outputs, or "starting point" and "ending point", of the original process with the inverse process is also evidenced early on in Randy's work and explanations of his solution to Task III. For this task he was told a process that I had applied to an unidentified number, given the final value after applying the process, and asked to determine the original unidentified number (see *Figure 9*). Randy's comparison of Task III with Tasks I and II provides evidence that Randy was reflecting on his previous actions and the products of those actions in the first two tasks, while explaining his thought process in the third task. Randy would have needed to abstract his actions and products from each task, and then reflect on those products and actions to be able to compare the three tasks. The first few tasks were intentionally designed to promote a repetition of similar actions and conclusions with the goal of allowing for the reflections and comparisons that Randy engaged in.



R: "Uh well its going back to Task I and II where we were using functions and inverse functions to get from our starting point [points to 4 1/3 in step 1 of process *f*] to the ending point [points to 6 in step 5 of process *f*] and then from our ending point [points to 6 in step one of process f^{-1}] back to our starting point [points to 4 1/3 in step 5 of process f^{-1}]."

Figure 9

By Task III, each of the three students spontaneously described the process they had constructed to reverse the original process and calculate the original number, given the final number, as the inverse. Their descriptions of both the original process and the inverse process suggest that the subjects were beginning to coordinate the input and output values of the original function with the input and output values of the inverse function. Consider, for example, the explanation that Tess wrote for Task IV (*Figure 10*), where the student was given a function named *h* that involving Step 1, Step 2, and Step 3 without any details about these steps, and was asked to describe the process needed to undo *h*. It appears that Tess has constructed and generalized her understanding that the input to the original function will be the output of the inverse function.

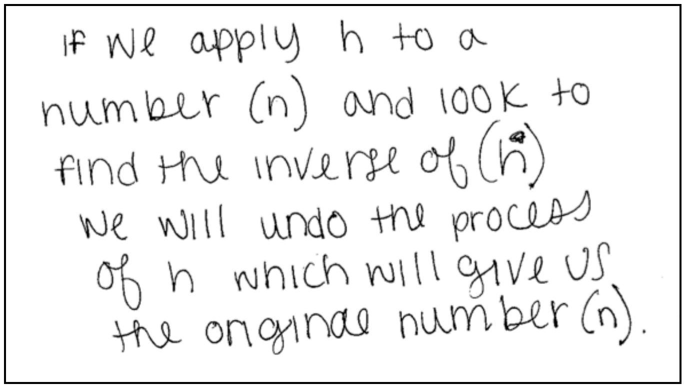


Figure 10

Like Tess, none of the students had difficulty generalizing their thinking in order to answer Task IV, and describe the general process of reversing a process. Madison even asked, “That’s it?” after successfully responding to this task. Their generalizations showed that the students had constructed a meaning for an inverse process, as well as reasoning behind an approach they could utilize to define an inverse process. Rather than a meaningless, memorized procedure, the students had developed their own procedures for defining an inverse process based on their understanding of reversing a set of actions. It was noteworthy that in Task II and III, Tess and Randy continued defining their processes by defining each step and numbering those steps to indicate their order of application in the process. This is consistent with how the original processes were provided to the student in Tasks I and II in order to accommodate an action conception of function. However, in Task III when Madison was told the process applied to some unidentified number and asked to represent this process for herself, she began to spontaneously introduce strings of operations. She chose to let the variable n represent the unidentified number that I started with. Her use of this variable and her strings of mathematical operations allowed her to construct the original function I had described to her. Once she had this function that accepted *n* as its input and expressed the final number as its output, she was capable of solving for *n* when given two example values for the output (i.e. the final, known number). *Figure 11* shows Madison's work for Task III, which can be compared to Randy's work shown in *Figure 9*.

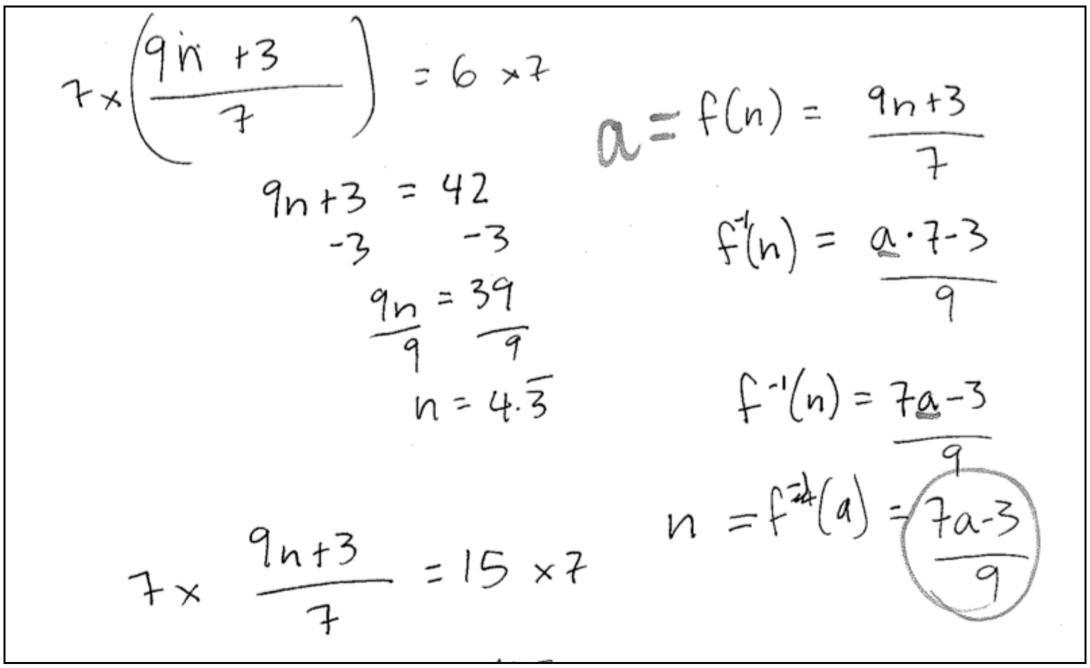


Figure 11

Once she was able to solve for the input given values of the output, I asked Madison if she could find the original, unidentified number given any final number. Madison decided to let the variable *a* represent the value of the final number, and modeled the process she had utilized to solve for the original number, *n*, to construct the inverse function. In the following excerpt (especially lines 27 and 29) Madison utilized the work she had written in the first subtask in order to describe how she would construct the function's inverse. She replaced the example's specific output value (6 in this case) with the variable *a* and was able to walk through the actions she had previously applied in order to solve for *n* in terms of any value for *a*. Thus, Madison's ability to construct the inverse function appears to be a product of her reflection on her actions she engaged in when solving for input values given specific output values. Her construction of this inverse function seemed to come naturally from her understanding of "solving for", and she was able to do so without being given some procedure to memorize. Her description of her thought process follows:

