Conceptions of Function Composition in College Precalculus Students

by

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ABSTRACT

Past research has shown that students have difficulty developing a robust conception of function. However, little prior research has been performed dealing with student knowledge of function composition, a potentially powerful mathematical concept. This dissertation reports the results of an investigation into student understanding and use of function composition, set against the backdrop of a precalculus class that emphasized quantification and covariational reasoning. The data were collected using task-based, semi-structured clinical interviews with individual students outside the classroom. Findings from this study revealed that factors such as the student's quantitative reasoning, covariational reasoning, problem solving behaviors, and view of function influence how a student understands and uses function composition. The results of the study characterize some of the subtle ways in which these factors impact students' ability to understand and use function composition to solve problems. Findings also revealed that other factors such as a students' persistence, disposition towards "meaning making" for the purpose of conceptualizing quantitative relationships, familiarity with the context of a problem, procedural fluency, and student knowledge of rules of "order of operations" impact a students' progress in advancing her/his solution approach.

DEDICATION

This dissertation is dedicated to three people who have influenced me more than they can imagine. My wife, Kimberly, has given me everything, and followed me everywhere; she deserves a great deal of the credit for anything I may ever accomplish. My son, Nick, serves as an inspiration to me every day, and makes me want to be the best role model I can possibly be. My father, Donald, taught me at a young age that I could do almost anything, if I worked hard enough toward it. This work is dedicated to the three of you.

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CHAPTER 1

INTRODUCTION

This dissertation describes an investigation into precalculus students' ways of knowing function composition and using function composition in solving novel problems. This chapter presents the motivation for selecting function composition as a topic for investigation and the research questions for the study.

Educational policymakers often agree on the importance of function composition in high school mathematics. The National Council of Teachers of Mathematics (NCTM) *Principles and Standards* (1989; 2000) advocate that students in grades nine through twelve should be able to "understand and perform transformations such as arithmetically combining, composing, and inverting commonly used functions" (p. 296). As students build this knowledge and capability, NCTM proposes that students will "come to understand the concept of a class of functions and learn to recognize the characteristics of various classes" (p. 296).

Little research has been published concerning students' understanding of function composition, despite NCTM emphasizing the importance and desirability of students building a robust understand of, and fluency using, function composition. This study addresses this research gap, analyzing student products and behaviors from task-based clinical interviews in an attempt to better understand how students know and use function composition.

Statement of the Research Questions

This study investigated precalculus students' conceptions of function composition, including the mental actions students perform when solving problems requiring (from the researcher's point of view) the understanding or use of function composition. This study was situated in the context of a precalculus course that emphasized quantitative reasoning, covariational reasoning, problem solving, and the concept of function. The primary research questions for the study were:

1. What is the nature of precalculus students' understanding of function composition?
2. What reasoning abilities and understandings support precalculus students in understanding and using function composition?
3. What factors are important facilitators of, or obstacles to, students' possessing a robust conception of function composition?

My motivation for studying precalculus students' understanding of function composition stems from my own experiences as a student and as a teacher, as well as from reports in research literature suggesting student difficulty with function composition (Carlson, 1998; Dubinsky, 2002; Engelke, 2007). As a teacher of first year college mathematics courses, I have observed that many students have difficulty understanding the concept of function composition. Many of these students subsequently have difficulty knowing when and how they might use function composition when attempting to solve novel problems. For many students function composition seems to be just another procedure, an end unto itself rather than a concept that can help students improve their understanding of other mathematical concepts and problem contexts. This is not surprising; many curricula introduce function composition, then proceed to show students procedures for computing answers to function composition problems.

Indeed, function composition can provide a powerful tool for students to understand topics in more advanced physics and engineering classes. For example, electrical engineering students in a communication systems class work extensively with sinusoidal waveforms, but the functions of interest are rarely of the form $f(t) = \sin(t)$. Rather, students find themselves studying properties of functions of the form $g(t) = A\sin(Bt + C) + D$, or even more complex combinations of such functions. For the student who has a fragile understanding of function composition, describing the period of *g* may be difficult and the student may rely on memorized procedures. However, the student who is able to work fluidly with the concept of function composition can approach this problem differently. He can reason that the output of *g* will complete one full period when the input to the sine function varies from 0 to 2π. The input to the sine function will vary from 0 to 2π when the output of $h(t) = Bt + C$ varies from 0 to 2π. The output of *h* will vary from 0 to 2π when the input to *h* varies from $t = -\frac{C}{B}$ to $t = \frac{2\pi - C}{B}$.

As a student, my own introduction to function composition came in a high school precalculus course, and it was not until many years later that I came to see connections between function composition and other mathematics topics. Function composition may facilitate a better understanding of a wide range of applied mathematics problems, from problems in which inputs are subjected to multiple transformations, to related rates problems, a connection which was investigated by Engelke (2007). In hindsight, I believe that a more robust understanding of function composition would have facilitated my understanding of function (and vice versa), inverses, and function transformations such as $f(x+a)$ and $f(kx)$ in relation to $f(x)$.

This study contributes to the growing body of research on precalculus students' mathematical knowledge and learning, by adding to the sparse research into knowing and using function composition. My research findings contribute new theory and it is my intent that my findings will inform curriculum and instructional revisions. It is my hope that curricula informed by my research findings will emerge as effective for promoting the development of more robust understanding of function composition among precalculus students.

CHAPTER 2

REVIEW OF THE LITERATURE

This chapter provides a review of select research literature relevant to this study, beginning with a review of prior research on student understandings of function and function composition. This review necessitates a discussion of research on problem solving, quantitative reasoning, and student understandings of variable, relative to conceptions of function composition. The chapter concludes with an examination of the treatment of function composition in several common precalculus textbooks.

Function

Before turning to function composition, it is worthwhile to examine existing research on student understandings of the concept of function. Much has been published on how students learn the concept of function and the level at which they understand function (Carlson, 1998; Dubinsky & Harel, 1992; Monk, 1992; Oehrtman, Carlson, & Thompson, 2008; Sfard, 1992; Sierpinska, 1992; Thompson, 1994). These studies not only emphasize the importance of understanding the concept of function for understanding more advanced mathematics, but they also illustrate many of the ways in which students' understanding of function is impoverished.

Sierpinska (1992) notes that for a person to make significant progress in understanding the concept of function, the person "has to notice changes and relationships between them as something problematic, worth studying" (p. 31). Sierpinska goes on to assert that "the notion of function can be regarded as a result of the human endeavor to come to terms with changes observed and experienced in the surrounding world" (p. 31). Mathematically, this suggests that success in coming to terms with observed changes requires, in part, an ability to quantify attributes of situations (an idea discussed later in this chapter), construct varying quantities and to see a purpose for relating them mathematically. Indeed, Sierpinska notes:

> In studying functions it is important to bring students to perceive and verbalize the subjects of changes: students should be able to say not only *how it changes* but also *what changes* (p. 57).

Exploring this notion further, Monk (1992) observed that students typically think about particular functions in a static sense (what he termed a pointwise view of functions), and have difficulty coordinating changes in one quantity with changes in another quantity (which Monk called an across-time view of functions). Monk also studied how students interpret graphs of functions, noting that to correctly interpret a graphical representation of a function, a student must be able to view a point on the graph as representing a specific pair of values of two varying quantities. “Moving along the curve” of a graph requires the student to coordinate the changing magnitudes of the values of two quantities. When a student is unable to coordinate the values of two quantities illustrated in a graph, the student may exhibit the behavior Monk called *iconic translation*. This inability to quantify problem attributes and/or develop dynamic mental models of the quantities changing in tandem is also referred to as “shape thinking” (Thompson, 1989).

Action and process views of function.

Since developing an understanding of functions as relating the values of varying quantities is a useful goal, it is important to understand the factors that are central to a student’s ability to understand the concept of function and reason about changing quantities. According to Dubinsky and Harel (1992), a key to the student's ability to reason effectively about functions is the student's development of a *process* conception, as opposed to an *action* conception, of function. As Dubinsky and Harel write,

> An *action*…conception of function would involve, for example, the ability to plug numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one step at a time (e.g., one evaluation of an expression). A student whose function conception is limited to actions might be able to form the composition of two functions given via algebraic expressions by replacing each occurrence of the variable in one expression by the other expression and then simplifying. Nevertheless, he or she would probably be unable to compose two functions in more general situations, e.g., when they were given by different expressions on different parts of their domains (p.85).

Students who have this limited view form various misconceptions of functions. These range from believing that a piecewise function is several distinct functions rather than a whole function or that two functions are only equal if they "look" the same. To such a student, $f(x) = x^2$ would not be the same as $g(x) = \sum_{k=1}^{x} 2k + 1$, on the natural numbers (Oehrtman, Carlson, Thompson, 2008). A student with an action view is unable to look past specific computations, or procedures, in order to see a function as simply accepting input and producing output, regardless of the algorithm. To generalize the above, a student with an action conception of function sees "a command to calculate" and thinks of the algebraic expression as only producing a calculated result.

We can contrast the *action* conception of function described above with the *process* conception of function. Again according to Dubinsky and Harel (1992),

> A *process* conception of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result. When the subject has a process conception, he or she will be able, for example, to combine it with other processes, or even reverse it. Notions such as 1-1 or onto become more accessible as the subject's process conception strengthens (p.85).

Note that process conception and action conception are not to be interpreted as independent of one another. Neither is it my intent to imply that the action conception is not useful. Instead, it is probable that an action conception can be extended in order to develop a process conception.

In contrast to an action view, a student with a process view sees an expression as "self-evaluating." In other words, the student can imagine that the input values are evaluated without actually doing the computations. This in turn may allow students to then imagine "running through" a continuum of values at once rather than one value at a time. Note that this does not mean the student can perform the calculations, but rather he realizes these computations could be performed. Furthermore, the process conception of function helps develop a conceptual view of function; that is, the view that a function is an input and output relation that exists without performing specific calculations.

Carlson (1998) and Oehrtman, Carlson & Thompson (2008) extend this idea of a process conception to include the ability to visualize a function over an interval of the domain. Once students are given a problem, they may not immediately reveal that they have a process conception of function. Instead, they may start with an action conception and build to a process conception. Initially, students can appear to have a process conception yet when probed further, it becomes evident that they may have a procedural, or action, view of the concept of function (Dubinsky and Harel, 1992).

Sfard (1992) expressed many of the same general ideas. In her study, most students were inclined to think of functions in highly algorithmic terms (which she termed an *operational* conception of function). She contrasted this idea with the *structural* conception of functions, an “object” conception that was similar to Dubinsky and Harel’s (1992) *process* view of functions. Sfard gave instructional recommendations that teachers not introduce new mathematical concepts (such as function) using rigorous definitions and the other common trappings of *structural* conceptions, “as long as the student can do without it” (p.69). Nonetheless, Sfard deemed a structural conception of functions as essential for subsequently allowing a function “to become a building block of more advanced mathematical constructs.”

The process conception of function, specifically the conception of a function as a process that accepts an input and produces an output, has been determined essential for the development of a rich image of function (Carlson, 1998). Indeed, Oehrtman, Carlson, and Thompson (2008) suggest the following as a description of a robust function conception: “a conception that begins with a view of function as an entity that accepts input and produces output, and progresses to a conception that enables reasoning about dynamic mathematical content and scientific contexts.”

Function Composition

In contrast with the wealth of research on student understandings of function, little work has been published which focuses on students' understanding of function composition (Engelke, Oehrtman, & Carlson, 2005). Most of what has been published focuses on students' understanding of function composition relative to their understanding of the concept of function (Carlson, Oehrtman, & Engelke, 2010; Oehrtman, Carlson, & Thompson, 2008).

Engelke, Oehrtman, and Carlson (2005) found that in the case of composition problems that require students to find the value of a composition of two functions at a point, 90% of the students in their study were able to do so, provided the functions in question were defined by algebraic formulas. Engelke et al. conjectured that the relatively high levels of success in such problems were related to the fact that correctly solving these problems required only an action view of function. Oehrtman, Carlson and Thompson (2008) addressed this issue further:

> Similarly, with an action view, composition is generally seen simply as an algebra problem in which the task is to substitute one expression for every instance of *x* into some other expression. An understanding of why these procedures work or how they are related to composing or reversing functions is generally absent. (p.32)

Engelke et al. (2005) further conjectured that the much lower student success rate (less than 50%) for problems presented using other representations (tabular, graphical, context) was the result of such representations requiring students to engage in process-level thinking about functions. In general, Engelke et al. found that students were most successful in solving function composition problems that could be solved using memorized "plug and chug" algebraic procedures. Taken together, these findings suggest that a process view of function may be very important to understanding function composition.

Carlson, Oehrtman, and Engelke (2010) reported on the use of the Precalculus Concept Assessment (PCA) as a means for studying the concept of function, including function composition. Student performance on the PCA has been linked to subsequent student performance in calculus courses (Carlson, Oehrtman, & Engelke, 2010). Carlson et al. included seven items on the PCA that assess student understandings of function composition. Some of those discussed in detail by Carlson et al. are particularly relevant to the present study, and warrant further discussion here.

One problem considered by Carlson, Oehrtman, and Engelke (2010) as a candidate problem for use in the PCA is especially interesting, because of the exceptionally poor performance students exhibited when answering this problem. The problem asked students to express the diameter of a circle as a function of its area. Analyzing the interview data for five students who failed to answer the problem correctly revealed that all five had written $A = \pi r^2$ and $d = 2r$, and had expressed a desire to write a formula for area (*A*) in terms of diameter (*d*). The fact that not one of these five was able to do so suggested that the students were not viewing a function as a general process that defines how the values of two quantities change together. Student performance on this problem was so poor (less than 5% correct responses) that the item was subsequently dropped from the PCA and replaced with another function composition problem.

A second relevant problem from the PCA is the ripple problem. Consider the following:

> A ball is thrown into a lake, creating a circular ripple that travels outward at a speed of 5 cm per second. Express the area, *A*, of the circle in terms of the time, *t*, (in seconds) that have passed since the ball hits the lake. (Carlson, Oehrtman, & Engelke, 2010)

The statements made in interviews by students who correctly answered this problem are especially interesting. Their comments indicated that they had all constructed a dynamic image of a circle rippling outward, and supported the conclusion that they had imagined time as an input with radius as an output, and then reconceptualized this radius as an input to the area function, allowing the students to create a function for computing the area for varying amounts of time. This idea of "reconceptualization of the output of the first function as a suitable input for the second function" will be discussed further in Chapter 3.

Yet another PCA problem asked students to create a formula for the area of a square in terms of its perimeter. This problem differed slightly from the ripple problem: attaining a correct solution required that students invert one function prior to performing a function composition. Despite these differences, as with the ripple problem, all students who answered the question correctly were students who demonstrated a process view of function when explaining why they chose the answer they did. This led Carlson, Oehrtman, & Engelke (2010) to suggest that a process view of function is required for correctly reasoning about composition problems.

Engelke (2007) studied students' understandings of related rates problems in calculus. These are problems that "require the student to investigate the relationship(s) between two or more changing quantities, one of which is unknown and needs to be found" (p.2). Engelke's description of related rates problems includes attending to the relationship(s) between two or more changing quantities, suggesting that Engelke's results are relevant to the precalculus function relationships studied here. Engelke goes on to describe related rates problems as typically falling into one of two general categories:

1) The variables are related through function composition; or

2) The variables are related parametrically through a third variable, usually time.

In fact, I suggest that function composition may play an important role in both of these types of related rates problems. For example, consider the "plane problem" discussed by Engelke (2007), a problem that she classifies as belonging to the second category of problems:

> A plane flying horizontally at an altitude of 3 miles and a speed of 600 mi/hr passes directly over a radar station. When the plane is 5 miles away from the station, at what rate is the distance from the plane to the station increasing? (p. 3)

The mathematicians in Engelke's (2007) study typically solved this problem by drawing a right triangle and then considering the lengths of each of the sides of the triangle as functions of time. However, further analysis suggests that this problem might also be conceptualized in a manner similar to function composition, as a "function of functions". At any specific instant in time, we have the relationship between the lengths of the sides of a right triangle as described by the Pythagorean Theorem, $c^2 = a^2 + b^2$, or

$c = \sqrt{a^2 + b^2}$. However, the problem statement describes a dynamic situation in which the relevant quantities are themselves functions of time, *t*, and we have $c(t) = \sqrt{a(t)^2 + b(t)^2}$. Thus, an understanding of function composition is helpful in conceptualizing this problem, even though it is not a problem that Engelke (2007) classifies as a "composition problem". Further exploration of student reasoning when completing this problem is needed.

Engelke's key finding concerning the successful solution of related rates problems was the importance of first conceptualizing the two quantities to be related; then focusing on finding two function processes where the input variable is the input to the both the first function process and the composed function, and the second function process' output variable is the output of the composed function. In doing so, students indicated that they were searching for a "middle variable" that would link the input and output values of the composed function.

Quantity and Quantitative Reasoning

Throughout the preceding review of prior research related to student understandings of the concept of function, I have described functions as relating two quantities whose values may be changing. This highlights the importance of looking closely at what I intend by the term, quantity. A meaning of *quantity* that provides a useful theoretical perspective is that of Thompson (1989), who defines a quantity as being an attribute of something (*e.g.*, a perceived situation as interpreted from a problem statement) that is conceived of as admitting a measurement process. This meaning of a quantity as a conceptual entity provides an additional perspective for analyzing students' problem solving behavior by allowing me to examine students' distinction of quantities and the process of quantification. Quantification is defined by Thompson to be the cognitive process of assigning numerical values to attributes of a problem situation. In order for an individual to quantify an attribute of a situation, the individual must conceive of the attribute as admitting an explicit or implicit act of measurement. It is not necessary that the individual actually measure the attribute, nor even that he explicitly imagines measuring the attribute; rather, the individual need only imagine that the attribute could be measured, if it were desirable to do so. It is in this process of quantification that an attribute becomes "a quantity" in the mind of the student. Note that a quantity is not an attribute of the problem external to the student, but is constructed by the student himself. That is, to comprehend a quantity, an individual must have a mental image of an object and attributes of this object that can be measured (*e.g*., a car in a race with attributes weight, height, speed, and distance traveled), an implicit or explicit cognitive act of measurement that produces the quantity (*e.g*., measuring distance traveled in miles), and a number, or value, which is the result of that measurement. The idea that this value need not be known explicitly implies that a person may want to have a way to reference the value of a quantity without knowing that numerical value. This in turn necessitates a discussion of select research literature on *variable*.

Variable

As Trigueros and Jacobs (2008) note, the term *variable* is typically used in a variety of contexts to refer to usages of letters that are quite different. Küchemann (1980) undertook an early important study of student usages of letters by secondary algebra students, and created a framework for classifying the ways in which students used letters in algebra. What Küchemann termed Letter as Variable is of primary interest in this study, because in problems involving function composition, I am interested in relationships between quantities that vary.

Trigueros and Ursini (1999, 2001, 2003) built on Küchemann's (1980) work to create a model for analyzing student difficulties, textbook treatment, and classroom observations, as well as for guiding instructional design. Ursini and Trigueros (1997) and Trigueros and Ursini (2003) found that students were able to interpret and manipulate variables as specific unknowns only at a very elementary level, and were unable to differentiate among different uses of variable. Ursini and Trigueros also found that students had difficulty using variables to relate quantities. As a result, it should not be surprising that students develop conceptions of variable that Trigueros and Jacobs describe as "narrow, limiting, and generally underdeveloped" (p.5). In an examination of the concepts that facilitate a student's understanding of function composition, this is a recurring theme: the need for students to be able to build a mental model of varying quantities that supports thinking about the variations dynamically.

Indeed, Jacobs (2002) reached similar conclusions after studying Advanced Placement BC calculus students. For example, when asked to discuss the meaning of the expression $\lim_{x \to a} f(x) = L$, students typically used dynamic imagery when describing x approaching a, but they exhibited little of that dynamic imagery in discussing the behavior of $f(x)$, using static descriptions of "plugging in". In general, Jacobs noted little to no indication that students held dynamic images of quantities varying in tandem when discussing topics like derivative, for example.