25.	M:	Alright, so you'd take 9*n*+3, all over 7 and then you're gonna, whatever number you have…can I call it like…
26.	I:	You can call it whatever you want.
27.	M:	I'm not used to coming up with my own variables. Call it *a*. On this side [points to 6 on right side of first equation in *Figure 11*].
28.	I:	So replace 6 with *a*. Ok.
29.	M:	So you multiply *a* times 7 [points to 6 times 7 in first line of *Figure 11*]. And can I call this [adds parenthesis around (9*n*+3)/7] something? Whatever. You do *a* times 7 and this side of the equation times 7 [pointing to left side of first equation in *Figure 11*]. So 9*n*+3 divided by 7 times 7. So then its 9*n*+3 equals *a* times 7.
30.	I:	I'll write down what you're saying. *a* times 7 equals 9*n*+3.
31.	M:	And then you subtract 3 from both sides and it would be *9n* equals *a* times 7 minus 3. And then you divide that by 9.
32.	I:	Ok. And what is *a,* again?
33.	M:	Whatever number you started with. Or, would that be like the output? Technically?
34.	I:	Its like the 6 and the 15, right?
35.	M:	Yeah.
36.	I:	Those were the numbers that I ended up with and you're trying to find the number that I started with.
37.	M:	Ok.
38.	I:	So how can you compare this process or this function [circles *n* = (7*a* – 3)/9] to this one that you started with?
39.	M:	Well, it would be the inverse.
40.	I:	Why so?

41.	M:	Because if you…well I don't know, but I know if I tested it…
42.	I:	Ok. Like how could you show some one that those two functions are inverses?
43.	M:	Well, you're gonna end up, like whatever *n* equals, like here $n = 4.3$, 4 and 1/3, and you go through that process [points to $n = (7a - 3)/9$], you're gonna end up with 6. And then, if you start with 6 as *a* and go through this process, you're gonna end up with *n* equals 4 and 1/3 again.
44.	I:	Is that true for only those two numbers or any two numbers?
45.	M:	Any two numbers in this *n* and *a* context.

While Madison was able to spontaneously construct a function and apply her understanding of "solving for" to determine the inverse function in Task III, Tess and Randy were both also able to engage in this type of reasoning in Task V: Part I. Again, all three of these students were able to determine the rule for the inverse function by solving for the input of the given function (temperature measured in degrees Celsius) in terms of the output of the original function (temperature measured in degrees Fahrenheit) without needing some outlined procedure.

In the case for each student, he or she was asked to solve for the input value of a function when given some specific output values of the function. Through the repetition of solving, it was intended that students would have the opportunity to reflect on those actions involved in solving and generalize them to create the desired inverse function. It is difficult to decipher whether Tess, Randy, and Madison engaged in reflective abstraction on their previous actions to produce this generalization, or if they engaged in pseudo-empirical abstraction by reflecting on the patterns in the work they had written. Regardless, either type of abstraction is arguably beneficial in allowing the student to produce the algebraic rule for an inverse function, while maintaining the proper input and output variables for a given context.

Further, although the question simply asked them to write a function that determined the temperature measured in degrees Celsius given the temperature measured in degrees Fahrenheit, each of the three students spontaneously referred to this function as the inverse of the given function ($F = (9/5)C+32$). Thus, it appears that the students had constructed an understanding that an inverse function expressed the original function's input in terms of the original function's output. Tess spontaneously chose to name the original function h and claimed that the function she had constructed must be h^{-1}. Upon my request, she was then able to describe the input and output of h and h^{-1} (see *Figure 12*).

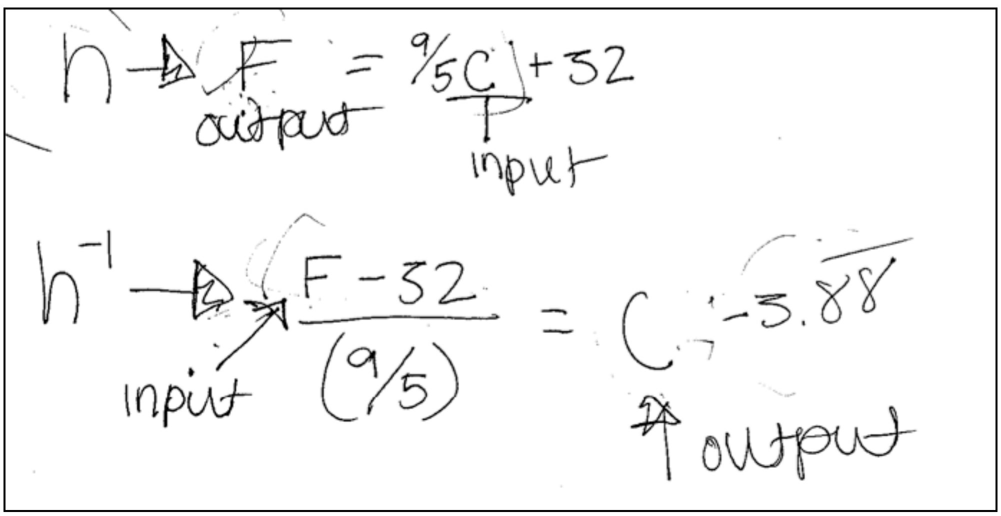


Figure 12

Although Tess' use of notation had not been perfected (i.e. she was not able to come up with notation such as $C = h^{-1}(F) = \frac{5}{9}(F-32)$), she expressed a clear distinction between the two functions and coordinated each function with its respective input and output quantities. Notation was also a struggle for Madison on multiple tasks. An example of her work from Task V is shown in *Figure 13.* Madison was able to verbalize the quantities that were represented by the input and output, and describe the rule of the inverse function. However, she was unable to explain how function (inverse) notation is utilized to show those connections.

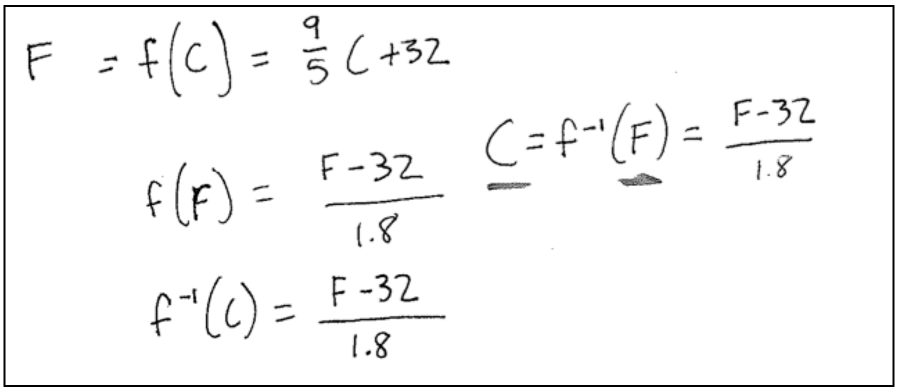
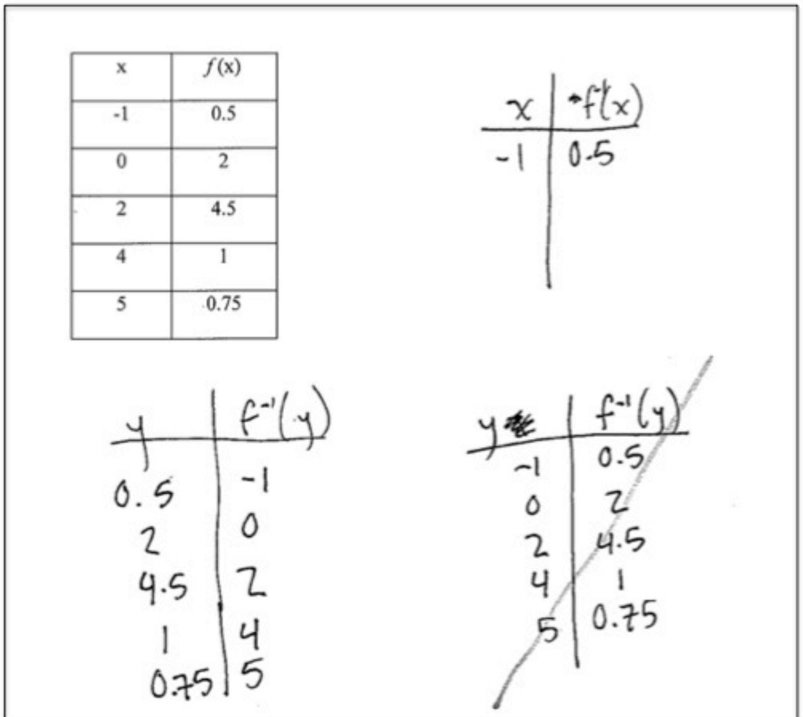


Figure 13

Up to this point in the teaching session, the focus had been on envisioning the reversal of a process, particularly a function, but this provided a good opportunity to stress the conventions of function inverse notation. As function notation is essential to any instructional sequence involving function inverse, it is important that students have a strong understanding of function notation before the complications of function inverse notation are added. This includes discussing what the name of a function is referring to, how one should indicate the input and output variables, and the algebraic rule of the function.

The misuse of function notation continued for each of the three students when attempting to construct a table of values for the function f^{-1}, given the rule or a table of values for the function f. In Task VI, the rule for the function f, which takes number of dollars, x, as its input and produces a number of euros, y, as its output, was provided. The student was asked to create a table of input and output values for f. The student was then asked to use that table to produce a table of values for the inverse of f. In Randy's first attempt, he labeled the input column as $f^{-1}(y)$ and the output column as x. In Tess's first attempt, she labeled the input column as x^{-1} ("because it's the input to the inverse function") and the output column as $f^{-1}(x)$. While both students could describe the reversal of the process of f, as well as the input and output quantities of the function f^{-1}, the tables that Randy and Tess individually produced did not reflect this understanding.

Likewise, Madison's three attempts to construct a table of inverse values (see *Figure 14)* revealed her struggle to coordinate her meanings with the desired notation. For this particular task, the student was provided with a table of values for some function f and then she was asked to use the given values to produce a table of values for f^{-1}. The following excerpt, is the conversation I had with Madison after her second attempt to construct the table, in which she labeled the left column x and the right column $f^{-1}(y)$. Madison (lines 53 and 55) expressed that that the phrase, "2 is the input to f" is more helpful for her than saying, "f of 2". Again, I claim that this is evidence that Madison was struggling to coordinate her meanings involving the relationship between input values and output values of a given function with the conventions used for function notation and the construction of tables.



Images starting at top left and continuing counter clockwise: given table, first attempt, second attempt, and third attempt.

Figure 14

46.	I:	I: So we want to make a table of output values of f^{-1} [writes $f^{-1}(y)$], so you're right, there, output values of *f* inverse, and then [points to left column of the second table] the input values to *f* inverse, which would be *y*.
47.	M:	[Scratches out label *x* in the table and replaces it with *y*].
48.	I:	Um, and then we have to reverse the table, right? If I know [points to column *x* in given table] this is the input to *f* and [points to *f*(*x*) column in given table] this is the output to *f*, 0.5 would be the input to *f* inverse.
49.	M:	[Begins constructing third table by labeling columns *y* and $f^{-1}(y)$.] Yeah, because you're putting in 0.5 and then you're going to get out -1.
50.	I:	Mhmm.
51.	M:	[Adds the pair (2,0) to the third table.]
52.	I:	You did it for this example [points to first subtask]. When 2 is the input to f^{-1}, the output to f^{-1} is zero.
53.	M:	I think I need to just like talk about it like that. Instead of saying like *f* of *x*, the input to *f* is…
54.	I:	The input to *f*, uhuh.
55.	M:	As far as helping kids learn, I think talking like that definitely helps.