These researchers found that students consistently tend to view a variable as representing a static unknown that needed to be found, and have difficulty imagining a variable as representing something that varies. These static conceptions of variable are often revealed when students are able to evaluate composite functions at a single fixed input value, but are unable to describe variation of output values over an interval of input values in the composed function's domain.

Covariational Reasoning

A student's ability to imagine the values of two quantities changing together while attending to how the values of the quantities are changing together has been called *covariational reasoning* (Saldanha & Thompson, 1998; Carlson et al., 2002). Thompson & Thompson (1992) also provided an early discussion of covariation, in describing cognitive differences between the notions of ratio and rate. Thompson & Thompson wrote of co-varying accumulations of quantities, and the notion of a rate as "a reflectively-abstracted conception of constant ratio" (p. 8).

Confrey and Smith (1994) provide a different perspective on covariational reasoning and its role in understanding functions. According to Confrey and Smith, the most typical approach to conceptualizing function relationships in the curriculum is the *correspondence approach*, in which the focus is on building a rule that allows one to determine a unique *y*-value for any given *x*-value. Confrey and Smith argue instead for what they term a *covariation approach*, which they seem to link quite closely with tabular representations of functions. They focus on coordinating movement from y_m to y_{m+1} with movement from x_m to x_{m+1}. The advantage of such an approach is that it does, indeed, draw the student's attention to coordinating changes in output with changes in input. However, the approach has definite weaknesses, as well. First among these is that in order to determine "what y_{m+1} goes with x_{m+1}", the student must attend to the "rule" that Confrey and Smith discuss as part of the *correspondence approach*. Thus, the *covariation approach* seems to be in large part inclusive of the *correspondence approach*, and certainly isn't an approach that can be considered "independently". A second weakness in Confrey and Smith's approach is that despite bearing the word "covariation" in its name, it still encourages a pointwise view of functions; that is, by its emphasis on filling in columns of a table, it focuses more on what might be called "frozen moments in the covariation" and less on a dynamic image of two covarying quantities. Such a conception of covariation does not support imagining continuous variation of one quantity and the corresponding continuous variation of the other, related quantity.

Saldanha and Thompson (1998) describe understanding covariation as "holding in mind a sustained image of two quantities' values (magnitudes) simultaneously." This image of covariation is considered developmental. In other words, one first coordinates two quantities' values (e.g., think of the first quantity, and then the other, think of the first quantity, and then the other, etc.). Then, as a student's image of covariation develops, her/his understanding of covariation begins to involve understanding time as a continuous quantity. Thus, the ability to imagine continuous changing quantities begins to form (e.g., as one quantity changes, an individual has the realization that the other quantity changes simultaneously). This parallels a process conception of function in that the student is able to imagine the simultaneous changes without having to determine the change in one quantity, and then the change in the other quantity. The two levels of development discussed also exemplify the *pointwise* and *across-time* views of a functional relationship described by Monk (1992). With a pointwise view, a student can only analyze a graph point by point, or when presented with a function *f*, he sees this as a call to evaluate a point (e.g., an action conception). Contrary to this is the idea of an across-time view of functions, with which the student is able to reason about a graph dynamically. This allows continuous movement across the independent variable while tracking continuous changes in the dependent variable. In other words, presented with a function $f(x)$, the student sees this as a general mapping (or relationship) between quantities that can be evaluated for any point within the domain (e.g., a process conception). According to Thompson (1994),

> Once students are adept at imagining expressions being evaluated continually as they "run rapidly" over a continuum, the groundwork has been laid for them to reflect on a set of possible inputs in relation to the set of corresponding outputs. (p.27-28)

Given the importance of the ability to reason covariationally, Carlson (1998) and Carlson, Jacobs, Coe, Larson, & Hsu (2002) investigated the complexity of students' images of covariation. Namely, the "construction of mental processes involving the rate of change as it continuously changes in a functional relationship" was investigated. The term "covariational reasoning" is used by Carlson et al. (2002) to mean, "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other." The covariational reasoning abilities of high-performing second-semester calculus students were studied, and during this investigation, a theoretical framework was created and refined. Initially, multiple behaviors of students involved in interpreting and representing dynamic function situations were identified (Carlson, 1998). In order to classify the behaviors exhibited, a framework that consists of five mental actions (MA1-MA5) and behaviors associated with these actions was developed.

Carlson et al. (2002) found that the mental actions alone were not adequate to describe a student's covariational reasoning ability, which can be inferred from the *collection* of behaviors exhibited when responding to a problem. In order to analyze the developmental nature of covariational reasoning in a graphical context, the covariation framework was extended to describe multiple levels (L1-L5) of covariational reasoning, resulting in a framework consisting of five distinct developmental levels based on the five mental actions (Carlson, Jacobs, Coe, Larson, & Hsu, 2002).

One's covariational reasoning ability is said to reach a given level when it supports the mental actions associated with that level and the mental actions associated with all lower levels. For instance, a student who is determined to exhibit L3 reasoning (quantitative coordination) is able to reason using MA3 (determining the amount of change of one variable with changes in the other variable) as well as MA1 and MA2. In other words, he or she is also able to coordinate the direction of change with one variable with changes in the other variable. Note that Carlson et al.'s usage of the term "levels" includes the idea that higher-level reasoning (*e.g.*, L5) implies the ability to engage in lower-level reasoning (*e.g.*, L3).

Problem Solving

Many of the mathematical tasks discussed in this study are intended to allow investigation of students' knowledge and use of function composition in the context of solving novel word problems. As a result, it is appropriate to review select research into students' problem solving behaviors and abilities.

Much of the problem solving research in recent decades builds on Pólya (1945). Schoenfeld (1983, 1992) revised and extended the body of research into problem solving behaviors, especially with regard to the affective domain. Schoenfeld took note not only of what mathematics a person knows and what strategies he might employ to use that mathematics, but also included in his framework metacognitive activities of monitoring and control, elements of beliefs and affect, and cultural elements that affect a student's perception of what mathematics and problem solving comprise.

Carlson and Bloom's (2005) multidimensional problem-solving framework significantly extended Schoenfeld's (1983, 1992) work. Carlson and Bloom discovered a highly cyclic pattern of problem-solving behavior among the mathematicians they studied. The typical pattern of behavior began with an Orienting phase, followed by multiple iterations of a Planning-Executing-Checking cycle. Carlson and Bloom also noted a Conjecture-Imagine-Evaluate sub-cycle often occurring within Planning phase of the problem-solving process. Carlson and Bloom's research is relevant to the study of students' knowledge of and ability to use function composition to solve novel problems for at least two reasons: (1) applications of function composition encompass problem solving, and (2) the *Orienting* phase, as described by Carlson and Bloom, engages students in making sense of the problem statement, and creating a mental model of the situation. This mental model is in part the basis for subsequent quantification, and also informs the students' creation of relationships among quantities.

Brief Textbook Analysis

Having examined the existing body of research related to student knowledge and use of function composition, it is reasonable to consider a brief review of the current treatment of function composition in mainstream precalculus textbooks. To better understand textbook treatment of function composition, this section examines three current precalculus texts (Hungerford & Shaw, 2009; Larson, Hostetler, & Edwards, 1997; Rubenstein, Craine, & Butts, 1997).

Two of these contemporary textbooks, by Hungerford and Shaw (2009) and Rubenstein, Craine, and Butts (1997) attempt to discuss the input/output view of functions in relation to function composition. The other text, by Larson, Hostetler, and Edwards (1997), focuses exclusively on algebraically evaluating the composition of two functions. However, in each of these texts, the authors present a procedural explanation of how to algebraically compute the composition of two functions that is based on substituting one expression into the other formula any place an x appears. The idea of functions as mathematical processes that accept inputs and produce outputs is not leveraged to facilitate conceptual understanding of why the given procedures work.

None of the texts surveyed give significant attention to fostering a process view of function. All three texts focus primarily on teaching students the procedures to compose two algebraically defined functions, with little explicit attention to considering the composed function as a function itself. Interestingly, both Hungerford & Shaw (2009) and Larson et al. (1997) discuss how useful it can be to be able to *decompose* a function into two simpler functions, although neither set of authors explain when or why it can be useful.

None of these texts attempt to foster covariational reasoning in situations involving function composition. The texts include a combination of functions defined algebraically, functions defined by tables, and functions defined by graphs. However, in every case, the function composition questions that the student is asked to answer require composition at a specific fixed input value, requiring no attention to covarying quantities.

Hungerford and Shaw (2009) pose some questions about covariation over an interval, in a problem in which the student uses the graph of a function *f* to fill in a table of values for the function *g* defined as $g(x) = f(f(x))$ and then sketch a graph. However, the table that the student completes is pointwise, and the student is not asked to reason about the graph of *g* beyond sketching it. Furthermore, Hungerford and Shaw do not address the question of why the student would want to compose *f* with itself. Perhaps the most troubling aspect of these contemporary treatments of function composition comes *after* the "function composition section" of each text. Although all three texts alluded to the importance and usefulness of function composition, none of the three texts make any mention of function composition in any subsequent sections of the text. All three texts use function *notation* in later sections of the textbook; however, they fail to discuss how and when function *composition* might be applied.

CHAPTER 3

THEORETICAL PERSPECTIVE

This chapter describes a theoretical perspective for analyzing students' understanding of function composition as applied to novel word problems. The chapter begins with a conjectured series of mental actions performed as a student solves an unfamiliar word problem, and a discussion of the conjectured knowledge that supports these mental actions. The second half of the chapter then presents a description of the exploratory study performed prior to the main study.

Conceptual Analysis of a Function Composition Problem

The theoretical perspective for this study was informed by a conceptual analysis (von Glasersfeld, 1995; Thompson, 2000) of the hypothesized mental actions a student with a robust understanding of function composition might perform to solve function composition problems. This approach serves to highlight the mental actions and ways of thinking that may be propitious to a robust understanding of function composition. Consider the ripple problem discussed earlier:

> A rock is thrown into a pond, creating a circular ripple that travels outward from the point of impact at 9 cm/second. Create a formula to express the area enclosed by the ripple as a function of the elapsed time since the rock hit the water.

To provide a correct response to this problem, the student must perform a series of mental operations, about which I make several conjectures. First, the student might develop a mental picture of the situation described in the problem. This mental picture need not be dynamic at first – it might be a static mental picture of a circular ripple at some moment in time, very much like a "mental photograph". However, in this particular problem, note that if the mental picture of the problem situation is static, it may be difficult for the student to articulate the role of "elapsed time" in the problem, because while time is typically a part of a *dynamic* conceptualization of this problem, time is not a necessary part of a *static* mental picture. Constructing a mental picture of aspects of the problem situation varying simultaneously is a key step, because it provides the basis for the student to subsequently *mathematize* the problem.

Mathematization refers to the process by which a mental model of a problem is reconceptualized in a way that makes it amenable to mathematical activity. A key component of this process is what Thompson (1989) describes as *quantification*. Quantifying involves identifying an attribute of a situation, conceiving of that attribute in a way that admits a measurement process (which may or may not be explicit), and conceiving of the *quantity*, with appropriate units, that is the result of that measurement process. In the ripple problem, the student needs to mentally construct quantities corresponding to the elapsed time since the rock hit the water, the radius of the circle formed by the ripple, and the area enclosed by the ripple.

It is not necessary that the value of a quantity be known explicitly, for the student to have imagined an attribute of a problem situation as something that could be measured. Indeed, for the student to solve the ripple problem, he or she must conceive of the elapsed time, the radius of the circle, and the area of the circle as quantities that can take on a range of possible values. However, as a basis for quantification, the importance of the initial mental image of the problem cannot be overstated. The attributes that the student uses to create quantities are not attributes of a problem external to the student, but rather are attributes of the problem as it exists in the mind of the student.

For the student to create formulas to express the function relationship requested by the problem statement, he or she may next conceptualize these quantities' values as being represented by what Trigueros and Ursini (1999, 2001, 2003) term *related variables*. That is, he or she must conceive of something (typically letters) representing the values (known or unknown) of the quantities created earlier, and begin to think about how the quantities might be related.

A student attempting to solve this problem is likely to remember a formula for the area of a circle. As a result, a first attempt to write a formula for the area enclosed by the ripple is likely to be $A = \pi r^2$, where A represents the area enclosed by the ripple and r represents the radius of the circle formed by the ripple. However, the student has been asked to relate the area enclosed by the ripple with an elapsed time, rather than with a radius. A key to the student's advancement toward a solution is the realization that his goal is to relate area and elapsed time.

To progress further toward a solution, the student must construct a relationship between elapsed time and the radius of the circle formed by the ripple (while attending to the units he is using to measure each quantity). The student must construct a relationship similar to $r = 9t$, where r is the radius (in centimeters) of the circle formed by the ripple and t is the elapsed time (in seconds) since the rock hit the surface of the water.

Construction of the algebraic relationships $A = \pi r^2$ and $r = 9t$ does not necessarily imply that the student has constructed a dynamic mental model of the problem situation. In fact he or she might still possess only a static mental model; such a person might be able to answer a question like "What is the area of the circle 5 seconds after the rock hits the water?" while remaining unable to describe how the area of the circle varies as the elapsed time varies. The student may also still not be able to express a relationship between A and t using a single formula. The ability to describe how the area and elapsed time vary together requires that he or she reason about the values of two quantities and how they change in tandem, which I described earlier as covariational reasoning (Carlson, Jacobs, Coe, Larson, & Hsu, 2002).

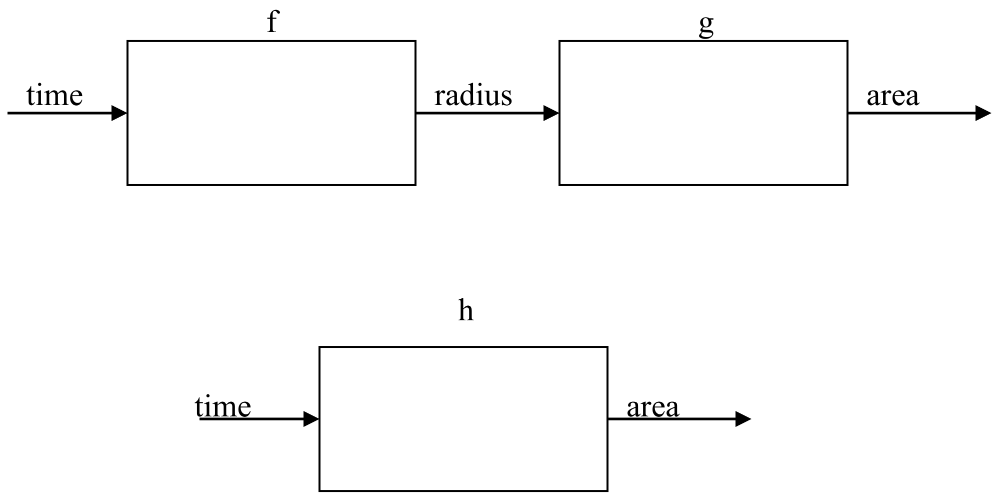
Carlson, Jacobs, Coe, Larson, & Hsu (2002) provide a useful framework for analyzing covariational reasoning, which provides an explanation for the types of descriptions students may give when answering the question "How does the area of the circle vary as elapsed time varies?" A person whose covariational reasoning is at Level 1 of Carlson et al.'s framework may be able to say "the area changes as time changes", but is unable to describe the direction or amount of variation. A person who is able to say "the area of the circle increases as elapsed time increases" may be engaging in Level 2 covariational reasoning. The person who is able to determine that "as the elapsed time increases from 2 seconds to 4 seconds, the area of the circle increases from 1017 square centimeters to 4069 square centimeters" is attending to how the output values change while imagining changes in the value of the input quantity, suggestive of the use of Mental Action 3 in Carlson et al.'s covariation framework.

The problem statement asked the student to relate the area of the circle and the elapsed time, and to use a *function* to express the relationship. To do so, the student must possess an understanding of the concept of function. Specifically, to provide an ideal solution to this problem, the student must possess a process conception of function (Dubinsky & Harel, 1992). A key notion in building a process view of function is that a function is viewed as something that accepts input values and produces output values.

With a process view of function, the student will likely be able to create two functions defined by algebraic formulas: one (say $r = f(t)$) that accepts elapsed time as an input and produces radius as an output, and a second (say $A = g(r)$) that accepts radius as an input and produces area as an output.

Prior to constructing the composite function $A = g(f(t))$, I conjecture that the student may conceive of the output of *f* as being a suitable input for *g*. Only then can he or she think about "connecting together" the two functions previously constructed – until a person conceives of the output of *f* as being the input to *g*, such a connection of functions makes no sense. After the student has mentally connected the functions *f* and *g*, the next step in the "ideal solution" is to think of the composite function as a single function (say *h*). A person who has conceptualized the composite function, *h*, as a single function, is able to imagine two processes being combined into a single process. He or she can imagine the covariation of time and radius, and the covariation of radius and area. He or she can imagine an amount of time being mapped to a radius and that resulting radius being mapped to an area. The student may speak about time being converted to area through a series of two processes. He or she might depict processes *f*, *g*, and *h* in a manner similar to that depicted in Figure 1.

Figure 1: Process Depiction of Composed Functions



Certain composition problems present added complexities that warrant further discussion. For example, consider a problem that asks a student to "express the area of a circle as a function of its circumference." Many of the mental actions required for the student to solve this problem are similar to those discussed above. However, this problem incorporates an additional source of complexity: it requires the solver to invert a function prior to composing two functions. Inversion is necessary because the solver is likely to remember the formulas $A = \pi r^2$ and $C = 2\pi r$, where A, r, and C represent the area, radius, and circumference of a circle, respectively. Written in this way, both formulas lend themselves to an interpretation of the radius as the input quantity, with area and circumference the output quantities. To conceive of a function that takes circumference as input and produces area as output, the student must first conceive of the inverse of the second relationship, before composing that inverse with the first formula to create a formula that gives area as a function of circumference.

Considering these conjectured student actions, I conjecture that students' understanding of, and use of, function composition is influenced by:

- Students' view of functions, including students' construction of mental imagery to support developing a process view of functions
- Students' understanding of the reversibility of functions and the construction of inverse functions
- Students' quantitative reasoning and quantification
- Students' ability to covary quantities over an interval of input values
- Students' understanding of function notation
- Students' problem solving behaviors and attitudes

A primary goal of this study was to characterize students' understanding and use of function composition. This study occurred in an instructional setting that emphasized quantitative reasoning, covariational reasoning, and the development of a process view of functions. I conjectured that the student's ability to use function composition to solve novel problems might reflect the student's quantitative reasoning, covariational reasoning, conception of function including their use of function notation, and problem solving behaviors. I further conjectured that other factors might emerge as significant in understanding the student's knowledge and use of function composition, factors that were not considered prior to the study. I also anticipated that the results of the study would lead to the refinement of the conjectures presented above, and would inform development of curricula.

The Exploratory Study

Prior to the main study, I conducted an exploratory study using an early draft of a portion of the materials to be used in the precalculus course in which the main study was to be situated. The exploratory study consisted of individual pre-interviews with three students, a single teaching session focused on function composition, and individual post-interviews with all three students. In this section I describe the design of the exploratory study, including a discussion of the methods of data collection and data analysis. I then provide a discussion of the results.

The participants and setting.

The exploratory study occurred in the context of a precalculus course at a large public university in the southwestern United States. The subjects of this study were three volunteers from a section of the precalculus course that had been redesigned to emphasize quantification and covariation. Students who volunteered were given monetary compensation for their time.

Rachel's story.

Rachel was an economics major. Rachel took prealgebra in high school, and qualified for high school Algebra II, but dropped out of that class after one week. In college, she took what she described as a "refresher course", a course that was her only college mathematics course prior to this precalculus course. She also attempted to take Brief Calculus in college, but withdrew after one week, because she "wasn't getting it" and because she was unable to understand the instructor because of his very thick accent. With regard to function composition, Rachel did not recall ever having heard the term or studied the topic. Relative to the rest of the students in her precalculus class, Rachel's performance on homework assignments, quizzes, and the PCA suggested that her ability level was about average.

During the semester, Rachel made comments that indicated that she felt she was “bad at math”. Also, several weeks into the semester, Rachel withdrew from the precalculus course, despite having a course average of over 80% at that time. Unfortunately, I was unable to determine the reason or reasons for her withdrawal.