This provides evidence to suggest that Madison was not considering the relationship between input and output values when viewing a table. Further evidence of this behavior in the context of graphs was evidenced in Randy's comments about the graph of the function g, with the x-axis labeled "Number of Years since Bill Made His Investment" and the y-axis labeled "Value of Bill's Investment (In Dollars)". The student was asked to determine the value of $g^{-1}(400)$, and the value of a so that $g^{-1}(a) = 8$ using the graph. Once Randy was able to correctly determine those two values, we had the following conversation:

56.	I:	How are you relating, I'm assuming this is what you had to do, you had to relate the output and the input of g^{-1} to the input and the output of g. How are those related? Like what's the input and output of g, what's the input and output of g^{-1}?
57.	R:	Ok, so this, the output of g^{-1} of a equals 8, I basically went to the uh, the output which, basically, in this graph, if I were to do the inverse it would be flipped [pointing to the two axes], but uh, I went from 8 and then found 475. And the same with 400 [pointing to the first subtask] I went from 400 [points to y-axis and traces finger over to graph and then down to x-axis] and found 6, I went down to find the number of years.
58.	I:	Ok, so what are… so for g, what is the input quantity and the output quantity?
59.	R:	g, for g it would be uh…number of years.
60.	I:	That's the input or the output?
61.	R:	Um…this would be the input [points to "Number of years since Bill made his investment"].
62.	I:	Of g?
63.	R:	Of g and… this [points to "Value of Bill's investment (in dollars)"] would be the output of g.
64.	I:	Ok. And what would be the input and the output quantity of g^{-1}?
65.	R:	g^{-1}, uh…um…the input would be the number of dollars Bill invested, and then the output would be the number of years that Bill made his investment.

Based on the number of pauses and "um's" in Randy's responses to my questions about the input and output quantities of *g* and *g* inverse, it appears that he did not have a clear conception of the covarying quantities in the situation. His response provides further evidence that he was manipulating symbols with tables and graphs by memorizing procedures that allowed him to reach correct answers without understanding that a function table and graph are a dynamic illustration of a functional relationship between inputs and outputs. I contend that if these students have not conceptualized a table or graph as a representation of a mapping from one varying quantity to another, there is little meaning that they can draw from that table or graph other than a set of coordinate pairs. With little meaning for graphs or tables, it seems unlikely for a student to have developed meaning behind the actions they perform with tables and graphs. Thus, such a student would be left with no other option than to cope by memorizing meaningless procedures.

I had not anticipated this obstacle when planning the instructional sequence, and therefore, argue that the extent of tabular and graphical problems in the instructional sequence did not provide enough opportunities for students to develop the desired understanding of a graphical or tabular representation of a function and its inverse. The results suggest that students may benefit from repeated opportunities to describe and interpret the relationships conveyed by function graphs and tables, and that these ways of representing function mappings should be discussed and practiced when the idea of representation of a function is initially introduced. Then, the student could potentially apply those understandings to graphs and tables when building his meanings for inverse functions.

In the last two tasks of the instructional sequence, which did not include tables or graphs, the three students were once again successful. In Task VIII, the students were told that if one were to start with some number x and apply a process A and then apply another process B, the result would be some number y. Without any additional information about the two processes, A and B, the student was asked to explain how to reverse the application of these combined processes. As each of the three students had been able to generalize their thinking in Task IV when they each described their general approach to reversing a process, this task would require them to further generalize their thinking to the reversal of a combination of processes.

In each session, I told the student that they could describe their ideas verbally, with notation, or with some diagram. Randy and Madison both chose to utilize function notation and were successful in expressing the original process as $B(A(x)) = y$ and the reversal of that entire process as $A^{-1}(B^{-1}(y)) = x$. Randy's response follows:

66.	R:	So, so to reverse that…start with… and apply A and B [he continues reading the task for the third time]. So, [writes out $A(B(x)) = y$] this would be the same, right? No, we apply A and then B [writes $B(A(x)) = y$].
67.	I:	Ok that sounds fine, if you're thinking of A and B as functions.
68.	R:	Yes.
69.	I:	So you apply A first, and then apply B to get from x to y. That looks fine to me. So, what I'm looking for, how could I start at y and end at x?
70.	R:	Mkay…
71.	I:	So I've not taught you this. I don't know if you've learned this. You don't have to use notation. You can, you can draw a diagram, or you can just verbally say what you would do.
72.	R:	So based on this, [writes $A^{-1}(B^{-1}(y)) = x$].
73.	I:	Ok, walk me through it.

74.	R:	So um, originally I said with function notation, A of x, originally I believe, reading this [pointing to $B(A(x)) = y$], I start with the process A of x and get that and apply B to get y. Um, I think that with doing the inverse or reverse function of B and then applying A inverse, I would get x. I would work my way backwards.
75.	I:	So, why do you do B and then do A? [Pointing to $A^{-1}(B^{-1}(y)) = x$]
76.	R:	Well because, um I'm just doing the backwards, I'm backwards pedaling basically, in which case I ended with B [pointing to $B(A(x)) = y$], so I should start with B in this case [points to $A^{-1}(B^{-1}(y)) = x$] and then end with A.

Unlike Randy and Madison, Tess chose to construct a diagram in order to represent how the application of process A followed by the application of process B could be reversed (see *Figure 15*). After she had constructed the first diagram on the left to represent the application the two processes A and B, we had the following conversation about the reversal:

77.	T:	Ok, so, if you're starting at y, you have to take, you have to start with y. You'd have to do, first, the inverse of B process, so B inverse process, which would give you ending number h. And then to find your original number [circles x in previous diagram] that we don't have, you have to apply the A process, the A inverse process, excuse me, to find some number x.
78.	I:	Perfect, um, is this h [in the second diagram] the same as this h [points to h in first diagram]?
79.	T:	Yes, it's the same h. Because, you're just in reverse. If you're looking at it, if there's two numbers in the middle [points to h in first diagram], depending on what process, you're always going to end up with the same middle number if you reverse those processes.

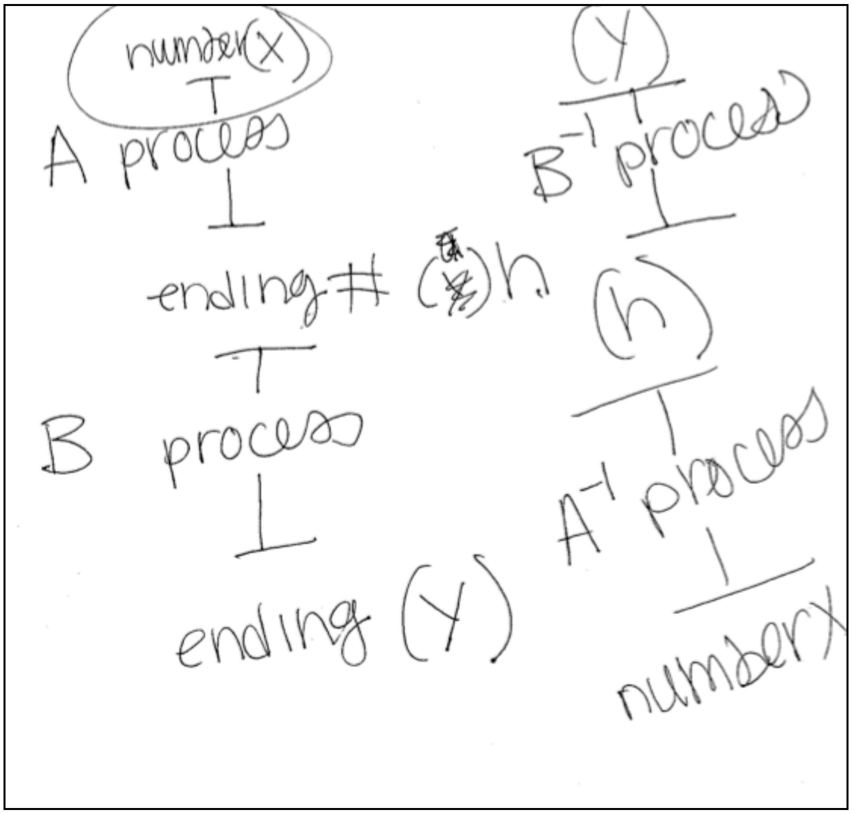


Figure 15

Each of the students' responses mirrored the reasoning in their responses to the first four tasks of the instructional sequence. When reversing a process, as the examples given in Tasks I-IV, the students had described reversing each step of the given process and applying those steps in the reverse order. In the case of Task VIII, the students had to first envision the steps and their order of application in the original combined application of processes A and B. In the case of Madison and Randy, this step was required in order for the students to know that the composition $B(A(x)) = y$ was appropriate, rather than $A(B(x)) = y$ (see Randy's explanation in line 66). This step was also required of Tess when designing her diagram and deciding when process to apply first.

Then, since the students had previously considered applying the reverse of each step in the reverse order, they were now able to conclude that they needed to apply the reverse of each process in the reverse order. The design of the instructional sequence had accommodated the students' engagement in the reversal of processes, reflection on the necessary actions involved in the reversal of processes, and the generalization of those actions. Then, the students were able to extend those generalizations to a context in which they envisioned reversing the combination of two processes.

The three students' responses provide further evidence that each of the students had developed a process conception of function. Dubinsky wrote that a process has been internalized when the subject has a total picture of the process and is capable of moving back and forth applying and reversing the mental actions involved (1991). Each of the students expressed the ability to anticipate applying a process to an input to result with some output, which could then serve as the input to a second process. This was the case with both the original application of processes A and B, as well as in the reversal of that entire process. With a process conception of function, the students were capable of reasoning about this reversal without being given the step-by-step details of each process.

Post-test

The first activity of the post-test asked the student to evaluate his or her work on the pre-test and express what they might change or approach differently after their participation in the teaching session. In doing so, all three students were able to correctly determine the value of f^{-1} for a given input value, when given the rule for the function f. While, Madison determined the algebraic rule for the inverse function first, Randy and

Tess chose to determine the value of $f^{-1}(25)$ before determining the rule for f^{-1}. They were able to do this by setting the rule for $f(n)$ equal to 25 (i.e. $25 = 0.45n+7$) and solving for the appropriate value of n. This was a significant shift for Tess, who, four days prior, had not distinguished between the two functions f and f^{-1} and had simply treated the input to f^{-1} the same as the input to f. Randy and Tess were now capable of relating a function and its inverse in such a way that the input to the inverse function is the output to the original function, and vice versa.

All three students were also able to correctly alter their definition of the inverse function, which included the correct rule of assignment with the appropriate input variable for the given context. Randy and Tess constructed the inverse function by solving for the input to f, which was the number of pounds of apples that the customer picked n, in terms of the output to f, the cost of the customer's load of apples. This was the approach to defining the rule of the inverse function that they had developed in the Celsius to Fahrenheit conversion task during the teaching session. Randy, who had utilized the approach of swapping the variables in his pre-test, claimed the he would no longer swap the variables "because its not always appropriate. Here we want $f^{-1}(c) = n$, not $f^{-1}(n)$." This is further evidence that Randy had shifted towards focusing on the relationship between input and output quantities.

On the other hand, Madison utilized her understanding of the actions involved in the reversal of a process. Madison initially wrote $f^{-1}(c) = (c/0.45) - 7$, and we had the following conversation:

80.	I:	How do you know that you divide by 0.45 first and then subtract 7? How did you decide that?
81.	M:	Well, if you're looking at this one [had previously written $f(1) = 0.45(1) +7$] you're going to multiply it by 0.45 and then you're going to add 7. But then, ooh, you added 7, so this would be c minus 7 [added parenthesis around $c - 7$] and then you divide. [This changed her response to $f^{-1}(c) = (c - 7)/0.45$.]
82.	I:	So you changed it. Ok, why did you change that?
83.	M:	Because you have to change the order, too. Since I multiplied by 0.45 first, I divide by 0.45 second because you have to reverse the order.

In her response, it is clear that Madison is considering the reversal of the given function's process. This shows that she now associates finding the inverse of a function with reversing the function's process. She has shifted from trying to remember some meaningless procedure, to now applying a generalization she developed during the teaching experiment. She illustrated that she understands that reversing a process involves reversing each of the actions of the original process and applying them in the reverse order. She also attended to how the two quantities of interest were being related by expressing that c, the output of the original function, was now the input to the inverse function.

Thus, the three subjects, at varying levels and from various educational backgrounds, were now capable of determining the rule for an inverse function without having to be taught a procedure. The method of determining the inverse function was no longer something that the student needed to memorize and recall, because it followed from their understanding of reversing a process.