Alicia’s story.

Alicia was majoring in Interior Design. In high school, she took Advanced Geometry, Algebra, Precalculus, and AP Calculus. However, she didn’t take the Advanced Placement test, so she did not get college credit for AP Calculus. Alicia performed well throughout the precalculus course, and received an “A” for a final course grade, with one of the highest course averages in the class.

Hayley’s story.

Hayley was majoring in animal physiology and behavior. In high school, she had taken algebra, geometry, precalculus, and Advanced Placement statistics. In college, she had earlier taken a college algebra course. Hayley earned a final grade of “B” in the course.

Methods of data collection.

Much of the data for the pilot study were drawn from semi-structured task-based interviews (Clement, 2000; Goldin, 2000). A single teaching session was also conducted in the classroom with three volunteer students. All interviews as well as the teaching session were videotaped, and all written student products were retained (or photographed, in the case of whiteboarded student work).

Task-based pre-interview.

The tasks used in the exploratory study pre-interview are shown in Appendix A. This interview was designed to gain insights about student knowledge of function composition and behaviors when faced with novel function composition problems, before participating in the teaching session. The tasks in the interview were not sequenced in any particular order, but were intended to provide insights into students' ability to solve function composition problems in a variety of contexts using a variety of function representations. The interview included tasks related to functions defined by formulas, tables, and graphs; some, but not all, were set in the context of word problems.

Teaching session.

The teaching session used the materials in Appendix B. The curriculum design team for the precalculus course created these materials; the design and evaluation of these materials is not part of this study. The exploratory study only included that portion of the course materials dealing explicitly with function composition. Students remained in their normal classroom for an initial exploration of functions and function inverses, and were removed from the classroom for instruction directly related to function composition.

The teaching session was designed to use preexisting course materials to influence students' presumed knowledge of functions as processes accepting input and producing output. Students were presented with a situation in which they were asked to develop a function relating two quantities that were not trivial to relate directly. Through interactions with each other and with the instructor, students were to build a conception of linking two processes together to create a composite function, which could then be explored, allowing students to reconceptualize this composite function as a single function.

The final sections of the teaching session were intended to allow students to explore function composition using functions defined by tables and functions defined using graphs. The activities using functions defined by graphs included questions requiring students to consider the variation of the output of the composite function in response to changes over an interval of input values.

Task-based post-interview.

The protocol used in the exploratory study post-interview is shown in Appendix C. This interview was designed to gather insights about student knowledge of function composition and ability to work novel function composition problems after participating in the teaching session. Just as in the case of the pre-interview, the tasks in the post-interview were not sequenced in any particular order, but were intended to provide insights into students' ability to solve function composition problems in a variety of contexts using a variety of function representations. The interview tasks included functions defined by formulas, tables, and graphs, and some tasks asked the student to reason about dynamic variation of output quantities in response to changes in the input quantity. As in the pre-interviews, some tasks were set in the context of a word problem, and others were not.

Methods of data analysis.

The videotapes of all interviews and the teaching session were reviewed and transcribed. The videos were then reviewed, and the transcripts analyzed in an attempt to gain insights into what the students might have been thinking. Conceptual analysis (von

Glasersfeld, 1995; Thompson, 2000) was used in an attempt to answer the question, "What mental actions in the person I'm observing would explain the behavior I seem to be observing?"

Results from pre-interview session.

In this section, I present an overall analysis of the three pre-interviews conducted prior to the exploratory study teaching session. This is presented with each student's pre-interview discussed separately.

Analysis of Rachel's pre-interview.

In her pre-interview, Rachel was not successful in solving novel word problems that involved (from the observer's point of view) function composition. For example, when asked to find the area of a square as a function of the perimeter, Rachel stated that she was "not that good at expressing things as functions yet", but that she knew how to find the area of a square. She eventually drew a square, labeled each side of the square as x, and wrote $p = 4x$. When asked about a formula for the area, she stated that it would be "x squared". However, even after probing, she was unable to construct a formula that related the area of a square with its perimeter.

Rachel was, however, able to meaningfully interpret function composition notation, despite claiming to have not seen it before and having been only briefly introduced to function notation. In a problem involving two functions defined using formulas, Rachel was able to find $g(h(2))$, by finding $h(2) = 5$ and substituting 5 as the input to g, but she did not spontaneously use input/output language. When specifically asked to use input/output language to explain what she had done, Rachel described her work as shown in excerpt 1.

Excerpt 1

Rachel:	To do this problem, I was given… I needed to find *g* of *h* of 2, so the input of $h(x) = 3x - 1$ is 2, so I would replace the *x* with 2 as the input, and then whatever I got would be the output, so 5 would be the output of the function *h* with the input of 2… um, and then basically once I got that, I was able to make the output of $h(2)$ the input of $g(x)$.

This pattern of reasoning appeared to continue when solving problems using other representations. When asked to use graphs of *f* and *g* to find $g(f(2))$, Rachel coordinated input and output values, first finding $f(2) = -2$, and then finding $g(-2) = 1$. Rachel explained her thinking in excerpt 2:

Excerpt 2

Rachel:	OK, well I know this line is *f*, so to find $f(2)$ I would find 2 on the *x*-axis, which is right here, so I know that $f(2)$ has an output of -2. So now it's asking me to use the output of… $f(2)$ as the input to *g*. So for this line *g* right here, I would find the input -2, which is right here, so I know that $g(f(2))$ has an output of 1.

When questioned about how she was able to find output values that corresponded to given input values, Rachel indicated that she found the appropriate value on the x-axis, moving vertically until she intersected the graph of the function, and then moving horizontally until she intersected the y-axis to determine the appropriate output value. This suggested that Rachel understood graphs as coordinating values of input quantities with values of output quantities.

Rachel was also able to solve problems involving composition using two functions defined in a table. When asked how she determined which function she evaluated first, Rachel indicated that it was because it was the part "in parentheses", and clarified that she was using "order of operations" rules to make this decision.

Analysis of Alicia's pre-interview.

The second student, Alicia, solved all the problems in the interview, but she was often not able to articulate the input and output quantities of composite functions. For example, when asked to express the area of a square as a function of the perimeter, Alicia immediately wrote the equations $A = s^2$ and $P = 4s$. She then solved for s in the second equation, "used substitution" (her own words), and wrote $A = \left(\frac{P}{4}\right)^2$. Her solution to this problem did not use function notation, or the word *function*, or input/output language. When asked to write this relationship as a function, she wrote $A\left(\frac{P}{4}\right) = \left(\frac{P}{4}\right)^2$. When asked to explain why she had written this function with $\frac{P}{4}$ as the input, she stated this was because "to get the side by itself, I'd have to divide 4 into the perimeter". This suggests that Alicia was not thinking of a composite function but was still thinking of the function $A(s)$, where s was replaced by an expression equivalent to s.

Even though she did not give a "textbook articulation" of function composition when solving these problems, Alicia was able to make sense of problems that used function composition notation. For example, in a problem that required evaluation of composite functions defined by algebraic formulas, Alicia was able to quickly find the correct answer. Her explanation suggests that she solved this problem by thinking of $h(2)$ as the input to g: "This just means that you have to do $h(2)$ first, because $h(2)$ is the input to $g(x)$". When asked to explain why she chose to evaluate the functions in the order she did, Alicia stated that this was because of the "order of operations", just as Rachel had stated in the previous interview.

Alicia was also able to solve problems that required composition of functions defined by graphs or tables, using specific input values. For example, Alicia was able to explain how to find $g(f(2))$ using the given graphs. She was also able to solve a series of three evaluations of function composition using functions defined by tables quickly and correctly.

Analysis of Hayley's pre-interview.

The third student, Hayley, demonstrated weaknesses in her function knowledge during her pre-interview. She was unable to make progress on problems asking her to express the area of a square as a function of its perimeter, or to express the diameter of a circle as a function of its area. Unlike Rachel and Alicia, Hayley was not able to use an "order of operations" interpretation to complete tasks that required her to interpret function composition notation. Instead, she demonstrated a detailed and consistent interpretation of function composition notation. She correctly interpreted the inner function as evaluating a function with a given input, but she interpreted the outer function as a command to multiply. Exactly what she was supposed to multiply varied. For example, one problem presented Hayley with two algebraically defined functions, and required her to find $g(h(2))$. She read this expression aloud as "*g* of *h* of 2", but described the requested operations as "you're multiplying the function *g* times the function *h* of 2". She then proceeded to find $h(2)$ correctly, and then she multiplied the result by the rule for function *g*, giving her an answer of $g(h(2)) = 5x^2$. These behaviors revealed that Hayley thought function composition notation represented multiplication, although she was able to interpret the meaning of $h(2)$. This suggests a weakness in Hayley's understanding of function inputs and outputs, possibly indicative of being in transition from an action view of functions to a process view of functions.

When asked to "use the graphs of *f* and *g* to evaluate $g(f(2))$", Hayley's first question was "What point do you want me to use?" Her subsequent response to the task suggested an incorrect understanding of what information the graph of a function is intended to convey. The graph of *g* had one point labeled with its coordinates, (-2,1), and the graph of *f* had two points labeled with their coordinates, (2,-2) and (4,3). Hayley wrote $g(f(2)) = (-2,1)[(2)(2,-2)(4,3)]$, and gave the following explanation of her response (see excerpt 3):

Excerpt 3

Hayley: g is -2 and 1. Those are the points on the graph, so that's right there, multiplied by f of 2, so you're gonna multiply these two points by the number 2, and that's why they're in brackets.

In Excerpt 3, Hayley believes the function f to be the two points from the graph of f whose coordinates are labeled, and understands $f(2)$ as a multiplication. She subsequently believes that evaluating $g\big(f(2)\big)$ requires multiplying $f(2)$ by g, which she believes to be the point (-2,1). Given this understanding of the problem, her otherwise bizarre response makes perfect sense.

Presented with a table that defined two functions and asked to evaluate composite functions at specific input values, Hayley correctly evaluated the inner (first) function, but her second step was to multiply each of these numbers by the name (f or g) of the outer (second) function. It appears that in some cases she was interpreting the names of the functions correctly, but in other cases she interpreted the names of functions as variables to be multiplied by whatever was in parentheses. This is fragile and inconsistent understanding of function notation and function inputs and outputs, as described previously.

Results from teaching session.

The teaching session focused largely on the ripple problem described earlier and presented in Appendix B. Working as a group, the students reached agreement on quantities that were meaningful and measurable attributes of the problem. Having identified quantities, the students created useful formulas relating the values of the quantities. However, students' descriptions of the formulas sometimes suggested that they had constructed vaguely defined quantities and incorrect quantitative relationships. For example, when explaining the correct formula, $f(t) = 0.7t$, Hayley explained that using this formula "you take 0.7, because that's the radius, and multiply it by the number of seconds".

The group members also voiced no concerns with the suggestion that to express the area of the circle as a function of the circle's radius, they could write $A = g(r) = \pi(0.7t)^2$. When probed, Alicia suggested, "You could just do πr^2". Hearing this, Hayley first wrote $A = g(r) = \pi(0.7)^2$, suggesting that she thought of 0.7 as being the radius. This response suggests that Hayley, and perhaps other group members, were not visualizing the radius as a function of time, and hence they were not visualizing the quantity radius as varying with variations in time.

When using the correct formulas to find the area inside the ripple 6 seconds after the rock hit the water, Alicia correctly described the input and output quantities of both functions, including a description of why she used the output of *f* as the input to *g*. Alicia consistently exhibited a better ability to talk about function input and output quantities than Hayley or Rachel.

When asked to use graphs of the functions *f* and *g* to find the area inside the ripple 6 seconds after the rock hit the water, Rachel did not refer to inputs or outputs; however, she did indicate with her finger how she would use the graph of radius as a function of time, locating $t = 6$ seconds on the input axis, moving up vertically until she reached the graph of the function, and then moving horizontally until she reached the output axis, where she could determine the radius. This suggests a degree of understanding of function inputs and outputs.

When asked to evaluate composite functions defined by tables but with no real world context, at a given input value, Alicia and Rachel were able to explain how they used the output value from the first function as the input value to the second function. However, Hayley exhibited difficulty similar to her pre-interview. Asked to evaluate $g(g(1))$, Hayley correctly found that $g(1) = 2$, but used the word "multiply" to describe what she should do next, and was unsure whether she should multiply 2 by *g*, or perhaps square 2, since she wanted to find $g(g(1))$. With help from other students in her group, Hayley discovered the correct way to solve this problem, and commented that she was seeing the problem differently than she had before. Her actions suggest that she had begun to see functions as accepting inputs and producing outputs. Hayley's ability to solve this and subsequent problems, and her fluency in describing function inputs, function outputs, and linking functions together, suggests that she had developed a more robust view of functions, moving toward a process view of functions and away from an action view.

The group was next asked to use the graphs of two functions to find the output of a composite function at various given input values. All three students were able to describe the process of using two graphs defined on the same set of axes to find specific outputs of a composite function, explaining how to find the output of the first function and use that value as the input to the second function. Each of the students demonstrated how to use the *x*-axis to provide the input value to the first function, how to move horizontally from the graph of the first function to the *y*-axis to determine the output of the first function, and how to repeat this process with the second function, using the output of the first function as the input to the second. The students were less successful in an activity in which they were asked to use graphs to evaluate the output of a composite function over an interval of input values. All of the students had difficulty attending to the ways function outputs changed in tandem with function inputs. Since the students had all experienced success in evaluating composite functions at a specific input value, their failure to describe variations over an interval of input values revealed weaknesses in their covariational reasoning ability. In addition, the students demonstrated difficulty attending to the directionality of change (increasing versus decreasing) when linking the two functions together, further supporting the idea that the students either had fragility in their covariational reasoning or were in transition from an action view of function to a process view of function.

Results from post-interview session.

A few days after completing the teaching session described above, the exploratory study participants were again interviewed individually. These interviews were semi-structured task-based interviews, and consisted of a combination of new problems and problems carried forward from the pre-interviews. The post-interview protocol is shown in Appendix C.

In her post-interview, Alicia was able to describe the input and output quantities for all of the functions used in the post-interview tasks. Alicia was also able to describe the output of the first function as becoming the input to the second function. This suggests that she had developed a more robust conception of functions as accepting inputs and producing outputs.

Just as they had done during the pre-interview, both Alicia and Hayley described the input and output quantities of a composite function in a way that suggests they were still having difficulty conceptualizing the composite function as a single function. For example, when $h(t)$ was defined as $g(f(t))$, both Alicia and Hayley stated that the input to h was $f(t)$. The significance of this warrants further exploration.

Hayley showed a clear improvement in her ability to use input/output language to talk about functions and function composition over the instructional sequence. In her pre-interview she typically evaluated what she referred to as “the first function” and followed this with a “multiplication” by the second function. However, in her post-interview, Hayley consistently described using the output of the first function as the input to the second, suggesting that she had progressed toward a process view of function and had developed the ability to articulate what it means to compose two functions.

Hayley also showed a significant improvement in her ability to make sense of a function defined by a graph. In the post-interview, she was able to perform evaluations of composite functions, including composite functions that required using the inverse of one of the graphically defined functions. She was able to use input/output language to describe using graphs to find output values for given input values, and vice versa. She was also able to describe using her intermediate results as inputs (or outputs, as appropriate) to the graph of the second function. Her actions and comments suggested that as a result of working with her groupmates to solve graphical problems during the teaching session, she had altered her understanding of functions and of the information conveyed by graphs of functions. However, she was still unable to determine how the output of a composite function varied over an interval of input values, suggesting a continuing weakness in Hayley's covariational reasoning.

In her post-interview, Rachel was successful on problems with real-life contexts or clear procedural solutions. However, she had difficulty completing tasks in situations that had neither a real-life context nor a clear procedural solution. For example, she remained unable to make meaningful progress when asked to "express the area of a square as a function of its perimeter", a problem that had also appeared in the pre-interview. She indicated that she didn't understand why a person would want to do that, since you could just express it based on the length of the side of the square.

Discussion of exploratory study results.

All three students exhibited behavior that suggested weaknesses in their covariational reasoning, their quantitative reasoning, and their views of function. In many instances they exhibited a weak understanding of inputs and outputs of functions, and in some cases, the students acted in procedural ways suggestive of an action view of function, rather than a process view. All three students exhibited difficulty reasoning about the behavior of functions over an interval of input values. This suggests impoverished covariational reasoning, as students showed an inability to attend to the relationships between changing input and output quantities in function composition problems. An observed weakness of the pre-interview design was that it did not include any tasks that required students to reason about dynamic variation in the output of a function in response to an interval of input values. As a result, the post-interview was expanded to include a problem that did require reasoning about dynamically varying quantities.

In many cases, the students in this study also showed a tendency to define quantities based on attributes that were either not measurable or were not well defined, such as Hayley's suggestion during the teaching session that 0.7 is "the radius" and 0.7*t* is also "the radius". This suggests that these students would benefit from a curriculum that emphasizes conceiving and reasoning about quantities, including activities that require students to use quantitative operations (Thompson, 1989) to create new quantities from quantities that have already been created.

Hayley exhibited behavior that suggests she had little understanding of what information the graph of a function conveys. During her pre-interview, her solutions for problems involving graphs were highly procedural, and suggested that her only understanding of graphs involved manipulating numbers that were used to label points on the graph. However, during her post-interview, Hayley demonstrated dramatic improvement in her ability to make meaning of a graph. This suggests that for her and perhaps other students, it would be beneficial to engage students in tasks to support their understanding of the meaning of a function's graph. For example, students should frequently be asked to describe what a specific point on a specific graph represents, and students should be asked to demonstrate how the values of input and output quantities are represented (by magnitudes measured on the horizontal and vertical axes, respectively). Such attention to the information conveyed by a graph can serve not only to improve student understanding of graphs, but can be used as opportunities to ask students to attend to the covariational between input and output quantities as well, positively impacting students' covariational reasoning.

Implications of the exploratory study for the main study.

The observations made in the exploratory study seemed to confirm many of my initial conjectures about factors that were important to students developing notions of function composition. Specifically, the exploratory study suggested that students' ability to construct robust mental pictures of problem statements, students' ability to reason about quantities and the relationships between them, and students' understanding of function representations were important factors in students' subsequent developing knowledge of, and ability to use, function composition.

Students in the exploratory study often tried to construct quantities using attributes that were either not measurable or not well-defined; as a result, tasks for the main study were designed to probe students' construction of quantitative structures and quantitative relationships (Thompson, 1989) and their mental images of the problem. This often took the form of probing interview questions such as "What are you thinking about when you talk about the height?" with the intent that students provide more details allowing insight into the object and attribute that the student was using to construct the quantity of height.

Since students in the exploratory study often struggled to reason about quantities and the relationships between them, interview tasks for the main study were chosen to include questions asking students to reason about pairs of quantities and the ways those quantities changed in tandem. As a result, I was able to explore students' covariational reasoning, while exploring their quantitative reasoning. This was important to explore, because many function composition problems require students to attend to the changing values of three quantities that are related through two function processes.

Hayley's actions in the exploratory study, and her impoverished understanding of functions defined by graphs, informed the development of interview tasks for the main study. Some tasks were expanded, and follow-up questions added, to allow the interviewer to ask questions intended to provide insights into what information the student believes is represented by a graph, table, or other representation. For example, students were asked to perform tasks using functions defined using a variety of representations, in an effort to gain insight into what the student believed the representation to "represent" relative to a specified functional relationship.

Finally, results of the exploratory study led me to realize that in some instances I had not thoroughly investigated what the student was thinking, but had instead probed "how well" the student was doing what I expected her to do. This realization informed the main study profoundly, leading me to shift my approach during task design, interviews, and data analysis toward being more responsive to student thinking.

CHAPTER 4

METHODS

This chapter describes the methods used in this study to investigate students' developing knowledge of function composition. The chapter begins with an overview of the design of the study, and a discussion of the data collection methods and sources. This is followed by a discussion of the methods for analyzing the data.