As mentioned in the analysis of the pre-test, none of the students had referenced the input or output quantities in the given context prior to the teaching session. However, in re-examining their previous work, two days after the teaching session, Randy and Madison showed signs of attending to the quantities by explaining that they would now feel the need to label the axes with the input and output quantities for the graphs of the functions f and f^{-1}. This suggests that these two students made shifts towards the perception of a graph as a coordination between the values of two varying quantities. This is likely a product of the instructional sequence in which the student was exposed to several contextual problems where he or she was repeatedly asked to describe how the given input and output quantities were being related.

Unfortunately, four days after the teaching session, Tess chose to draw the graphs of f and f^{-1} on the same axes and did not label the axes with the quantities being represented. When asked why she had only used one axis rather than two axes as she had before, she claimed that her decision was made for convenience and that she could also have drawn the graphs on separate axes. She chose to type the two functions into her calculator and sketch the graphs that she saw, rather than reasoning about how the linear functions could be graphed. Her responses to this question of graphing further supports my conjecture that her understanding of a graph was just some shape in the plane, rather than a representation of the relationship between input and output quantities. While I argue that Tess had developed a process conception of function during the teaching session, action and process conceptions of functions are not mutually exclusive, and Tess may be more inclined to apply one conception over another in various contexts.

To further support the claim that each student had developed a process conception of function (including inverse functions) during the teaching session, each of the three students, when given that f was a function, where able to conclude that $(f^{-1})^{-1}$ would be the original process f. When Randy answered this task, he said he was "100% confident" in his answer.

As shown in the following conversation with Madison, she was able to imagine a process from input values to output values, as well as the reversal of that process and the effect it would have on the input and output values.

84.	M:	I think you're gonna end up with your original, like your function.
85.	I:	Ok. Are you guessing or do you have a reasoning?
86.	M:	Well because, uh, I was like doing this, well, I think I was getting confused because when you have like an inverse you put in an input and you get out an output, which is going to be like the other one's output and input. Those are flipped. So I was like, well, you'll end up with those same values, but they're still going to be flipped, but then if you take the inverse of it that's what's going to get you back to the original. So that's [$(f^{-1})^{-1}$] going to get you just to f.

In her response, Madison appears to understand that the input quantity of the original function will be the output quantity of the inverse function, and vice versa (line 86). In Madison's perception, taking the inverse of some function causes the input and output quantities to be swapped. She was able to utilize this understanding of the input and output quantities of $(f^{-1})^{-1}$ to conclude that it would be equivalent to the original function f.

In attempting this same task, Tess and I had the following conversation:

87.	T:	So you're taking the inverse of the inverse function. Wouldn't that just put it back to the original? [Points to f in problem statement.]
88.	I:	Ok.
89.	T:	When you take an inverse of an inverse, you get back to your original function, right?
90.	I:	Ok. Can you tell me why?
91.	T:	Um, because…

92.	I:	Or, how could you argue to some one why you're right?
93.	T:	So, when you take the inverse of some function [writes f], you're just undoing that same function [writes f^{-1}]. So, if you undo the inverse function, you'd come back to your original. So, its kind of like taking one step forward [draws an arrow from f to f^{-1}] and then one step back [draws an arrow from f^{-1} to f] (See Figure 16).

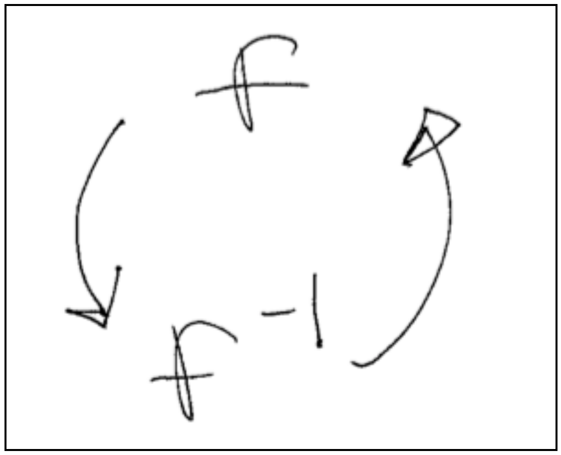


Figure 16

I claim that the students would need a process conception of function to be able to consider the meaning of $(f^{-1})^{-1}$, given that f is some function. If a student merely had an action conception of function, the student would not know what actions to carry out, and would not be able to see that composing a function with its inverse always returns the original input value. Instead, the student has to envision a general process, the reversal of that process, and then a second reversal. APOS theorists claimed that the ability to envision inverting a function required a process conception of function. It may also be argued that the student has an object conception of function since he is able to consider applying a transformation to the function f^{-1}.

As foreshadowed in the teaching session, students continued to have difficulty interpreting the quantitative relationships displayed by function graphs during the post-test. As I had not expected the students' lack of meaning for a table of function values, I did not design any table tasks on the post-test. In the first task of the post-test, the student was given the graph of the function *f* and was asked to determine the value of *a* so that $f^{-1}(a) = 2.5$. Tess and Randy were able to correctly use the graph of *f* to solve for this value of *a*. This required these students to understand that the output of the inverse function would be the input to the original function, and vice versa. Then, the students had to apply this understanding to use the graph of the function *f* to determine the value of *a*.

Madison, on the other hand, was less successful. She seemed to either not recognize or ignore the fact that the given graph was for *f* and, therefore, that the input and output of f^{-1} would have to be found in an alternative way. Rather than viewing the output of f^{-1} as the input to the function *f*, she viewed the output of f^{-1} the same as one would view the output of *f*. While describing inputs and outputs in her answer, Madison never referred to the function whose inputs and outputs she was discussing.

94.	M:	I think *a* would be 3.
95.	I:	Ok, can you tell me how you got that?
96.	M:	Well, *f* inverse, well its saying like when you input *a*, you're getting 2.5. Or is it saying that *a* is… No, no, when you input *a* you get 2.5. So if *a* is the input and the output has to be 2.5. So the input's on the *x*-axis. The output is on the y-axis. So, I went to 2.5 on the *y*-axis and went over to see where it touched the line and got 3.

Although she seemed to understand the conventions of locating the input quantity on the x-axis and the output quantity on the y-axis, she had not coordinated the input and output quantities of the function f^{-1}. Therefore, her knowledge of these conventions was not enough for her to successfully answer the question.