Overview of Study

This study investigated three students' understanding of function composition, and use of function composition in novel contexts. The precalculus class in which this study was situated met two days each week for fifteen weeks. Each class session was approximately seventy-five minutes long, giving approximately forty hours of total classroom contact time. In the classroom, students sat in groups of four to six students, an arrangement conducive to frequent group work and inter-student discussion. The students participating in this study sat at the same table. Each class session was videotaped; the instructor wore a microphone and was tracked using one video camera, and a second ceiling-mounted video camera and table-mounted microphone were dedicated to recording the actions of the students participating in this study. This study included data from individual interviews and written artifacts of student work; both of these data sources are discussed in greater detail later in this chapter.

The design of the precalculus course (which is outside the scope of this study) was informed by current research into ways of reasoning that lead to improved student performance in calculus classes. Much of this research was discussed in an earlier chapter. Informed by this research, the precalculus course was characterized by an explicit focus on quantitative reasoning, covariational reasoning, the concept of function, and problem solving in novel contexts. This study does not attempt to evaluate this curriculum, but I believe it is very important to understand that this study was not set against the backdrop of a randomly chosen “typical precalculus class”.

Individual student clinical interviews were conducted nine times during the semester, with each of the three student subjects. The objective of these interviews was to gain insights into students’ knowledge of function composition relative to their quantitative reasoning, covariational reasoning, views of function, and problem-solving skills, as well as other unanticipated emergent factors. In addition, these interviews investigated students’ use of function composition to solve problems in various mathematical content areas, such as linear functions, exponential and logarithmic functions, and trigonometric functions. The interviews were semi-structured, task-based interviews (Goldin, 2000), approximately one hour in length. All interviews were recorded using a wall-mounted camera focused on the student, a ceiling-mounted camera focused on what the student wrote, and a table-mounted microphone. All interviews were fully transcribed for later analysis.

Written artifacts of student thinking were also gathered throughout the semester. Students completed the Precalculus Concept Assessment (PCA) (Carlson, Oehrtman, & Engelke, 2010) both at the beginning and end of the semester. The results of those assessments provided data to situate the subjects’ performance within the class and provide quantitative data about students’ understanding of function, covarying quantities, and function composition.

Each of these data sources contributed to the intended outcome of the study: gaining insights into student knowledge of function composition and use of function composition when solving with novel problems.

The role of the researcher.

Radical constructivism (von Glasersfeld, 1995; Thompson, 2000) provides a useful lens for attending to the actions of the researcher relative to the actions of the students in this study. A main tenet of radical constructivism is the notion that other people's realities (including their mathematical knowledge) are fundamentally inaccessible to us (von Glasersfeld, 1995). This has the important implication that regardless of the "hat" the researcher is wearing at any given time, he must repeatedly ask himself the question, "What mental actions in the person I'm observing would explain the behavior I seem to be observing?" By doing so, the researcher is able to minimize the possibility of confounding his own perspective with that of the student being observed.

The participants.

The participants in this study were three student volunteers from a precalculus class at a large public university in the southwestern United States. Students were selected from the pool of volunteers to represent varying ability levels, based on PCA pre-test scores and performance on homework assignments early in the semester. Participants were compensated for their time while participating in this study. All participants were at least 18 years of age, and were required to sign the consent form in Appendix E prior to participation in the study.

Patricia's story.

Patricia initially entered college with the intention of becoming a nurse; however, she had recently switched her major to sustainability and conservation, with the eventual goal of pursuing a career in naturopathic medicine. Patricia made comments in early interviews indicating that she did not believe she was "good at math"; however, her final grade for the course was a B. As a college student, Patricia had completed courses in statistics, intermediate algebra, and college algebra. She had been out of high school for six years, and didn't remember what mathematics classes she completed in high school, although she did recall that she didn't take any mathematics her senior year. On the Precalculus Concept Assessment (PCA) pre-test (see Chapter 2 for additional discussion of the PCA and tasks used on the PCA), Patricia scored seven correct out of twenty-five. Her score on the post-test given at the end of the semester was fifteen out of twenty-five; this included the same seven correct answers from her pre-test, as well as eight correct answers to PCA items she answered incorrectly on the pre-test.

Karen's story.

Karen was majoring in kinesiology, with the stated intention of pursuing a career in physical therapy or occupational therapy. In high school, Karen completed Beginning Algebra, Intermediate algebra, Geometry, and Topics of Algebra. She was in her fourth semester as a university student, having completed College Algebra during her second semester and having attempted Precalculus her third semester. Karen intended to enroll in a 200-level statistics course in a future semester, because it was a requirement for her academic major.

On the PCA pre-test, Karen scored nine correct out of twenty-five. Her score on the post-test given at the end of the semester was seventeen out of twenty-five, including ten correct answers for PCA items that she had answered incorrectly on the pre-test and two incorrect answers for PCA items she had answered correctly on the pre-test. Karen's final grade for the course was an A.

Bridget's story.

Bridget was a second-year university student, having graduated from high school two years prior to this study. In high school, Bridget reported completing geometry, second-year algebra, and precalculus. As a university student majoring in civil and environmental engineering, she had already completed brief calculus and mathematics for business analysis, and intended to complete three semesters of calculus in the future. On the PCA pre-test, Bridget scored five correct out of twenty-five. Her score on the post-test given at the end of the semester was thirteen out of twenty-five, including nine correct answers for PCA items that she had answered incorrectly on the pre-test and one incorrect answer for a PCA item she had answered correctly on the pre-test. Bridget's final grade for the course was an A, with the second highest overall course average in the class.

The classroom setting.

This research occurred in the context of a precalculus class with approximately twenty students enrolled. Students sat in groups of three to five students at round tables conducive to group work and discussion among students.

Methods of Data Collection

This section provides a rationale for the data collection methods, including a discussion of both their general relevance and their relevance to a study of how students know and use function composition in particular.

Semi-structured task-based interviews.

The primary source of data for this study was a series of clinical interviews (Clement, 2000) with individual students conducted throughout the semester. These interviews were conducted at irregular intervals beginning a few weeks into the semester and continuing until near the end of the semester. Relative to the design of clinical interviews, Clement distinguishes between generative and convergent purposes in the overall data analysis. For studies in which the purpose is generative, the goal is to interpret the data in a reasonably “broad” manner, letting the data suggest models and theories that might be new. When the purpose of a study is convergent, more mature models and theories are brought to bear on the data, with the goal of coding the data with high reliability and using the results to either support or modify the theoretical lens. In this study, both approaches were employed. The goal of this study was to generate theory about student understanding and use of function composition, and subsequently test the theories that were generated. For that reason, individual clinical interviews were an appropriate method of data collection for this study.

Indeed, Clement (2000) notes that both the generative and convergent approaches might be used at different points in the same overall study. During the early stages, it might be helpful to view the data through a generative lens, to help identify appropriate theoretical conjectures. Later, when the theory is being "tested", a more convergent approach might be used as a tool to either refine the tool or support its validity.

The interviews used in this study were semi-structured task-based interviews (Goldin, 2000). "Task-based" refers to the focus on specific mathematical tasks on which the subject and the interviewer both focus much of their attention, rather than the interview just being a dialogue between the subject and the interviewer. The term "structured" refers to the explicit attention given to a number of factors: mathematical content is carefully chosen; interview contingencies are discussed and planned for; controllable and uncontrollable variables affecting the interview are considered and documented; theoretical perspectives are made explicit; and data analysis methods are presented in detail.

Goldin presents ten guidelines for designing "quality interviews", which were used to help design the interviews for this study:

1. design task-based interviews to address advance research questions;
2. choose tasks that are accessible to the subjects;
3. choose tasks that embody rich representational structures;
4. develop explicitly described interviews and establish criteria for major contingencies;
5. encourage free problem solving;
6. maximize interaction with the external learning environment;
7. decide what will be recorded and record as much of it as possible;
8. train the clinicians and pilot-test the interview;
9. design to be alert to new or unforeseen possibilities; and

10. compromise when appropriate.

When designing the interviews for this study, I attempted to address Clement's (2000) and Goldin's (2000) concerns wherever possible, and to heed their advice when feasible.

Each student in the study was interviewed multiple times during the semester. The tasks used in Patricia's, Karen's, and Bridget's interviews are given in Appendix D. These interviews were intended to serve two purposes: first, to allow me to characterize students' quantitative reasoning, covariational reasoning, problem solving abilities, and understandings of function; and second, to give insights into students' knowledge and use of function composition when solving novel word problems in a variety of mathematical settings. Both of these purposes are discussed in the following paragraphs.

The first purpose of the interviews was to allow me to characterize students' quantitative reasoning, covariational reasoning, problem solving abilities, and understandings of function. For this purpose, many interview tasks were designed to probe students' covariational reasoning, quantitative reasoning, problem solving, and views of function. For example, consider the following problem (Task 1 from Appendix D):

> Two friends that live 42 miles apart decide to meet for dinner at a location half way between them. The first friend, Tom, leaves his house at 6:05 and drives an average speed of 34 miles per hour on his way to the restaurant.
>
> Task1a: If the second friend, Matt, leaves at 6:10, what average speed will he need to travel to arrive at the same time as Tom?

Task 1b: If before leaving Matt knows that he averages driving 15 miles per hour to the restaurant, what time would he have to leave to arrive at the same time as Tom?

This problem was intended to provide insights into students' quantification, quantitative reasoning, and problem solving. However, decisions made in the moment by the interviewer were of critical importance. For example, in some instances, the interviewer needed to ask the student what the student believed the tasks were asking her to do, what her goals were for each calculation she performed, and what specific results represented (which distance, which time, whose speed). Such questions provided insights to not only what the student was doing, but also to what her goals were, and what quantitative structures she had mentally constructed. This in turn yielded clues to the student's mental imagery when performing the task.

The second purpose of the clinical interviews was to allow me to characterize students' knowledge and use of function composition in different mathematical contexts. For this purpose, interviews included tasks specifically designed to investigate students' use of function composition. Some tasks used function represented algebraically, while others used graphical, tabular, or verbal representations of functions. Some tasks involved "real world" settings, while others were more abstract.

Written artifacts.

The secondary sources of data for this study were examples of students' written work. All written interview products were photocopied and retained for each student in this study. These written interview products were intended to aid my ability to characterize student knowledge relative to the factors presented in earlier chapters –

covariational reasoning, quantitative reasoning, problem solving, and view of function – as well as other emergent factors.

The participants in this study also completed the Precalculus Concept Assessment (PCA) at the beginning and end of the semester. For this study, I used individual student PCA responses to aid my ability to situate the results of the study relative to student covariational reasoning and knowledge of function.

Methods of Data Analysis

This section presents the methods used in this study for data analysis. This includes a discussion of how each method was used, at what point in the study it was used, and why its use was appropriate.

Overview.

The methods of data analysis used in this study were consistent with a grounded theory (Glaser & Strauss 1967; Strauss & Corbin, 1990) approach to qualitative analysis. In general, the approach I used to analyze the data in this study followed a three-step iterative cycle:

1. Examine data with the intent of identifying episodes that may plausibly provide insight into student conceptions and reasoning, and formulate initial hypotheses based on these episodes.
2. Re-examine data to search for evidence that either supports or contradicts the initial impressions formulated in the previous step.
3. Accept, revise, or reject initial hypotheses to address the supporting or contradictory evidence identified in the previous step.

Stage 1: Reduction of data.

This study generated approximately 27 hours of video recordings of individual student interviews. During the semester, these videos were transcribed in their entirety, as soon as possible after each video was recorded, yielding approximately 600 pages of transcripts. Timely transcription of interviews served two purposes. First, this avoided the accumulation of a large amount of video at the end of the semester that had not yet been transcribed. Second, this facilitated changes to the clinical interview designs during the semester in response to student actions in earlier interviews.

After video transcription was complete, the transcripts were analyzed using both generative and convergent approaches (Strauss & Corbin, 1998) to identify and categorize episodes that plausibly relate to students' knowing and learning function composition or students' quantitative reasoning. These portions of the transcribed interviews formed the basis of the second stage of data analysis described below.

Stage 2: conceptual analysis.

Conceptual analysis (von Glasersfeld, 1995; Thompson, 2000) was used extensively in the analysis of data from this study. After the transcribed interviews were analyzed, I performed a conceptual analysis using the transcript excerpts, to infer mental actions that plausibly explain students' actions and products.

After conceptual analysis was complete, the results were examined together with the factors that I initially conjectured to be important when students solve novel function composition problems (as presented in the third chapter). This allowed conclusions to be drawn regarding the conjectured connections between student covariational reasoning, quantitative reasoning, and understanding of function and student knowledge and use of function composition. It also allowed refinement or rejection of factors conjectured to be important, in cases where refinements were suggested by student actions and products. When deciding whether to retain, revise, or reject conjectures about student knowledge, I followed two general principles. First, I was careful not to attribute more knowledge to a student beyond what was necessary to explain the student's actions. This was necessary to avoid jumping to unfounded conclusions and making unsubstantiated claims. This also helped me to avoid confounding my own (observer's) frame with the student's perspective. The second principle I followed was that when a body of evidence seemed to support a conjecture, substantial contradictory evidence was required in order to reject the conjecture; in cases where outright rejection was not warranted, I sought to revise conjectures to account for the contradictory evidence. This process should not be confused with formal statistical hypothesis testing, which was not a part of the data analysis for this study.

CHAPTER 5

RESULTS

This section presents selected data and results of this study. In the interest of readability and accessibility, this section does not present a full analysis of every task from all twenty-seven interviews. Instead, a focused analysis is presented that highlights episodes that were representative of the thinking observed and were most pertinent to answering the research questions for this study.

The results presented in this section were obtained from analyzing the videos and transcripts of nine clinical interviews with each of the three subjects. These results inform characterizations of each student's understanding and use of function composition, which together allow conclusions to be drawn relative to the research questions for this study and the framework of Conjectured Function Composition Mental Actions (Table 1).

From coding and conceptual analysis of the student interviews, several factors emerged as significant relative to students' conceptions and use of function composition. In this section, I present results relative to these factors:

- Problem solving behaviors
- Quantitative reasoning
- Covariational reasoning
- View of function
- Other emergent factors

In each of the following three sections, I present results illustrating students' reasoning or behavior relative to these factors (Problem solving behaviors, Quantitative reasoning, Covariational reasoning, View of function, or Other emergent factors). Results are then presented illustrating corresponding characterizations of the students' understanding and use of function composition. Because these factors are not typically independent and isolated from all of the others, I examine each of these factors - Problem solving behaviors, Quantitative reasoning, Covariational reasoning, View of function, and Other emergent factors – together, while analyzing student comments and actions in response to selected interview tasks.

Patricia

In this section, I present selected results of Patricia's task-based interviews. The focus in this section is on Patricia's understandings, reasoning, and behaviors, followed by a discussion of how these understandings, reasoning, and behaviors are reflected in her understanding and/or use of function composition when solving a novel problem.

As will be elaborated in the subsections below, Patricia was able to create detailed mental pictures that supported her creation of quantities and rudimentary quantitative structures from problem descriptions. Her view of function could most frequently be classified as an action view, with occasional indications that she was in transition toward developing a process view of functions. When solving most problems requiring covariational reasoning, she demonstrated behavior consistent with Level 3 covariational reasoning. She was aided in many of her solutions by certain helpful problem-solving behaviors and dispositions, such as a focus on sense-making and goal-setting, and high levels of persistence toward completing the tasks.

Patricia's solution to the dinner problem.

To begin characterizing Patricia's reasoning and behaviors, I present an analysis of her solution to the Dinner Problem (Task 1). This task was part of Patricia's first interview of the semester.

Table 1: Task 1 – The Dinner Problem
Two friends that live 42 miles apart decide to meet for dinner at a location half way between them. The first friend, Tom, leaves his house at 6:05 and drives an average speed of 34 miles per hour on his way to the restaurant. Task 1a: If the second friend, Matt, leaves at 6:10, what average speed will he need to travel to arrive at the same time as Tom? Task 1b: If before leaving Matt knows that he averages driving 15 miles per hour to the restaurant, what time would he have to leave to arrive at the same time as Tom?

Patricia began by reading Task 1 aloud, making notes on a separate sheet of paper about items that initially seemed important. This included noting the initial distance separating Tom and Matt, the times at which each person departed his house, and the speed at which he was traveling. For Matt, she denoted this speed as "*x*". With respect to this variable, Patricia's next step in solving the problem was to set a goal for her solution, which she described in Excerpt 4.

Excerpt 4

1	Patricia:	So this – my focus here is, um, arriving at the same time. That's what
2		we're trying to figure out… Oh, I get it. So he (*indicating Matt*) left a
3		little later, so he has to go faster to get there the same time as Tom.
4		Okay, so we know that *x* is gonna be larger than… 34.

Note that Patricia appeared to be simultaneously orienting herself to the problem, setting a goal for her solution, and creating criteria for evaluating potential solutions. In orienting herself to the problem, she examined the distances each person must travel, as well as the time each person left home, understanding that Matt was going to have to travel a little faster than Tom, a realization signified by her comment, “Oh, I get it”, and her subsequent explanation in lines 2 through 4. She decided that her goal was to find Matt’s required speed, subject to the constraints that the two friends must arrive at the restaurant at the same time. She also indicated that the answer she obtained must be greater than 34 miles per hour. Relating these behaviors to Carlson and Bloom’s (2003) Multidimensional Problem Solving Framework (MPSF), Patricia was working in the Orienting phase of the MPSF, while making conjectures and constraints that might be useful in the Planning and Checking phases, as well.

Relative to Patricia’s quantitative reasoning, note that in lines 2 and 3 of Excerpt 4 she demonstrated an awareness of a relationship between the speeds and times in the problem. Her comments are not indicative of an awareness of the quantitative operations creating speed from accumulations of distance and time; rather, it is likely that Patricia’s statement that “*x* is gonna be larger than… 34” is rooted in real-life past experiences that a higher speed gets you to your destination in less time.

At this point, Patricia exhibited some procedural uncertainty. Having set a goal and oriented herself to the problem situation, she found it difficult to decide whether she should multiply, divide, or perhaps perform some other operation using the numbers she had been provided:

Excerpt 5

Patricia: Um, see this is 34 miles per hour. So if we're gonna go 21 miles, this time here is gonna be less than… less than an hour. I just kinda get confused with, like, what goes multiplication, what goes, I guess, um, division. And so usually I just try to plug in a whole bunch of 'em and, like, see which one looks the closest.

As a result of her uncertainty about what mathematical operation to use to calculate the amount of time it would take Tom to drive 21 miles, Patricia considered discontinuing her pursuit of an answer. She also stated, “I have no idea how you would solve this without looking at a similar problem and seeing how they do it.” This suggests a strong procedural orientation. This is also consistent with her earlier behavior in Excerpt 4, where she intuited a relationship among distance, time, and speed yet did not conceptualize speed as representing corresponding proportional accumulations of distance and time. However, her conviction that the time she wanted to calculate must be less than one hour does suggest that she conceptualized distance and time as being somehow related.

At this point, the interviewer asked Patricia to talk about the various times involved in this problem, and asked if she had any time-related goals. In response to this prompt, Patricia referred to an illustration she had drawn earlier, and stated that the two friends need to arrive at the restaurant at the same time. Pointing to different distances and times that she had noted on her diagram, she stated:

Excerpt 6

Patricia:	Because this time, like, um – hmm. Because he had to get there the same time. This time, this time. Like this distance and this distance. So this distance (*indicating the distance to be traveled by Tom*) is the same as this distance (*indicating the distance to be traveled by Matt*, and so is the time. Even though they left at different – right? Um, okay. So basically, then, I would have to figure out this part here, and then I could come over here and build off of that one.

Note that when discussing the distances and times on her diagram, Patricia used words that suggest a rudimentary quantitative structure, such as "this part" and "build off of that one". This was not a fully-formed quantitative structure in the sense described by Thompson (1989), because Patricia did not seem to have an awareness of how to relate the distances, times, and speeds in this problem. However, her words suggest that she imagined relationships existing among attributes of the problem situation.

Patricia next set up a proportion, equating $\frac{34\text{ miles}}{1\text{ hour}}$ and $\frac{21\text{ miles}}{x\text{ hours}}$, before using cross-multiplication to conclude that Tom's trip required 0.617 hours. Note that this again suggests that Patricia had conceptualized a situation involving constant speed. She again exhibited some uncertainty about the operation to perform, but chose to multiply 0.617 hours by 60, determining that Tom's trip required about 37 minutes. With this piece of information, Patricia calculated that Tom arrived at the restaurant at 6:42PM.

At this point, Patricia paused to re-orient herself to the problem, because she said she wasn't sure what to do with the 6:42PM she just calculated. She re-read the problem statement, and concluded that she was trying to determine how much time Matt was allowed to complete his trip. She tentatively decided to subtract 6:10 from 6:42, concluding that Matt must complete his trip in 32 minutes. Patricia next checked to see if this partial solution was reasonable, and concluded that it was, referring to the diagram she had drawn of the problem situation earlier, as shown in Excerpt 7.

Excerpt 7

Patricia: If it takes Tom 37 minutes, we know that it's gonna have to take this guy a shorter amount of time, so we'd say it takes 32 minutes for Matt – Matt has to do the same amount of distance in 32 minutes.

Having calculated what she considered to be a reasonable amount of time for Matt’s trip, Patricia mentioned that she next needed to “do what I did previously, backwards.” By this, Patricia meant that she needed to divide 32 minutes by 60 minutes, to determine what fraction of an hour Matt traveled. However, this does not seem to be indicative of Patricia possessing a process view of function that allowed for ready inversion, because after concluding that Matt traveled for 0.533 hours, she looked for clues for what procedure she had followed earlier:

Excerpt 8

Patricia: What am I doing with this? I've lost my place. Okay. What did I do here? Did I do 20 – I should've recorded what I did. Maybe my calculator has it known. I think I did 21 divided by 34. Yeah, that's what I did. And then I times'd it. Oh, God. What am I doing? This is where I'm just gonna plug in to see what I can – what I can do with these numbers. It's not right. Go back to the question here. Trying to find... oh, miles per hour… his speed.

Patricia was trying to reverse the procedure she had used before. Since she was unable to recall her procedure, she returned to the problem statement, to remind herself of the goals she had set.

Noting that her goal was to determine Matt’s speed, Patricia was able to finish the problem and arrive at what she considered a reasonable solution. To do so, Patricia again made use of a number line analysis, commenting “I had it in – I had it in the wrong format here. Because this is only a portion of an hour.” Using number lines, she again relied on an assumption of constant speed, determining how far Matt would have traveled in a full hour, explaining that the speedometer “tells you in miles per hour. So this is what he would have to maintain on his speedometer, okay, to make it to the rest – the 21 miles to the restaurant by 6:42PM.” When asked if Matt would be traveling for a full hour, Patricia replied, “No, he doesn't. It's only gonna take, um, 32 minutes.” This suggests that while Patricia’s understandings of the operations involved in relating distance, time, and speed were weak, she did not believe that traveling at a speed of 39.37 miles per hour implied that Matt traveled for a full hour.

Turning to Task 1b, Patricia followed a problem solving approach that was similar to that used in completing Task 1a. She again began by reading the problem statement aloud, making notes of details that she believed to be important, and making comments that helped her set goals for solving the problem and criteria for evaluating possible solutions (Excerpt 9).

Excerpt 9

Patricia: Um, if before leaving, Matt knows that he averages driving 15 miles per hour to the restaurant – what? If before leaving, Matt knows that he drives – oh, so it's asking you to, um, speed up. Or – oh, oh, no. Change the time. Okay. So 15 miles in one hour. Crud. So changing this number right here. Instead of 6:10, he would have to leave much, much earlier than, um – much earlier than – 37 minutes? It's taking Tom 37 minutes, if Tom left, um, at 6:05.

Note that Patricia immediately decided that her goal was to determine the new time that Matt would need to leave home, with the evaluation criterion that for a solution to be reasonable, the departure time must be "much, much earlier" than 6:10. When asked to explain why she felt this way, Patricia replied, "Because he's going only 15 miles an hour. He's driving very slowly."

Noting a similarity between this task and the previous task, and indicating her desire to remember and apply the correct procedure, Patricia stated, "I think I can use an equation that I've already done here." However, she remained uncertain about what operations to perform. Patricia seemed confident using number lines to approach the problem, from which she set up and solved the proportion $\frac{15\text{ miles}}{60\text{ min}} = \frac{21\text{ miles}}{x\text{ min}}$, concluding that Matt required 84 minutes to travel 21 miles. However, her confidence seemed to come more from the belief that this was the “correct procedure” than from an awareness of corresponding accumulations of proportional quantities.

Patricia next converted 84 minutes to 1 hour and 24 minutes, but then struggled to subtract this from 6:42PM. When asked to explain the meaning of 1 hour and 24 minutes, however, she explained “It represents, um, before how much, like – if I'm here at 6:42, I have to leave, um, an hour and 24 minutes before 6:42.” This is important to later interviews, because it indicated that even in a case where Patricia struggled to calculate a quantity’s value, she was able to explain the meaning of the quantity in the context of the problem.

Patricia’s covariational reasoning was assessed throughout the sequence of interviews, beginning with the first interview. When attempting to complete Task 1, Patricia drew parallel number lines representing miles traveled and elapsed time. However, this does not mean that she had conceptualized a corresponding accumulation of time and distance. Rather, further probing revealed that she had seen this technique used in the classroom when solving an example problem involving distance and time, and believed this to be part of the correct procedure for solving this problem.

In her subsequent actions, Patricia reveals that she interpreted the given statement that Tom “drives an average speed of 34 miles per hour on his way to the restaurant” in such a way that she conceptualized a situation in which Tom traveled at a constant speed of 34 miles per hour. When she was solving problems, this meant that one hour was “equal” to 34 miles, a view which allowed her to solve many problems involving distance, speed, and time without conceiving of speed as comprising a corresponding accumulation of distance and time.

Patricia's solution to the box problem.

In a second problem, Patricia was asked to consider a box constructed by cutting equal-sized squares from each corner of an 8.5" by 11" sheet of paper and folding the sides up (Tasks 2a through 2d). This was a problem that had been discussed in class, and formed the basis of a group activity as well as individual homework. Thus, Patricia was familiar with the task (Table 2).

Table 2: Task 2 – The Box Problem

Starting with an 8.5" x 11" sheet of paper, a box is formed by cutting equal-sized squares from each corner of the paper and folding the sides up.

Task 2a: Describe how the length of the side of the cutout and the volume of the box covary.

Task 2b: Write a formula that predicts the volume of the box from the length of the side of the cutout.

Task 2c: Given the graph below, how does the volume change as the length of the side of the cutout varies from 1.8 inches to 1.9 inches?

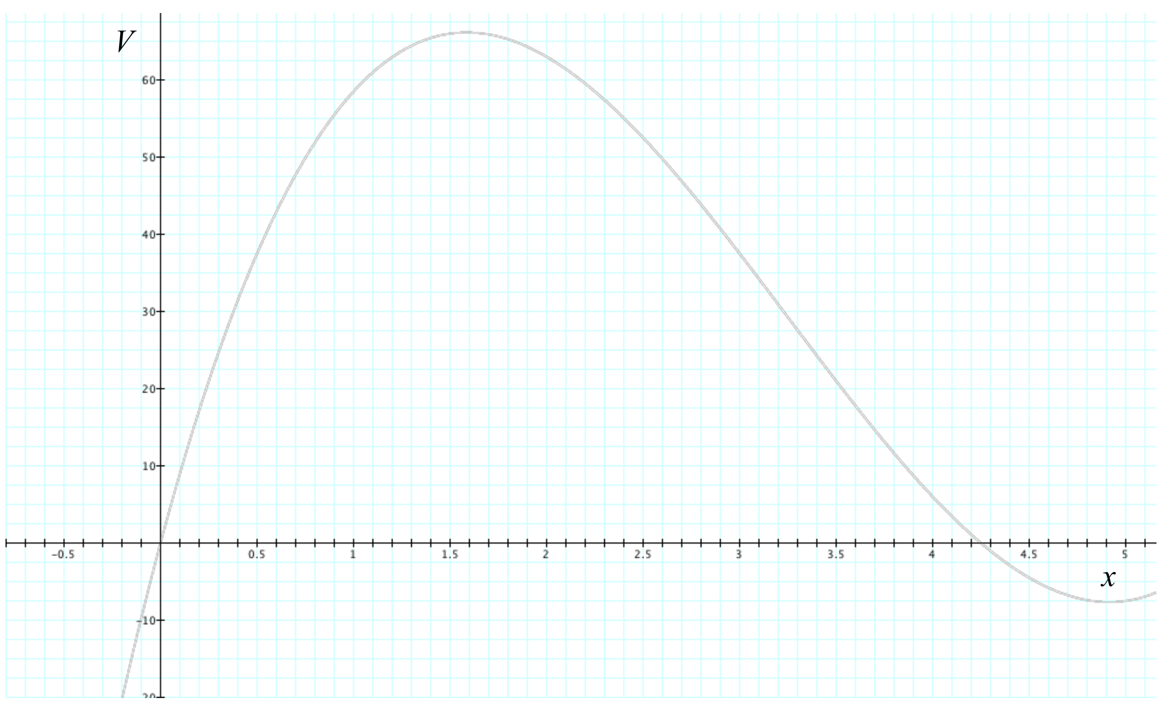


Table 2: Task 2 – The Box Problem
Task 2d: Use a formula to determine how much the volume changes as the length of the side of the cutout varies from 1.8 inches to 1.9 inches.

When completing Task 2a, Patricia had difficulty interpreting the problem statement. This inability to orient to the problem may have been the result of an inability to conceptualize the quantities involved in the problem; her previous exposure to this problem suggests that she would have had some experience with this problem situation, so the wording of the problem statement shouldn't have been unfamiliar to her. She was initially unsure what two quantities she was being asked to relate, and was also unsure if she was being asked to use specific values or just discuss the covariation of the quantities in general terms. Further probing revealed that the phrase "length of the side of the cutout" was difficult for her to process. Also, she was not certain whether or not the interviewer wanted her to calculate specific values and discuss those. After some clarification, Patricia noted:

Excerpt 10

Patricia: As those cutouts – the cutout lengths get larger – Your volume will hit a certain – I guess the number is like 108, something. The volume will hit a maximum, um, and then it'll decline because then the larger, I guess, the cutout your volume will then decrease.

Patricia was able to reason about how the cutout lengths and volume change together, although her reasoning suggests that she was only considering the direction of how the quantities change – as the cutout gets larger, the volume will get larger, then smaller. This thinking can be characterized as Level 2 covariational reasoning.

When solving Task 2b, Patricia immediately wrote $V(x)=(x)(8.5-x)(11-x)$, before correcting the formula to $V(x)=(x)(8.5-2x)(11-2x)$, which she explained was necessary because "there's two actually – there's two on each." Unfortunately, Patricia was not asked to clarify what she meant by this statement; as a result, I cannot determine whether her correction was based on quantitative reasoning, or was based on remembering the correct formula from class, along with some general wording about the reason the 2 was required.

Task 2b only asked Patricia to provide the equation described above, but she noted, "We can talk about this more. I have more things to say about this." Patricia proceeded to sketch the graph of the equation she had just written, stating that her graph was based on remembering the shape of the graph created by the calculator. She once again described the covariation of length of the side of the cutout and the volume of the box, this time using the graph to illustrate what she said. It is noteworthy that she described the graph almost as if it were an attribute of the formula, stating, "It comes with a little graph that goes like this." This was the first indication of something that she elaborated in later interviews – the notion that for representations like graphs and tables to describe functions, an algebraic formula must exist that defines the relationship. For Patricia, the algebraic formula "is" the function; graphs and tables are only depictions of the function, and cannot define a function themselves.

When completing Task 2c, Patricia indicated the correct points on the graph, and made correct statements about the covariation of cutout length and volume of the box.

However, she had difficulty explaining how she was using the graph to determine corresponding volumes and cutout lengths, although she eventually articulated that she was referring to values on the vertical axis to determine the volume. Patricia did not see how a change in volume might be represented on the graph. This suggests weakness in her ability to describe what a function's graph conveys about how two quantities change together.

Later in the interview, Patricia was prompted (Task 2d) to determine the change in volume that resulted from an increase in the cutout length from 1.8 inches to 1.9 inches. Patricia described how she would solve this problem, and described the meaning of quantities she wanted to calculate, without calculating them:

Excerpt 11

Patricia: Okay, so, um, we're doing 1.8, and then 8.5 minus 2, 1.8 and then there's two lines here. Now 11 minus 2, 1.8. And then you plug that in, and then you subtract it from, I guess, this one, right? I don't know. Maybe? Determine how much the volume changes. So – 8.5 minus 2, 1.9; 11 minus 2, 1.9. And so we would take this and this, and then subtract this and this. This (*indicating the unevaluated expression for the volume corresponding to a cutout length of 1.8 inches*) is the volume of the smaller cutout or volume of the box – of the box with the cutout, with the length of the side at 1.8 And this (*indicating the unevaluated expression for the volume corresponding to a cutout length of 1.9 inches*) is volume of the box with length of the side at 1.9. So – Subtracted this from this, and that is like, I guess, the change in volume, which is what you're looking for.

This provides another example of a situation in which Patricia was able to describe quantities, explain the meaning of quantities, and imagine using the values of those quantities in later computations, all without explicitly calculating the quantities' values.

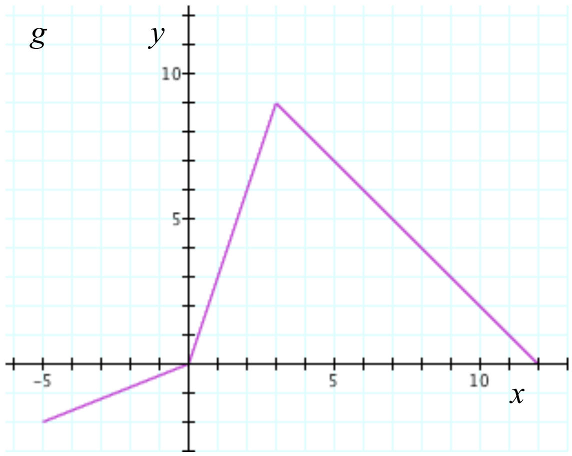
Patricia's interviews provided information about her approach to solving novel word problems (Carlson and Bloom, 2005). She demonstrated several helpful problem solving strategies, such as her tendency to create detailed, quantity-rich mental pictures of problem situations. In addition, Patricia continued to express a desire to identify memorized procedures and apply them, and was uncomfortable when she was unable to do so. This suggested that she believed a weakness in procedural knowledge inhibited her problem solving. Despite this apparent belief, Patricia's persistence kept her from giving up, and led her to complete tasks that she had initially expressed doubt about being able to complete.

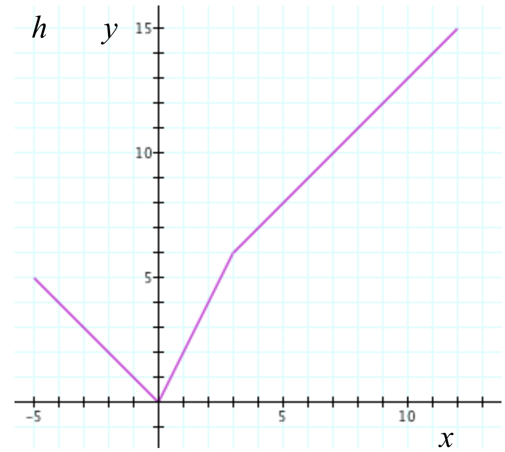
Patricia's solution to the graphical composition problem.

Task 3 (Table 3) allowed me to begin to explore Patricia's understanding of function notation and her approach to solving function composition problems; Task 3a required finding the output of composed functions at a given input value, while Tasks 3b through 3d required covarying the input and output quantities for individual functions and composed functions. Standard function notation was used in Task 3, and the functions were defined by graphs. By this point in the semester, the class had been exposed to standard function notation.

Table 3: Task 3 – The Graphical Composition Problem

Functions g and h are defined by the graphs below.





Task 3a: Determine each of the following:

i) $h(g(1))$

ii) $g(h(5))$

iii) $h(h(2))$

Task 3b: How does the output $h(g(x))$ vary as x varies from 5 to 9?

Task 3c: How does x vary as $h(x)$ varies from 10 to 15?

Task 3d: How does x vary as $h(g(x))$ varies from 6 to 10?

Patricia began by reading Task 3 aloud. She was initially confused about which graph in Task 3a represented which function, because she did not notice the function names in the upper left of each graph. After receiving clarification from the interviewer, she described how she would solve the problem, as shown in Excerpt 12.

Excerpt 12

Patricia: So you plug in 1, um, to *g*. Right? And then you find where it hits, which I would say is right here. That looks like that's 5, that's 10 – one, two, three, four, five – two, three – so it's 3. So *g*1 equals 3. And you plug it into *h*, which would be here. One, two, three up here, and it looks like five, six. So I would say, um, this would be 6.

While saying this, Patricia pointed to the graphs, indicating that she would count over 1 unit on the *x* axis of the first graph, go up vertically until she hit the graph of *g*, and then move to the left until she hit the *y* axis (at 3). She then counted over 3 units on the *x* axis of the second graph, moved her finger up vertically until she hit the graph of *h*, and then moved to the left until she hit the *y* axis (at 6). She did not use the words input or output, but she showed no hesitation about what actions to perform. When asked to explain how she knew what to do to solve the problem, she stated:

Excerpt 13

Patricia: I guess, right when you say order of operations, you start with this middle part here, and so *g* of 1. So you have to go to the *g* function. And, um, why do I look on this side? Because that's what I've been told to do.

Patricia attributed her ability to solve this problem relied on an understanding of rules of "order of operations" and a memorized procedure. Note that this is not indicative of a robust understanding of the concept of function, and could be characterized as pseudo-analytical (Vinner, 1997) behavior, and indicative of an action view of function (Dubinsky and Harel, 1992).

Task 3b required Patricia to reason about the covarying quantities x and $h(g(x))$, as x varied from 5 to 9. Patricia approached this problem by determining $g(5)=7$ and $g(9)=3$, and then $h(g(5))=10$ and $h(g(9))=6$. However, she did not attend to the direction of covariation, stating that as x varies from 5 to 9, $h(g(x))$ varies from 6 to 10. Again, this suggests that Patricia likely held an action view of function, or was in transition from an action view to a process view.

Task 3c required Patricia to attend to the changes in the input to a function, given a specified range of output values. However, Patricia interpreted the problem statement as telling her how the value of the input quantity, represented by the variable x, varied. This task revealed that Patricia did not interpret $h(x)$ as representing the output value of the function. It was only after repeated questions that Patricia concluded "*h* of *x* is actually – let's say *y*." After this renaming, she gave a correct answer to the problem, although her solution was reached by looking only at the endpoints of the specified interval, rather than attending to the changing values between the two endpoints. This again suggests an action view of function, as well as possible weakness in her covariational reasoning.

Task 3d exposed several of Patricia's weaknesses even more clearly. She was asked to explain how x varies as $h(g(x))$ varies from 6 to 10. From Task 3c, Patricia knew that she was being asked to find input values, and that she was being given output values. However, she also knew that in solving the problems in Task 3a, she used order of operations rules to determine what to do first. Combining these two approaches, Patricia looked first at $g(x)$, determining what values of x would give output values of g equal to 6 and 10. Noting that there was no value of x for which $g(x) = 10$, I allowed her to change the problem statement from 10 to 9, to see where she would go with her reasoning. Having determined that x varying from 5 to 3 would give $g(x)$ varying from 6 to 9, Patricia was unsure what to do with these values, although she knew she needed to use the graph of h. She decided to use these as input values for h, determining that the corresponding output values from h would vary from 6 to 8.

Several issues are demonstrated by Patricia's approach to this task. First, she did not view $h(g(x))$ as representing the output value of the function h. Second, she ultimately determined an interval of output values of h as her answer, despite the problem asking for an interval of values of x. Third, Patricia did not attend to the direction of covariation in her last step. Taken together, these actions suggest an action view of function, a weak understanding of function notation, and low level covariational reasoning.

Patricia's solution to the salary problem.

To better understand Patricia's view of function and understanding of commonly-used function notation, I present excerpts from Patricia's solution to Task 4 (Table 4).