In addition to finding the value of a so that $f^{-1}(a) = 2.5$, I asked the students if it was possible to determine the value of b so that $f^{-1}(f(b)) = 2$, or $f^{-1}(f(b)) = 3$. During the teaching session, each of the students had been able to describe, in their own words, that the product of the composition of a function and its inverse would be equivalent to the identity function. The idea of applying a function followed by its inverse (or in the opposite order) had been considered in Tasks I-V. For example, in Task I with the grid, each of the students could visually illustrate that if we started at the "Start" point, applied the original process and then applied the reverse process, that we would end up back at the "Start" point. The students were asked to consider this type of question for the fifth time in Task V where, given a particular temperature measured in degrees Celsius, the student could apply the original function, and then the inverse function, and the output would be the same original temperature measured in degrees Celsius. None of the students had difficulty answering these questions.

This task on the post-test was designed to determine if the students could generalize their reasoning about the composition of a function and its inverse. The three students' responses to this task when given the graph of f were diverse. Unfortunately, after a few attempts, Tess claimed that she couldn't remember how to do this. Rather than attempting to utilize her understanding of function inverse, it appeared that Tess had reverted back to her old habits of trying to remember some procedure. On the other hand, Randy and Madison both argued that the presence of the graph of *f* made them think differently about $f^{-1}(f(b)) = 3$ than if the graph had not been given.

I anticipated that, in trying to determine the value of *b* so that $f^{-1}(f(b)) = 3$, the students would note that the graph was unnecessary since they were applying a process and then subsequently applying the inverse of that process (granted, of course, that we were describing values in the domain of each function). This was not the case. In Randy's initial attempt to answer the question, he utilized the graph to find that $b = 3$. I then chose to ask whether he could determine the value of *b* so that $f^{-1}(f(b)) = 10$. I chose the value of 10 because the values on the graph did not lend themselves to Randy's method. As I had anticipated when posing this question, Randy responded by saying that it was impossible to determine this because there was not enough information given in the graph since the input values to *f* were cut off after 5. When I moved on to the next task, Randy whispered, "Is that right?" We had the following conversation:

97.	I:	You can…Do you want to keep thinking about it?
98.	R:	[Points to $b = 3$.] That one. Is that correct?
99.	I:	What do you… so what do you think about the inputs and outputs of *f* and f^{-1}?
100.	R:	Well I mean [points to $f^{-1}(f(b)) = 10$]…if I wasn't using this function [points to given graph of *f*] it would still be 10 for the *b*.
101.	I:	Why?
102.	R:	Because if the…if I'm inputting something and then I'm doing the opposite…trying to get the reverse function of that, then I'm going to get 10 in this case. But in using this scenario [points to graph of *f*] it wouldn't work that way.
103.	I:	Ok. Interesting. Did you use that reasoning…so you said whatever I plug in is what I'll get because you're doing the opposite…did you use that reasoning when you were trying to solve for this [points to $f^{-1}(f(b)) = 3$] or you just used the graph?
104.	R:	Yes, I used the graph.

105.	I:	So why did the graph change…you were using a different way of thinking…so if I didn't give you this [covers up the given graph] and I gave you the same question…
106.	R:	It would be 3.
108.	I:	But giving you the graph,
109.	R:	Made me think of it differently because its in a scenario [points to the labels on the graph's axes.]

Madison and I had a similar conversation regarding the same task, except that she had been asked to determine the value of b so that $f^{-1}(f(b)) = 2$:

110.	M:	Alright, this one is like…you take the inverse of a function…you just end up back with 2…yeah?
111.	I:	When you take the…
112.	M:	Well because if you have *f* of *b*, taking the inverse of that is just gonna undo it again. So b would be… your input and 2 would be your output. (Pause.)
115.	I:	You apply the inverse and you said that "undoes" it or reverses it.
116.	M:	So you get that *b* is gonna be the output *a*. [Writes $b = a$.] And when you input *a,* the output you get is…(pause)
117.	I:	Sorry, what was *a*?
118.	M:	Well like I'm going off of, since like this one [points to $f^{-1}(a) = 2.5$ in the previous task] the inverse of this was like *a* was the input. Then like if you input *a* you must get *b*.
119.	I:	Input *a* to what?
120.	M:	Well like you're inputting *f* of *b*… so you're inputting *b* and then you're going to get an output, which is going to be *a*. Then you take the inverse of that, which is going to land you back at *b*. So $b = 2$. But *b* was the input…originally…so then it must be [goes to graph] 2 as input…so 5 would be…5 would be the output. 5 would be *a*, so the input just would be 2.
121.	I:	So *b* would be equal to 2. Ok. So is it just a coincidence that these are the same [points to $b = 2$ and output of $f^{-1}(f(b)) = 2$] 2 and 2?
122.	M:	Oh I was just like…well yeah because like I was just overthinking it. I mean the inverse… like if you take the inverse of a function you're just gonna end up with the original because you're doing something and then just undoing it.
123.	I:	Ok. So you said you were overthinking it. Ok. So you could have just wrote $b = 2$?
124.	M:	Yeah.
125.	I:	Ok. Why do you think you did all that extra work?
126.	M:	Because I was looking at the graph.

In their responses, both Madison and Randy conveyed that when considering an expression like $f^{-1}(f(b))$, they would start with an input, apply the process f, get some output, apply f^{-1} to undo the previous process they applied, and end up with the original input value. For example, consider line 106, when Madison said, "you're inputting *b* and then you're going to get an output, which is going to be *a.* Then you take the inverse of that, which is going to land you back at *b*." Here she was describing the output of f becoming the input to f^{-1}, which would result in a return to the original input value. However, both of Randy and Madison felt that they needed to utilize the graph to determine their answer. While neither Randy nor Madison spontaneously applied their conceptual knowledge about applying a function and applying a function's inverse, they each not only described their understanding of this composition, but they also displayed their ability to coordinate inputs and outputs of the composed function with the inputs and outputs displayed in the given graph of the function f.

CHAPTER 6

CONCLUDING REMARKS

The evidence presented in this paper supports the claim that this proposed instructional sequence has the potential to help students construct meanings for function inverse. This was shown by each of the student's ability to, not only, determine a value of the inverse function when given the original function, but to also define the algebraic rule for the inverse function. All three students were also able to consider the inverse of an inverse function. And, two of the three students were able to describe the result of composing a function and its inverse.