Table 4: Task 4 – The Salary Problem
Using function notation, suppose $S(m)$ represents the monthly salary, in hundreds of dollars, of an employee after m months on the job. What would the function determined by $R(m) = S(m + 12)$ represent? a) the salary of an employee after $m + 12$ months on the job

Table 4: Task 4 – The Salary Problem
b) the salary of an employee after 12 months on the job c) $12 more than the salary of someone who has worked for *m* months d) an employee who has worked for *m* + 12 months e) The salary after *m* months is the same as the salary after *m* + 12 months.

Patricia begins by considering what she has been given, and what the components of the given information represent in the context of function notation, as shown in Excerpt 14.

Excerpt 14

1	Patricia:	Okay. Use the function notation S of *m*, and that's a monthly salary.
2		Monthly salary in hundreds. Uh, okay, so *m* is months on the job.

Note that Patricia correctly identifies $S(m)$ as representing a salary, in hundreds of dollars, and m as representing a number of months. This suggests that she is imagining the two quantities in the situation to be related, but she may or may not be conceiving of the function as accepting inputs and producing outputs.

She next examines what the task is asking her to find, and makes an initial attempt to answer the question, as shown in Excerpt 15.

Excerpt 15

1	Patricia:	What would the function determined by R… R of *m* equals S of *m*
2		plus 12 represent. So you would say, um, 12 months after however
3		many months they had been on the job, what their salary was.

Note that Patricia appears to be on the verge of correctly answering the question. However, she might also have simply been examining the first of the possible answers given. Patricia's explanation in lines 2 and 3 of Excerpt 15 suggests that she is still considering the functions in this task to be describing something about an employee who has been on the job for *m* months.

Patricia continues to consider other options for interpreting the "12" in Excerpt 16.

Excerpt 16

1	Patricia:	So the salary of employee after 12 months on the job would be \$12.00
2		more – oh, okay, hold on, let me write down what I think it is. Okay,
3		so it's, um, yeah, 12 months, so however many months they were on
4		the job, plus 12 months… and their salary. Yeah, I think it's B, but…
5		"the salary of employee after *m*+12 months on the job" - that doesn't
6		make sense (*indicating answer choice (a)*). Salary of employee after
7		12 months on the job. I think it's B, um, because it's not \$12.00 more
8		than the salary of someone who has worked…

In lines 5 and 6 of Excerpt 16, Patricia's comments indicate that she is unable to make sense of *m+12* as a quantity, which causes her to reject choice (a). She also rejects choice (c), explaining in Excerpt 17 that whatever is in parentheses is an amount of months, not an amount of dollars.

Excerpt 17

1	Patricia:	My first inclination is B, because S of *m* equals a dollar amount.
2		[*Pauses*]… this is – in the parentheses is an amount of months, so if

3	I'm going to throw six plus twelve, that's still an amount of months.
4	Yeah, I think it's B.

Note that Patricia was guided to her solution by her belief that whatever was in parentheses is an amount of months, and "*S* of whatever was in parentheses" represented a dollar amount. This allowed her to successfully eliminate three of the four distractors, but her inability to make sense of $S(m+12)$ and relate it to her understanding of $S(m)$ left her unable to answer the question correctly, as she ultimately chose to ignore the *m* and focus on "12 months" as the input value.

Patricia's solution to the circle problem.

Having examined Patricia's quantitative and covariational reasoning and problem-solving behaviors, I turn to her understanding and use of function composition. I examine her actions when attempting to solve a novel function composition word problem, and note manifestations of the factors explored in earlier sections. In a later chapter, I re-examine my conjectured framework of mental actions, in light of Patricia's solution to this problem. I begin by presenting Patricia's solution to Task 5 (Table 5).

Table 5: Task 5 – The Circle Problem
Express the circumference of a circle as a function of the area of the circle.

As she often does when solving a novel problem, Patricia begins by reading the Circle Problem aloud. However, in contrast to her behavior when solving other problems, when she would next set goals for her solution and for evaluating the reasonableness of potential solutions, in this task Patricia first tries to recall "the formula", as shown in lines

1 through 3 of Excerpt 18. This provides some insight into Patricia's view of mathematics, and is consistent with comments and actions she made throughout her interviews. When faced with a problem that she believes she recognizes as a problem that has a correct formula or procedure, Patricia tends to reduce her efforts at creating coherent mental pictures and sense making, focusing instead on remembering the "right" formula or procedure.

Excerpt 18

Patricia: Express the circumference of a circle as a function of the area. Okay, so – I don't remember the formula, right. It's like – Area equals pi R squared, no?

Int: What's R?

Patricia: Radius.

Int: Okay.

Patricia: And is circumference two pi R? Okay, so express the circumference of a circle. So when it says express and a function of, so I'm not quite sure. Would it be F of A? Yeah? Is that what that's saying? You're expressing the circumference so that's – the "expressing", that would be your output, correct? (*Underlines "Express the circumference" and writes "output" below it*)

In Excerpt 19, Patricia does attempt to make sense of the problem, and create a goal for solving the problem, which she later uses to evaluate potential solutions.

However, her fixation on remembering and immediately using the formulas results in her losing focus on the input and output quantities.

Excerpt 19

Patricia: Now what am I doing here? So we want to get circumference … and then you want me to plug A into that? So um (*writes* $f(A) = C$) – oh, crap. (*Attempts to combine the formulas* $A = \pi r^2$ *and* $C = 2\pi r$, *writing "*$f(A) = 2(\pi r^2$*", leaving the parentheses open, before continuing*)… So you're plugging this in, but then you would have to get rid of the two, so I have no idea.

Note that in Excerpt 19 Patricia wants to combine the two formulas she wrote down at the start of her solution, but is unable to do so in a way that satisfies her. She knew she wanted a solution of the form $C =$"an expression involving *A*", but her first attempt – substituting her area formula into her circumference formula – was not successful. Following this, Patricia considered substituting her circumference formula into her area formula, as shown in Excerpt 20. However, she is able to use her knowledge of which quantities are the input and output quantities to reject this idea. This is consistent with a student who is in transition between an action view and a process view of function.

In Excerpt 20, Patricia reached a point where she was unable to progress any further. She expressed her belief that $A = \pi r^2$ was "area as a function of radius", and $C = 2\pi r$ was "circumference as a function of radius", but she did not recognize the need to determine the inverse of the area function. This observation is consistent with the action view of function that Patricia had demonstrated in earlier interviews.

Excerpt 20

Patricia:	So this is why I – this is where I would stop and be like, "Okay, so let's see what it looks like when I do it the opposite way because that just looks a little easier." But then this to me is um, a function – this is area as a function of circumference. This is – and this is circumference as a function of – I don't know. Anyway, so let's just see what – how this looks like. So if C equals… – I don't see how this would work either…

Suspecting that she viewed a function as a "command to perform a calculation", the interviewer prompted her with a specific possible value for the area of the circle, to see if this would help her toward a solution, as shown in Excerpt 21.

Excerpt 21

Int.:	Suppose I told you that you know the area is 12.
Patricia:	Yeah, yeah, okay. Okay. [*Laughs*]. Yes, see I can do that. Yes. Okay, so – so 12 – area equals 12. You would do – you would say um, divided by pi. [*Yells*] oh, there we go. I feel it coming back. [*Chuckles*].
Patricia:	You divide by pi and then – so this is 12 divided by pi and then square root of it, no? Wait. That would help us find the radius… And then once you have R you would shove it in this here. So you would have two um, pi 12 – would it be – is that better? Equals

circumference.

Int.: So um, if the area wasn't 12, but was just –

Patricia: Anything. Oh, yeah, yeah, yeah. Okay, good deal. Good deal, good deal. I can do that. So then you would do – so – okay. Let's go back here. (*Writes* $C = f(A) = 2\pi\left(\sqrt{A/_{\pi}}\right)$ *and puts down pencil.*) Confident about that. [*Laughs*].

Int.: Well, why – why does that feel better to you?

Patricia: Um because trying to put – put like – expressing the circumference of a circle as a function of the area, you have to be very comfortable with flipping area and circumference back around with each other – you know what I mean? Kind of like undoing them and redoing them to come up with this. Do you know what I mean?

Excerpt 21 reveals that Patricia was much more comfortable with the problem when presented with a specific value for the quantity she had identified as the input. This reinforces the notion that she possessed an action view of function, seeing functions as a "command to calculate". However, when prompted to focus back on the general situation where the input is not assigned a specific numerical value, Patricia was quickly able to invert the area function and solve the problem (lines 12 through 15). This again suggests that Patricia is in transition from an action view to a process view of function.

Note how Patricia describes the process of inverting the area function before combining it with the circumference function: "flipping area and circumference back around with each other", and "undoing them and redoing them". Ultimately, she was able to perform the inversion for the function formula, but only after doing so at a specific value for area, consistent with a student possessing an action view of function, possibly beginning to transition to a process view.

Patricia's solution to the giraffe pen problem.

As another example of Patricia's approach to solving novel function composition problems, consider Task 6 (Table 6):

Table 6: Task 6 – The Giraffe Pen Problem
Bryan has decided to open a wildlife park, and he would like to build a square pen for some giraffes. Bryan hasn't decided how many giraffes to acquire, but he has been told he should allow 10,000 square feet for each giraffe to graze. Help Bryan design his park – find a way to determine how many feet of fencing he must buy to enclose his square pen, based on the number of giraffes that will live in the pen.

After reading the problem aloud, Patricia sketched the pen and labeled her sketch with the dimensions that she believed would be required if Bryan were to acquire one giraffe, as shown in Excerpt 22. This was consistent with her approach to solving other problems, in which she had found it to be powerful to consider a specific case before looking at the more general case described in the problem.

Excerpt 22

Patricia: So 10,000 square feet, that's something times something. [*Hums*]. Square pen (*sketches a pen*). Well, I want to make my little pen, so – Um, this is 10,000 square feet, wouldn't you do 10,000… divided by two and you would do – okay, so 5,000 feet on this side and 5,000 feet on this side. Um, for one giraffe.

6	Int.:	One giraffe, okay.
7	Int.:	So is that – a pen of 10,000 square feet, is that what you're saying?
8	Patricia:	Yes.

Note that Patricia has conceived of a mental model in which *something* about the pen "is 10,000 square feet". However, she has not constructed a quantity that would be considered "correct" (the area enclosed by the square pen, with a value determined by multiplying together the lengths of two sides of the pen). Rather, Patricia's constructed quantity seems to be something that consists of the *sum* of the lengths of two sides of the square.

In Excerpt 23, Patricia further explores her own understanding of the 10,000 square foot pen. In lines 1 through 3, and again in lines 7 through 9, she considers the operations required to calculate the area of the pen. She understands "5000 times 5000" to be different than "5000 to the second". However, in lines 14 and 16 she does demonstrate that she has a mental picture of a situation in which the pen remains square, with sides whose lengths grow by an equal amount in response to additional giraffes. Patricia's actions and comments in Excerpt 23 are consistent with earlier inferences about her understanding – that she constructs rich mental pictures, possesses weak operational understandings, and fixates on "the right formula" when she believes one to exist.

Excerpt 23

1	Patricia:	Hold on one second. So maybe it's not just 5,000 times 5,000 equals
2		10,000. Maybe it's like something to the second. So maybe this –

square root of ten would tell you one side? I don't really know. Hm.

Int.: All right, so this 10,000 square feet – what about that pen is it?

Patricia: Um, the area it covers. Square feet is area.

Int.: Okay.

Patricia: Which – area equals length times width, maybe? I don't know. Um, so then that tells you that I think 5,000 by 5,000 – I mean 5,000 times 5,000 equals 10,000 square feet. I feel okay with that… But say I want six giraffes, how many square feet is it gonna be? You know what I mean?

Int.: Well, if – um, if he decides he wants more giraffes, what's gonna happen to this – the pen he needs to build?

Patricia: Going to increase. In square footage.

Int.: Okay. How about the – like length and width of it? Are they gonna –

Patricia: They're gonna increase equally because it has to be a square.

[*Long Pause*]

In Excerpt 24, Patricia demonstrates that she does understand the concept of perimeter, and recognizes that is being the quantity whose value she is trying to find (lines 2 through 9 of Excerpt 24). However, in lines 16 through 19, she again demonstrates operational fragility, stating that the length of one side of a square can be found by dividing the area by two. This provides further evidence that her understanding of area is fragile and incorrect.

Excerpt 24

Int.: And what's – what's the problem asking of you do you think?

Patricia: Um, to find a way to determine how many feet, so um – that he has to buy. Okay. Okay, so that kind of now confuses me just a little bit because square foot usually covers an area.

Patricia: But um, if you're actually just buying the actual fencing, that confuses me because then you would actually have to not only – you'd – you'd have to take one side and times it by four. Do you know what I mean? You couldn't say, "Oh, I want –" Do you know what I'm saying? Because the side is just part of the perimeter.

Int.: And where's the four coming from?

Patricia: Because there's four sides.

Patricia: So for a ten – I mean I don't know if you meant to do this, but this –

Patricia: This would be my issue would be okay, well, if you're saying um – if 10,000 square foot equals the area for them to graze –

Patricia: Would that necessarily mean you need 10,000 square feet of fencing?

Patricia: Oh, wait, wait, wait, wait… Oh, I get it. So you would do – so 10,000 is for one giraffe, divided by two, and that tells you what one side equals, and then you times it by four because you have 10,000 square feet and you wanna figure out what one side is – yeah?

Patricia: But then you need – Four sides. Is that not right?

I explored Patricia's understanding of area in Excerpt 25. After being led to the realization that 5,000 times 5,000 does not equal 10,000, Patricia quickly returns to the notion (originally expressed in line 3 of Excerpt 23) that the length of one side is found by taking the square root of the area. In lines 13 through 16 of Excerpt 25, Patricia uses this new procedure to correctly solve the problem, for the case in which Bryan has only one giraffe.

Excerpt 25

Int.:	I have one question. If I told you that you had a square pen and it was – you know, 5,000 feet on a side – or five – you know, five – each side was 5,000 feet long – how did you say you would find the area of it?
Patricia:	Oh, I said area is length times width.
Int.:	Okay. So what it – what would that be in that case?
Patricia:	Hm. 10,000, correct?... Maybe not. Okay, 5,000 times 5,000 is not 10,000. It's much larger than that. [*Chuckles*].
Patricia:	Yes, so then – hold it – okay, it – that's what I was trying to do before. I was trying to get the square root of 10,000 to get –
Int.:	Oh, okay. Well, then that's different. You – you divided 10,000 by 2 here.
Patricia:	So let's do 10,000… square root… and that would equal one side. Just say it's one side – I don't know what that one side is – and then you times it by four. And then you would have it – instead of having

it in square feet, you would have it in um, feet, and then by how many
of those sides you have.
Patricia: Hold on one second. I think – mm, okay. Um, but you don't want to
put 10,000 here, you wanna put like giraffes. You know what I
mean? You wanna say –
Patricia: Based on the number of giraffes. Um, [*hums*]. And I don't
necessarily know how to do that.

Notice that in lines 18 through 22, Patricia pauses, and then reorients herself to the problem and her goal for a solution. This frequent self-evaluation of her progress and her goals was an important factor in her ability to correctly solve this problem, as well as others throughout her interviews.

In Excerpt 26, Patricia introduces a variable, *x*, to represent the number of giraffes Bryan decides to acquire, and concludes her solution.

Excerpt 26

Patricia: [*Hums*]. I basically do like sort of like X and then –
Int.: So what's X?
Patricia: X would be – Giraffes. Number of giraffes. Hm. And so you want
say one giraffe equals 10,000. I don't really know. [*Hums*]. Because
like X times what – would I – so 10,000X?
Int.: What's that? X times –
Patricia: X times 10000, uh huh. Okay. Square root, and then multiply this by

four? [*Chuckles*].

Int.: What are all these things? What – what is – what is 10,000x?

Patricia: Ten-thousand X, okay so say I have two giraffes –

Int.: Yeah.

Patricia: That means I – I need um, 20,000 square feet, and then –

Int.: So this 10,000 – or the 20,000 you said – the 10,000x is a –

Patricia: X times 10,000 would be how many total square feet.

Int.: Okay, so what is that? What about the pen is that?

Patricia: Grazing area. Okay. And then doing the whole square root thing would say – Change it – I think – change it from square feet – um, getting rid of the square feet and saying it in just feet. Getting rid of the square foot and um, saying the – the length of one side. Correct?

Int.: The length of one side of what?

Patricia: Of the pen. It could be any side 'cause it's a square.

Patricia: And then multiplying it by four – multiplying that equation by four, that's telling you the perimeter of the pen.

Int.: Okay.

Patricia: Perimeter of pen – in feet that he would have to buy.

Int.: Okay.

Patricia: I like this equation. I like this one.

Patricia: Did I get it right?

Patricia: [*Laughs*]. Close enough?

30 Int.: Yeah, that's what I'd end up doing.

31 Patricia: Nice! I really like that.

Several features of Excerpt 26 are noteworthy. Patricia very quickly builds a correct formula (lines 7 and 8). However, this does not indicate a robust understanding of all aspects of the problem. In lines 14 and 16 she articulates that $10000x$ represents the total grazing area required for x giraffes, but in lines 16 through 19 she describes the square root operation as "changing it" and "getting rid of the square feet and saying it in just feet". She correctly understands this result as being the length of one side of the pen, in feet, but it is not clear that she possesses a useful understanding of the concept of area. Finally, in lines 27 through 31 Patricia expresses happiness and excitement at having correctly solved the problem; this was consistently seen as an important aspect of her approach to solving problems, and her persistence in doing so.

Summary Characterization of Patricia

Patricia was able to understand and use function composition to build new functions in applied contexts, to a limited extent. This can be seen, for example, in Excerpts 19 through 21, and 24 through 26. Several factors were revealed in this study as either facilitating or complicating her ability to make progress on the function composition problems, as described in the following paragraphs.

Action view of function, in transition to process view.

The function image that Patricia commonly exhibited was an *action* view (Dubinsky and Harel, 1992), although in some cases she demonstrated behavior that suggested she was transitioning to a *process* view, such as when she engaged in the sort of “black box” reasoning described above. Her action view manifests itself in her inability to “invert” one function when solving the Circle Problem (see Excerpts 19 and 20), as well as in her tendency to evaluate functions at specific input values and difficulty describing the behavior of functions over an entire interval of possible input values.

Patricia demonstrated a fragile understanding of function notation in some situations. Patricia was unable to use symbols as a means of expressing how two quantities are related. She did not view the independent variable as representing the values of a quantity, nor did she appear to understand that the symbol $f(x)$ was a representation of all the possible output values that the dependent quantity can assume. This was most evident in her solutions to the Graphical Composition Problem and the Salary Problem. While her difficulties in solving both of these problems can be attributed to her possessing an action view of function, it is worth noting that when solving these problems Patricia did not view $f(x)$ as representing an output value, but as a “statement” about the function.

Ability to conceptualize quantities and relationships.

One of the most remarkable aspects of Patricia’s problem solving behavior was her ability to create detailed mental pictures of problem situations. When given a task that involved a real world context, Patricia was able to provide a detailed description of the situation. The descriptions she provided during the problem solving process suggest that she was sometimes successful in using these mental pictures to conceptualize the quantities and their relationships, and that she built productive quantitative structures of these relationships by reading (and re-rereading) the problem statements. This is illustrated, for example, in Excerpts 4 through 8, in which Patricia repeatedly returns to the problem statement as an aid to setting goals, evaluating partial solutions and intermediate results, and deciding when she was done and satisfied with her solution.

In other instances, spanning a variety of problems, Patricia believed relationships existed between quantities, but was unable to conceptualize how they were related – that is, she was unable to conceptualize the relationship correctly, and was uncertain what mathematical operation or operations comprised that relationship (for example, see Excerpts 5, 8, 23, and 25). In such cases, she was unable to create meaningful formulas to express the quantitative relationships in the problem context. This is consistent with Thompson's (1989) and Moore and Carlson's (2012) finding that meaningful formulas emerge from both conceptualizing quantities and how they are related. While this was clearly frustrating to Patricia, in many cases it facilitated deeper insights into her quantitative reasoning and the quantitative structures she had constructed. Her uncertainty about the mathematical operations that related different quantities also led her to view some functional relationships as "black box" processes without concerning herself with their inner workings.

Constructing images of the quantities and her ability to reason about those quantities and the relationships among them were crucial to her ability to understand and reason about function composition problems. This can be seen most clearly in her solution to the Giraffe Pen Problem, in which she demonstrates awareness of relationships between multiple pairs of quantities long before she was able to articulate the correct operations to perform. She described the existence of a relationship between the area of the pen and the length of one side of the pen (in Excerpt 23), a relationship between the number of giraffes and the area of the pen (in Excerpt 23), and a relationship between the length of one side of the pen and the perimeter of the pen (in Excerpt 24), which she states is the quantity whose value she ultimately wants to determine.

Patricia had a weak understanding of the concept of “area”. This proved to be significant, because many function composition word problems involve area (in addition to concepts such as volume, perimeter, or circumference,). In this study, both the Circle Problem and the Giraffe Pen Problem involved area as a key attribute of the situation described.

The two novel function composition problems discussed in this study, the Circle Problem and the Giraffe Pen Problem, differ in at least one key attribute: the Giraffe Pen Problem statement describes a detailed “real world context”, while the Circle Problem, in which she was asked to relate the circumference and area of a circle only required the conceptualization of quantities in a formula. In this and similar problems she immediately started searching for the “correct” formula or formulas with little or no attention to conceptualizing the quantities related by the formula. However, when given a context that was novel to her, she created detailed mental pictures that supported her “discovery” of relationships between quantities where previously learned formulas were not available.

The next chapter will include further discussion of the results presented above, concerning Patricia’s knowledge and use of function composition. For now, I turn to a discussion of the second student, Bridget.

Bridget

In the following subsections, I present selected results of Bridget’s task-based interviews. The focus in this section is on Bridget’s understandings, reasoning, and behaviors, followed by a discussion of how these understandings, reasoning, and behaviors are reflected in her understanding and/or use of function composition when solving a novel problem.

Bridget's solution to the dinner problem.

To begin to understand Bridget's understandings, reasoning, and behaviors, I present representative episodes from her solution to Task 1, the dinner problem.

Table 7: Task 1 – The Dinner Problem
Two friends that live 42 miles apart decide to meet for dinner at a location half way between them. The first friend, Tom, leaves his house at 6:05 and drives an average speed of 34 miles per hour on his way to the restaurant. Task 1a: If the second friend, Matt, leaves at 6:10, what average speed will he need to travel to arrive at the same time as Tom? Task 1b: If before leaving Matt knows that he averages driving 15 miles per hour to the restaurant, what time would he have to leave to arrive at the same time as Tom?

Bridget begins by orienting herself, which she does by sketching the relative positions of Tom, Matt, and the restaurant, and labeling the distances between them, as described in Excerpt 27. She then makes notes of "6:05" and "6:10" beside Tom and Matt's names, respectively, and labels the 21-mile distance between Tom and the restaurant as "34 mph". She then decides on her goal:

Excerpt 27

Bridget:	Okay. Okay, so I'm gonna approach this problem by drawing a picture.
Bridget:	And I'm gonna say Tom lives here and Matt lives here, and there's 42 miles between them, and they need to meet here at 21 miles. He's leaving at 6:05 and he's leaving at 6:10.

Bridget: He (*indicating Tom*) drives 34 miles per hour, and we don't know what he (*indicating Matt*) wants to drive. Okay.

With the sketch and her comments in Excerpt 27, Bridget demonstrated that she had constructed a mental image that supported her ability to identify attributes of the problem that she believed were relevant to her solution. This excerpt does not support significant conclusions about Bridget's quantitative reasoning, except to say that she has constructed a mental image of the problem.

In Excerpt 28, Bridget sets up a proportion, $\frac{34\ mi}{60\ min} = \frac{21\ mi}{x\ min}$, that she hopes to use to determine how long it will take Tom to reach the restaurant. This suggests that she has some level of awareness of speed as involving corresponding accumulations of distance and time. This also suggests some ability to quantify problem attributes and create basic quantitative structures relating distance and time to form speed. In lines 3 through 6, notice that Bridget demonstrates helpful problem solving behaviors, when she revisits her ultimate goal (determining how fast Matt needs to travel, to arrive at the same time as Tom) as well as the intermediate goal (determining how long it will take Tom to get to the restaurant) she believes will allow her to finish the problem.

Excerpt 28

Bridget: Okay, so if he goes 34 miles per hour for 21 miles – 34 miles per 60 minutes is – how many minutes per 21 miles? I'm thinking.

Bridget: Um, I'm trying to figure out how long it's gonna take him (*indicating Tom*) to get here. So that I can determine how fast he (*indicating*

	Matt) needs to go to get there in the same amount of time.
Bridget:	He's going 34 miles in 60 minutes – but he needs to get 21 miles in a certain amount of minutes.

Bridget reduces the left side of her proportion and correctly determines the number of miles per minute Tom traveled (on average), as shown in lines 1 and 2 of Excerpt 29. However, she is then unable to correctly solve the proportion to find the unknown time corresponding to 21 miles, as shown in the remainder of the excerpt. She determines an answer, but says it doesn't look right, based on what she knows from her engineering classes about unit cancellation:

Excerpt 29

Bridget:	I don't know if they're necessarily equal to each other. 34, 60 – so he's going – 0.567 miles in one minute, and he needs to go 21 miles, so this isn't a proportion. Um – so I guess it takes him 11.9 minutes to get here.
Int.:	Okay, so where did the 11.9 – where did that come from?
Bridget:	Thirty-four divided by sixty equals 0.567 miles in a minute, and then I – Multiplied it by 21 m – um, miles, which is how far he needs to drive. [*Whispers*] 0.567 miles. [*Normal voice*] I don't know if that makes sense. The conversion is not right.
Int.:	What conversion do you mean?
Bridget:	Like in science you make like conversion factors, like here would be

	0.56 miles per minute –
Bridget:	And if I multiplied by 21 miles, the miles wouldn't cancel.

Bridget's conviction in Excerpt 29 that something doesn't make sense does not appear to reflect any quantitative reasoning beyond a pseudo-conceptual awareness that the units should "cancel" to leave her with an answer in minutes.

Bridget eventually decides 11.9 minutes might be reasonable, because of what she understands of the corresponding accumulations of distance and time. Namely, she is certain that the "correct" amount of time should be something less than 60 minutes (the amount of time it would take Tom to travel 34 miles), since he only needs to travel 21 miles. Since 11.9 minutes is less than 60 minutes, it passes her test of what is reasonable. As a result, she decides to accept this result and to continue forward in her solution, as shown in Excerpt 30.

Excerpt 30

Bridget:	Hm. I'm kinda confusing myself a little bit. [*Sighs*]. It still sounds right to me though. He's gotta go – if he can get 34 miles in an hour – He can get… 21 miles in 11.9 minutes. That's what that was. Or – no, that's right. Okay. Now I have to figure out – that's gonna put him here at – I'm gonna call this 12 minutes. He gets here at 6:17.
Bridget:	And he (*indicating Matt*) needs to get here in seven minutes, so how many miles per hour does he need to go?

Bridget's actions in Excerpt 30 require her to coordinate several different starting times and elapsed times, which she does effectively. This suggests that she possesses the ability to engage in basic quantitative reasoning with additive structures of quantities.

Proceeding with her solution, Bridget calculates that Matt would need to drive 180 miles per hour to reach the restaurant in time. She believes that can't be a correct answer, for two reasons – first, she believes that is extremely fast for a typical car to travel, and second, it is much faster than what Tom drove to cover the same amount of distance. This is illustrated in Excerpt 31. Bridget consistently engaged in this type of problem solving behavior throughout her interviews, checking her results against other quantities' values and against what she knew from "real life" to be reasonable or unreasonable.

Excerpt 31

Bridget:	Hm, no. Hm. I'm not sure how I figure that out.
Bridget:	I was seeing if I could do like miles per minute like I did over here.
Bridget:	But like 21 miles divided by seven minutes is 3 miles per minute.
Int.:	What are you not liking?
Bridget:	Well, it just doesn't make sense because if I did – if I try to convert it to miles per hour –That'd be 180 miles an hour, so he's not gonna be going that fast.
Bridget:	I kinda don't trust my own thinking. I kinda think I'm doing this wrong. If he's (*indicating Tom*) getting 21 miles in 12 minutes, he (*indicating Matt*) needs to get 21 miles in 7 minutes. He's (*Tom*)

going 34 miles an hour, to go in 12 minutes. So yeah, it definitely can't be 180.

After a long pause, Bridget decides she should revisit the calculations that led her to conclude Tom reached the restaurant in 12 minutes. She does this by setting up a different proportion than before, $\frac{21\, mi}{34\, mi} = \frac{x\, min}{60\, min}$, as described in Excerpt 32, which this time she solves correctly. Notice that this proportion entails different reasoning than the proportion she used in her initial attempt to solve the problem (although both proportions are mathematically "correct"). In her second attempt, she eliminates the need to reason about "speed", as she is able to focus on corresponding proportional accumulations of distance and time (which are, admittedly, the accumulations that form "speed"). By noting that traveling six-tenths of the distance should require six-tenths of the time, Bridget is able to determine how long Tom's trip takes, without needing "distance equals speed multiplied by time".

Excerpt 32

Bridget: Hm. Thirty-four miles in one hour. He's getting 21 miles. So – [*calculates*] I think this should be taking him 37 minutes.

Int.: Now, why do you say that? This is a big change from 12 minutes.

Bridget: Because if I look at it proportionally –

Bridget: Um, 21 goes into 34 point six times, so it's like –

Bridget: Twenty-one is six tenths of 34.

Bridget: And then if I… multiply that times an hour, then it'd be 37 – 37 is six tenths of an hour.

Bridget: If that meant that this was 37 minutes, then he (*indicating Tom*) would get here at 6:42.

Bridget: And that would mean that he (*indicating Matt*) has 32 minutes to get 21 miles.

Bridget: And 32 minutes is like a half an hour – pretty much.

Bridget: And he's going 21 miles in one hour, so he'd have to go 42 miles an hour to get there in 32 minutes 'cause if I double this, then he's going twice as far in an hour and he could go 21 miles in half an hour.

Bridget: That's what I want to say.

When completing part 2 of the same task, Bridget was able to quickly reach a solution that was mostly correct, except for claiming that 40 minutes is four-tenths of an hour. She again used proportional accumulations of distance and time to find the answer, setting up the proportion $\frac{6\ mi}{15\ mi} = \frac{x\ hr}{1\ hr}$, as described in Excerpt 33. Notice that she completes this task without using speed as anything but a distance that requires one hour to travel; that is, the 21-mile distance that Matt needs to travel is comprised of one 15-mile distance (requiring one hour) and one 6-mile distance (requiring 6/15 of one hour).

Excerpt 33

Int.: Well, let's move on to part 2 then. Have a look at it. What's that say to you differently than the first part said?

Bridget: This time it’s telling you exactly how fast he’s going, and it wants to know what time he has to leave. So instead of solving for this (*indicating Matt’s speed*), I’m solving for this (*indicating Matt’s starting time*).

Bridget: So here’s Tom at 6:05, and here’s Matt, and we don’t know what time he’s leaving. He’s going 15 miles per hour… So 21 minus 15 is 6 because he’s going 15 miles – there’s one hour. And then – He’s got six miles left to go and six divided by 15 is 0.4, so that’s one hour and 40 minutes.

Int.: Okay. What – where did the 40 minutes come from again?

Bridget: Because 6/15 equals 0.4.

Int.: Now what – what are you hoping to find here with this?

Bridget: Well, I was kind of breaking it down like to make it simpler in my head. I just take out an hour of it ‘cause I already know it’s gonna take him at least an hour. He’s going 15 miles per hour. He’s –

Bridget: And one hour has gone by, he’s gone 15 miles, and he still has 6 miles to go, so how long is it gonna take him to get those six miles.

Bridget: So if I calculated it right, I think it’s gonna take him an hour and 40 minutes, which would mean if – assuming that this is right – [*chuckles*] – and Tom’s getting there at 6:42 –

Bridget: That means he’s gonna have to leave at… 5:02. An hour and 40 minutes – or 6:42 minus an hour and 40 minutes. 5:02.

Bridget's solution to the box problem.

To better understand Bridget's covariational reasoning, I present representative episodes from her solution to Task 2 (Table 8).

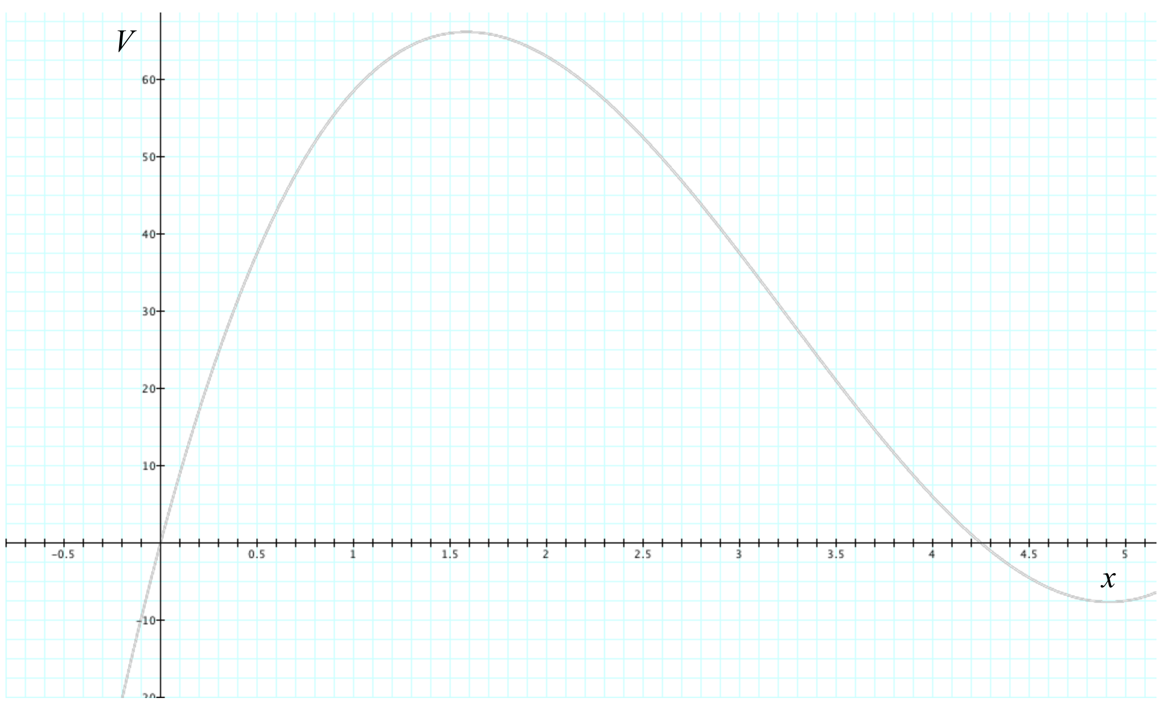
Table 8: Task 2 – The Box Problem

Starting with an 8.5" x 11" sheet of paper, a box is formed by cutting equal-sized squares from each corner of the paper and folding the sides up.

Task 2a: Describe how the length of the side of the cutout and the volume of the box covary.

Task 2b: Write a formula that predicts the volume of the box from the length of the side of the cutout.

Task 2c: Given the graph below, how does the volume change as the length of the side of the cutout varies from 1.8 inches to 1.9 inches?



Task 2d: Use a formula to determine how much the volume changes as the length of the side of the cutout varies from 1.8 inches to 1.9 inches.

In Excerpt 34, Bridget gives immediate insight into her covariational reasoning, as she demonstrates an ability to use Level 2 covariational reasoning when talking about the Box Problem. In lines 1 through 3, she describes corresponding changes in the quantities (MA1). Then in lines 4 through 10, Bridget describes the direction of change in volume as the length of the side of the cutout changes (MA2), and explains her reasoning.

Excerpt 34

Bridget: When it says, "Describe to me how the length of the side of the cutout changes – or co-varies with the volume of the box," I wrote that as the length of the cutout changes, so does the volume.

Bridget: I didn't say specifically that as the length of the side of the cutout increases, so does the volume because the volume only increases to a certain point, and then it decreases again.

Bridget: It has a peak. So I can't always say that as the length of the cutout increases so does the volume because once the length of the cutout's getting here (*sketches a graph of a function that increases to a peak and then decreases, and points to the peak*), the volume's decreasing.

In Excerpt 35, Bridget responds to the second part of the problem, in which she was asked to create a formula for the volume of the box based on the length of the side of the cutout. She had seen this problem, so her comments in lines 1 through 3 may not reflect anything more than a memory of the "correct" formula from class. However, in the remainder of Excerpt 35, Bridget is able to explain the meaning of each of the three factors in her formula, as well as the elements that comprise each of the factors. This indicates, at a minimum, awareness that the length of the box (for example) is somehow formed by removing two corners from the entire sheet of paper; her explanation suggests that she knew that the length of the box could be determined by subtracting two cutouts lengths from the length of the paper.

Excerpt 35

Bridget:	Okay, the volume equals um, length minus 2x where x is the length of the side of the cutout in inches, and the volume – V is volume – in cubic inches. Times width minus 2x, times height.
Int.:	What are these three factors? What do those have to do with the volume of a box?
Bridget:	Well, this is length of the box – you need to know the length times the width, and then the height to find the volume of a box.
Bridget:	And you say length minus 2x because we have a piece of paper and we cut out corners of it, and the length of the paper is changing as you cut out different size corners. So x represents the corner, so you have to subtract two corners to find the length, and you also have to subtract two corners to find the width.
Bridget:	And then height is gonna be whatever the length of the cutout is.
Int.:	So then is –what is that L?
Bridget:	L is the length of the whole paper.
Bridget:	And then x is the length of this little corner right there.
Bridget:	This is L minus 2x, and this is W minus 2x. And this is H or x.

Part c of this problem gives insights into Bridget's understanding of the information conveyed by the graph of a function, as well as additional insight into her covariational reasoning. In Excerpt 36, Bridget described how to use the graph to find the change in volume that corresponds to the length of the side of the cutout increasing from 1.8 inches to 1.9 inches. In lines 10 through 15 of the excerpt, she demonstrated an ability to find a given value (1.8, and then a second time using 1.9) on the horizontal axis, move up vertically until she hits the given graph, and then move horizontally to the vertical axis (the Volume axis) to find corresponding volumes (65 and 60, respectively). When questioned further about what a specific point on the graph means, Bridget indicated that it tells her a specific volume that corresponds to a specific cutout length. She also indicated that she is "a graph person", and made comments suggesting that she found graphs to be very helpful when solving problems.

Excerpt 36

Bridget: Okay, given a graph, how does the volume change as the length of the side of cutout varies from 1.8 to 1.9 inches? Well, this line represents the volume changing as the length of the cutout is changing. So given the picture of this graph, the volume is decreasing as the length of the side changes from 1.8 to 1.9.

Int.: How could you use that graph to say how much it's changing by?

Bridget: Um, you can plug this point into the equation and this point into the equation and find the difference.

Int.: Pretend you don't have the equation, but you just have the graph.

Bridget: Well, then I could find this point and this point and find the difference.

Int.: Okay, so you're just going over to the –

Bridget: Yeah, you just go to – the axis – Volume.

Bridget: Well, we'll say volume here is about 60, and volume here is about 65.

Bridget: So the change is about five cubic inches.

Part d of this problem asked Bridget to determine the same thing she determined in part c, but using the formula rather than the graph. Her response to this task is illustrated in Excerpt 37. She completes this task easily and correctly, and expresses no uncertainty or difficulty. Bridget demonstrates a confident understanding of the meaning of the letters in the formula she created, and a robust understanding of the meaning of her results relative to her actions and products from part c of this problem.

Excerpt 37

Bridget: Okay. Using the formula – okay. That's two times – oh, length. How – how big was this paper – 8.5 by 11?

Bridget: Hm. [*Uses calculator, plugging 1.8 into the formula*]. And this is V of 1.8. Sixty-four point two nine.

Bridget: [*Uses calculator, plugging 1.9 into the formula*]. Okay, so I was a little off on this one (*indicating her estimate of 60 using the graph in part c*), but the volume changes about one cubic inch. In order to find the exact point that V is at, you have to plug in 1.8 for x and 1.9 for x.