I conjecture that the shifts in their understanding was achieved by posing questions that promoted reflective abstraction, and the questions and interactions led to all three students moving toward a process conception of function. The instructional sequence was designed to engage the student in the repetition of particular reasoning patterns and conclusions in a variety of contexts in order to provide an opportunity for him to reflect on his actions and the products of those actions when comparing the tasks. I hypothesize that the reflection on actions, or the act of reflective abstraction, is what allowed the students to make the generalizations that they did.

One such generalization, articulated by each of the three students during the teaching session, was that when reversing a process, the reverse of each action must be applied in the reverse order. The first three tasks were specifically designed so that the student could repeatedly engage in the mental actions of reversing a given process and then generalize those actions to reverse an unidentified process in the fourth task. By repeatedly constructing the reversal of a process and then naming that new process using function inverse notation, the evidence suggests that the students began to construct a meaning for a function's inverse as the process that reverses, or undoes, the original function process.

I also argue that the students were able to generalize the results they constructed to conclude that the input and output quantities of a function and its inverse will be swapped. This idea played a role in each of the nine tasks and I repeatedly had the students identify the input and output quantities of the original function, as well as its inverse. This focus on quantities during the teaching session is likely what allowed all three students to shift towards attending to and reasoning about the input and output quantities of a function and its inverse.

And, finally, I hypothesize that reflecting on and then generalizing the actions that they engaged in during the numbers game and the Celsius to Fahrenheit conversion is what allowed each of the three students to construct their own approach to determining the algebraic rule for a function's inverse. The instructional sequence was designed so that the student would solve a given function for its input value when given specific output values. After this was carried out multiple times, the student was then asked to extend his thinking to consider solving for a function's input value for any output value. In this way, the design was intended to prompt the student to reflect on his previous actions and generalize those actions in order to construct the given function's inverse. In doing so, the student was able to utilize their understanding of solving to construct their own approach to determine the algebraic rule of a function's inverse.

It is important to note that for two months prior to the study, the three subjects had been students in a conceptually oriented pre-calculus course, where the development of reasoning abilities was prioritized over “answer-getting” strategies. In addition, it is also important to note that this was not the first time these students were introduced to the concept of inverse function. So, while the students may have had the benefit to construct powerful meanings about function inverse prior to the teaching session, there was also the potential that the students came in with undesirable meanings and procedures. Further research would need to be conducted in order to assess whether this instructional sequence could produce the same effects with students who had never been introduced to the concept of function inverse.

As reported in the results, the three subjects initially struggled to coordinate their conceptions about an inverse function and its input and output quantities with function notation. They also had difficulty constructing and interpreting values in tables and graphs. If this instructional sequence were to be implemented in a classroom, I recommend that more tabular and graphical tasks that prompt them to express what the tables and graphs are relating be added to both the instructional sequence and homework set. It is not clear how much opportunity for repeated practice is needed for particular students. Further investigations are needed to understand the role of homework and the amount of practice that is needed for students at various ability levels to construct the desired meanings. I also argue that further research is needed to learn about students’ understanding of function notation, graphs, and tables as representations of a functional relationship between two covarying quantities.

REFERENCES

Asiala, M., Brown, A., De Vries, D. J., Dubinsky, E., Mathews, D., & Thomas, K., (2004). A Framework for research and curriculum development in undergraduate mathematics education. *Research in collegiate mathematics education II* by Kaput, Jim, Schoenfield, Alan H. & Dubinsky, Ed (Eds.), CBMS Issues in Mathematics Education, 6, 1-32.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics,* 23*,* 247-285.

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. *Research in collegiate mathematics education III* by E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), 114-162.

Carlson, M., Oehrtman, M., and Engelke, N. (2010). The Precalculus Concept Assessment: A Tool for Assessing Reasoning Abilities and Understandings of Precalculus Level Students. *Cognition and Instruction* 28, no. 2, 113-145.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking*, 95-126.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In G Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy*. *MAA Notes, 25,* 85–106.

Dubinsky, E. & McDonald, M.A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. *New ICME Studies Series, 7,* 275-282.

Esty, W. (2005). Teaching about inverse functions. *The AMATYC Review*, 26(2), 4-10.

Glasersfeld, E. von (1991). Abstraction, re-presentation, and reflection: An interpretation of experience and Piaget's approach. In L.P. Steffe (Ed.), *Epistemological foundations of mathematical experience*, 45-67.

Hunting, R. (1997). Clinical interview methods in mathematics education research and practice. *Journal of Mathematical Behavior,* 16 (2), 145-165.

Moore, K., Weber, E., and Thompson, P.W. (submitted) Graphs are more than shapes.

Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics*, 27-42.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education*, 267-307.

Thompson, A. G., Philipp,R. A., Thompson, P. W., and Boyd, B. A. (1994). Calculational and conceptual orientations in teaching mathematics. In A. Coxford (Ed.), *1994 Yearbook of the NCTM*, 79-92.

Van Dyke, Frances. (1996). The Inverse of a Function. *Mathematics Teacher* 89, no. 2, 121.

APPENDIX A

HUMAN SUBJECT CONSENT FORM: IRB # 1108006730

Consent Form

Dear ______________________(Participant):

I am a professor in the School of Mathematical and Statistical Science at Arizona State University.

I am conducting a research study to learn about teacher's learning and its impact on teaching. I am inviting your participation, which will involve using data that is collected as part of coursework for research purposes.

Your participation in this study is voluntary. You can skip questions if you wish. If you choose not to participate or to withdraw from the study at any time, there will be no penalty. It will not affect your grade. You must be 18 or older to participate in the study.

The results of this study may be used in reports, presentations, or publications but your name will not be known/used (whichever applies). Results will only be shared in the aggregate form.

If you have any questions concerning the research study, please contact the research team at: (480) 964-6188. If you have any questions about your rights as a subject/participant in this research, or if you feel you have been placed at risk, you can contact the Chair of the Human Subjects Institutional Review Board, through the ASU Office of Research Integrity and Assurance, at (480) 965-6788.

This instrument was designed by Marilyn Carlson's research group. If you have any questions about this instrument, you may contact Dr. Marilyn Carlson by e-mail at: marilyn.carlson@asu.edu

By signing below, you give permission for my responses to these questions to be used for the purpose of validation of this assessment instrument. I understand that all data is confidential. The researcher will not associate my name with my score and my identity will not be disclosed to any party not associated with this research project.

***Name:*____**