---

9 Int.: Okay. And then you found the change in volume how? In –

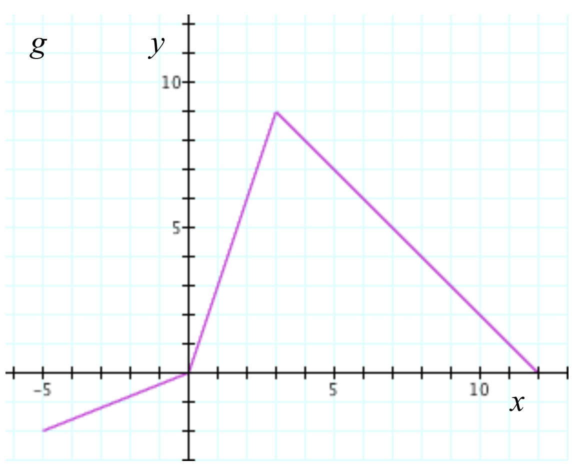
10 Bridget: Um, in relation to the two you just find the difference.

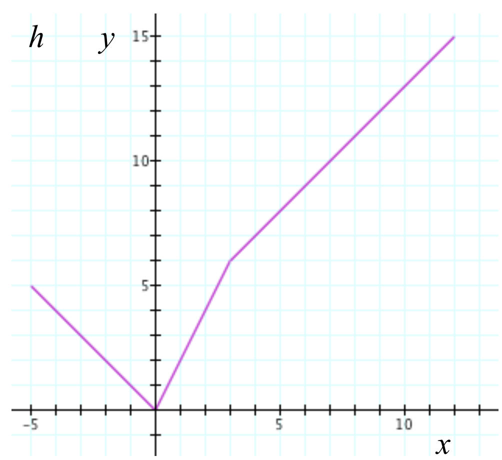
Bridget's solution to the graphical composition problem.

Bridget's solution to the Graphical Composition Problem allows further exploration of her understanding of graphs, function notation, function composition, and covariational reasoning, as well as her view of function. The problem consists of several tasks, and is illustrated in Table 9.

Table 9: Task 3 – The Graphical Composition Problem

Functions g and h are defined by the graphs below.





Task 3a: Determine each of the following:

i) $h(g(1))$

ii) $g(h(5))$

iii) $h(h(2))$

Task 3b: How does the output $h(g(x))$ vary as x varies from 5 to 9?

Task 3c: How does x vary as $h(x)$ varies from 10 to 15?

Table 9: Task 3 – The Graphical Composition Problem
Task 3d: How does x vary as $h(g(x))$ varies from 6 to 10?

In only nine lines of text (Excerpt 38), Bridget correctly completes all three subparts of part a. It is noteworthy that Bridget is comfortable using the output of one function as the input of another, or even as a subsequent input into the same function, when finding $h(h(2))$. This by itself is not indicative of a process view of function, but she clearly demonstrates an ability to interpret function notation and carry out the actions needed to determine the output from composing two function processes for a specific values of the input variable.

Excerpt 38

Bridget:	Okay. So the first one says H of G of 1.
Bridget:	So the first step is to find G of 1. So you go on the G graph, you find 1 on the X axis, and it equals 3 on the Y axis. So then 3 becomes the input for H and on the H graph, you find 3 on the X axis, which is 6 on the Y axis. So H of G of 1 is 6.
Bridget:	And then, G of H of 5. So H of 5 is 8. And so you put 8 into the G graph. And G of 8 is 4. So G of H of 5 is 4. And then H of H of 2, you find H of 2 on the graph, which is 4. And then you put 4 into the H graph. And so 4 is 7. H of H is 7.

Bridget's responses to parts b through d of this problem provides information about her ability to apply covariational reasoning further insights into how she is viewing a function in the context of this problem (excerpt 39).

Bridget approached part b by inputting several different values into the composed functions, to determine the corresponding output values. She does this successfully, but in the process, fails to coordinate the changing values of the quantities described in part b. To complete this task successfully, there are three pairs of quantities whose values Bridget must coordinate: x and $g(x)$, $g(x)$ and $h\big(g(x)\big)$, and then x and $h\big(g(x)\big)$. She must first consider function g, coordinating changes in the output, $g(x)$, with changes in the input, x. Second, she must consider function h, coordinating changes in the output of h, which is $h\big(g(x)\big)$, with changes in the input to h, which is $g(x)$. Finally, she must combine the results of her reasoning about functions g and h, coordinating changes in the composite function's overall output, $h\big(g(x)\big)$, with changes in the composite function's input, x. In the end, her answer only reflects a coordination of the values of $g(x)$ and $h\big(g(x)\big)$.

In lines 8 through 13 of Excerpt 39, Bridget demonstrates that she does not possess a robust understanding of the "reversibility" of functions that would characterize someone possessing a process view of function. Weaknesses in her covariational reasoning abilities were revealed by the fact that she was unable to complete either part c or part d, despite having previously demonstrated an understanding of function notation.

Excerpt 39

1	Bridget:	How does the output H of G of X vary as X varies from 5 to 9? Okay.

G of 5. So I'm just going to take a few samples and see what they are. G of 5 is 7. G of 6 is 6. G of 8 is 4. Okay. And then H of 7 is 10. H of 6 is 9 and G of 8 – or no, H of 4 is 7. So as G of X decreases, H of G of X decreases.

Int.: Okay. From what to what?

Bridget: From – well, I guess from 10 to 7.

Bridget: Okay. How does X vary as H of X varies from 10 to 15? How does X vary as H of X varies? Hm. X goes from 10 to 15?

Bridget: I don't know. It seems too easy or like I'm missing something.

Int.: Okay. So it seems like X and H of X are the same thing to you, or what?

Bridget: Yeah. When there's not another graph involved then X is H of X.

Bridget's solution to the salary problem.

Before turning to Bridget's solutions to function composition problems, I present her solution to the Salary Problem. This problem is illustrated in Table 10.

Table 10: Task 4 – The Salary Problem

Using function notation, suppose $S(m)$ represents the monthly salary, in hundreds of dollars, of an employee after m months on the job. What would the function determined by $R(m) = S(m + 12)$ represent?

a) the salary of an employee after $m + 12$ months on the job
b) the salary of an employee after 12 months on the job
c) \$12 more than the salary of someone who has worked for m months
d) an employee who has worked for $m + 12$ months
e) The salary after m months is the same as the salary after $m + 12$ months.

Select excerpts from Bridget's explanation of her solution to the Salary Problem are presented in Excerpt 40. In line 2, she comments that she doesn't know what to think of the "12" given in the problem statement. She recognizes that whatever is in parentheses should be an amount of months (line 3), but she has difficulty conceptualizing m+12 as a suitable input quantity. Based on the position of m+12 inside the parentheses, she decides the correct answer should have something to do with an employee who has worked m+12 months (line 5), which allows her to reject choices b and c. However, until prompted by the interviewer, Bridget does not notice that both choices a and d refer to an employee who has worked for m+12 months. Once prompted, Bridget is able to read both choices and quickly choose the correct one, based on her understanding that the output of the function S is a salary, not an employee.

Bridget's solution to this problem revealed weaknesses in her ability to initially conceive of m+12 as a quantity. Her responses further suggest that she had difficulty conceptualizing an argument as an input to a function. It is noteworthy that she briefly considered (in line 3 and 4) that m might represent a salary.

Excerpt 40

Bridget: Hmm, I don't like the wording on this. I feel like there's not enough information. I don't know what to think of 12. I don't know if that's an amount of months because it's in the parentheses and M represents the monthly salary, or S then represents monthly salary and M is the months on the job…. I'm thinking of D. They've worked M plus 12 months.

Bridget: And I think 12 is representing like an additional amount of months or something. But I don't know why they would do that if they would just add that to the M months.

Int.: All right. Can – can you tell me the difference between D and A? Notice those both talk about m plus 12 months.

Bridget: (*Reading aloud*) Salary and employee salary is... An employee… oh.

Bridget: Um, A is the salary of the employee after M plus 12 months and D is just the employee after M plus 12 months, so it has to be A is what I'm thinking.

Bridget's solution to the circle problem.

Having examined Bridget's quantitative and covariational reasoning and problem-solving behaviors, I turn to her understanding and use of function composition. I examine her actions when attempting to solve two novel function composition word problems, and note manifestations of the factors explored in earlier sections. In the next chapter, I re-examine my conjectured framework of mental actions, in light of Bridget's solutions to these problem. I begin exploring Bridget's understanding and use of function composition by presenting Bridget's solution to Task 5 (Table 11).

Table 11: Task 5 – The Circle Problem
Express the circumference of a circle as a function of the area of the circle.

Much like Patricia had done, Bridget began by immediately trying to recall previously learned formulas for area and circumference. This required some assistance, as seen in Excerpt 41. Eventually, Bridget wrote correct formulas for circumference and for area, both of which were in terms of the radius of the circle: $C = 2\pi r$ and $A = \pi r^2$.

Excerpt 41

Bridget:	The circumference of a circle as a function of the area of a circle. [*Pauses*] It would be helpful if I could remember the formulas. Circumference… What's the circumference of a circle? I forgot.
Int.:	What IS it, you mean, or what is the formula?
Bridget:	The formula. Like 2 pi something.
Int.:	Well, draw something and show me what it is first.
Bridget:	That's the circumference.
Int.:	I see the circle. What about the circle?
Bridget:	The actual length of this line [*tracing around the outside of the circle*].
Int.:	Length of that line, okay, is the circumference. Yeah. And you think it's like 2 pi something?
Bridget:	It has something to do with diameter. I know it has pi, maybe a square.
Int.:	How about "2-pi times the radius"? Would that help? The circumference is 2-pi r.
Bridget:	But it's not 2-pi r squared, because that was kind of in my head for

some reason. I don’t know. 2 pi r. Yeah.

Int.: 2-pi r is the circumference. Okay.

Bridget: Um hm. And this is the radius, which is what r is.

Bridget: And the area, maybe that’s 2-pi r squared. Is the area 2-pi r squared?

Int.: It’s actually pi r squared.

Bridget: Area equals pi r squared. When you don’t use it you forget. Okay. So express the circumference of a circle as a function of the area. So this is where I always get confused if I’m supposed to input C into A or A into C.

Int.: That’s what I was going to ask you next.

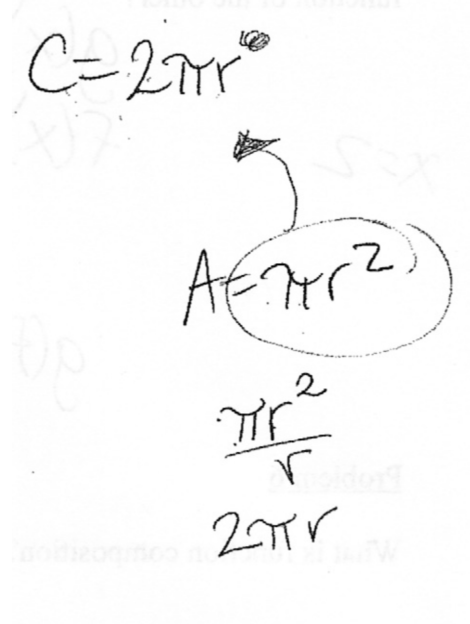
Bridget: Yeah. That’s where I always get flip-flopped.

Int.: With this wording, “function of the area” means area is the input for this function. And then circumference is the output.

Bridget: Okay. So it’s like C of A.

Armed with correct formulas, Bridget next started thinking about how she might “combine” the formulas. In line 31 of Excerpt 41, she had set a goal of finding a formula of the form, $C(A)$, or a function that takes area as an input and produces circumference as an output. Her subsequent work is shown in Figure 2, which she discussed in Excerpt 42.

Figure 2: Bridget’s Formulas for the Circle Problem



Excerpt 42

1	Bridget:	So if C is 2-pi r, and A is pi r squared, then how would I combine
2		those two?
3	Int.:	Why do you want to combine those two?
4	Bridget:	Because I have to find circumference using area. And if area’s pi r
5		squared… I don’t know. [*Pauses*] Um. I know I’ve done this before. I
6		just can’t remember how to do it. So you want to find – I know what
7		I’m looking for. Like I want to find the circumference given the area.
8		But as far as actually finding it goes. It’s probably easy and I just
9		can’t see it.
10	Bridget:	Well, r squared. Well, I don’t know. The thing that I’m kind of
11		thinking but I think that I’m wrong, is like trying to convert this to
12		that. So I could turn this into pi r by like diving by r and getting pi r

and then multiplying by two and then I have circumference.

Bridget: That's the way that I'm thinking of it. But I'm thinking I'm making it too complicated, that it's easier than that. I know there's a way to do it, I just don't know.

Bridget was confident that she needed to somehow "combine" her two formulas, as evidenced by her use of the word in line 1 of Excerpt 42, and circling part of the area formula and drawing an arrow toward the circumference formula. However, she was unable to find a way to combine the formulas that satisfied her. She had previously demonstrated an ability to perform substitutions of one formula into another, but the difference in this case was the requirement that she solve her second formula for radius in terms of area, which someone with a more robust understanding of function might describe as "inverting" the function. In lines 10 through 13, she discusses the possibility of making the two formulas more compatible by dividing the area formula by r and then multiplying by two, but she decides in lines 14 through 16 that this approach would be "making it too complicated", and she subsequently gets stuck.

Bridget remained unable to progress any further, so I chose to explore whether giving her a specific input value might help her progress toward a solution. In Excerpt 43, Bridget discusses an example in which the area is given as "ten".

Excerpt 43

Int.: If I told you the area was ten, can you use either of your formulas to find out anything else?

Bridget: Well, that’s the reason I’m having a hard time visualizing this because I can’t just input ten in for r, you know.

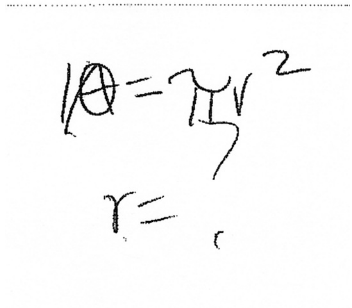
Bridget: I wonder if I could input… no, that wouldn’t work either. If area equals ten. Maybe I would solve for r and then input r into here. So it would be like ten equals pi r square and then I’d solve for r and the r equals whatever it is. And then I’d put it into there and find the circumference. That’s what I would do.

Int.: Okay. What about in terms of formulas. If you don’t know that A is ten - it’s just “A”?

Bridget: Then I would solve for r with A here instead and input them into the – I don’t know. If I didn’t know the area, and area’s the input, then how would I know? [*Pauses*] Then I would write not enough information given.

As evidenced by lines 3 and 4 of Excerpt 43, Bridget is still thinking of both of her formulas as having radius as the input. She discusses inputting 10 for the radius, but knows that can’t be right. Bridget then writes her area formula, and then replaces *A* with 10, as shown in Figure 3.

Figure 3: Bridget's Formulas for a Circle with Area = 10



This proves helpful to Bridget, as she is now looking at a formula with only one unknown. Consequently, in lines 6 through 9 of Excerpt 43, Bridget describes a correct solution to the problem for the case in which $A = 10$. However, in lines 12 through 15, she is unable to generalize and write a formula that answers the original problem, and again is unable to progress further. This strongly suggests that her success in solving the problem when the area was given as 10 is not the result of a new, robust understanding of the invertibility of functions. Rather, her success is attributable to the notion that when looking at $10 = \pi r^2$, solving for the single unknown was the only action apparent to Bridget. Having done so, she was then able to calculate the circumference.

In terms of the conjectured mental actions, Bridget had difficulty conceiving of the "reversibility" of one or both functions. Like most students, Bridget had memorized the formula for the relationship between the area of a circle and the radius of that circle as $A = \pi r^2$, and not as $r = +\sqrt{A/\pi}$, although both are equally correct. This left Bridget with two formulas, both of which had radius as the input. Her view of function, which was demonstrated earlier to be not yet a process view, had the result that she was unable to create the inverse of her area function, which left her ultimately unable to complete the problem.

Bridget's solution to the giraffe pen problem.

As a second example of Bridget's understanding and use of function composition when solving novel problems, consider Task 6, the Giraffe Pen Problem, as shown in Table 12.

Table 12: Task 6 – The Giraffe Pen Problem
Bryan has decided to open a wildlife park, and he would like to build a square pen for some giraffes. Bryan hasn't decided how many giraffes to acquire, but he has been told he should allow 10,000 square feet for each giraffe to graze. Help Bryan design his park – find a way to determine how many feet of fencing he must buy to enclose his square pen, based on the number of giraffes that will live in the pen.

In Excerpt 44, Bridget reads the problem aloud, describes her understanding of the problem, sketches a square, and states a solution based on her understanding. She initially voices two misconceptions about that problem – that the pen is to be 10,000 square feet in area regardless of the number of giraffes acquired (lines 6 and 7), and then that each giraffe will get its own pen (line 14). This does not seem to have been the result of difficulties in reading the problem or creating a mental image; rather, this was the last problem in that particular interview, and Bridget was tired and anxious to be done for the day, and failed to read carefully. Whatever the source of her misconceptions, by the end of Excerpt 44 Bridget possessed an understanding of the problem situation that was compatible with my own. Her incorrect (and so far unjustified) arithmetic in lines 6 through 11 will be explored further in Excerpt 45.

Excerpt 44

Bridget: Help Brian decide how to design the park and determine how many feet of fencing he must buy to enclose his square pen based on the number of giraffes that will live in the pen. But he doesn't know how many giraffes.

Int.: He hasn't decided how many giraffes, no.

Bridget: That doesn't have anything to do with it because he's having a 10,000 square feet thing despite how many giraffes he has, so he needs 4,000 square feet of fence. Because he said he's building a square pen. So that leaves out the possibility that the, the width and the length are different. So that – 10,000 square feet's the area, so that means these (*indicating the length of each side of the pen*) are each 1,000 – 4,000.

Int.: Okay. Um. Except, he wanted 10,000 square feet *per giraffe*.

Bridget: For each giraffe, oh. Oh, okay. So he needs 4,000 feet of pen per giraffe. [*Pauses*] If they each get their own pen.

Int.: They don't each get their own pen, though. He wants to put them all in one pen.

In Excerpt 45, Bridget explains how she decided 4000 feet of fencing would be needed. She was aware that the length and width of a square are equal, and the area of the square was equal to the length times the width. Line 2 shows that Bridget's incorrect answer was the result of faulty mental calculations, believing that 1,000 times 1,000 equals 10,000. She had previously demonstrated (lines 10 and 11 of Excerpt 44) that she wanted to find the perimeter of the pen, and would do so by adding the lengths of the four sides. As a result, once her faulty arithmetic was corrected, she quickly gave the correct answer for a pen designed for one giraffe (lines 4 through 7 of Excerpt 45).

Excerpt 45

1	Int.:	Let me back up a bit here. The 1,000 comes from…?
2	Bridget:	One thousand times 1,000 equals 10,000.
3	Int.:	No it doesn't. One hundred times 100 does.
4	Bridget:	One hundred – yeah. So 400 (*indicating perimeter of pen*).
5	Int.:	All right. So if he had, um, one giraffe, you say he'd need 400 feet of
6		fence. And why do you say 400?
7	Bridget:	Because that's the perimeter.

Excerpt 46 is fairly long, but it gives a coherent description of Bridget's complete solution to the original problem. Prompted to consider the case in which two giraffes are acquired, Bridget quickly solves the problem with her calculator (lines 3 through 7), and then describes (lines 7 through 10) what she did, and what she would do for other numbers of giraffes. It is noteworthy that Bridget did all of this without writing any formulas, and indeed without writing anything except "20000". This suggests that her reasoning was not based on manipulating letters and numbers on a page, but was based on a conceptualization of the quantitative relations between area and length of a side, and between length of a side and perimeter.

Excerpt 46

1	Int.:	Okay. Well, what if he decides he wants two giraffes? What's his pen
2		going to look like then?
3	Bridget:	Oh, yeah, because it has to be square. So then he's having a 20,000

square feet pen, right? It's 10,000 per giraffe, so…um…what's the square root of that (*indicating 20,000, and referring to calculator*)? One hundred forty one something. So that means…in that case he would need 565.69 feet of fence. You just take – based on how many giraffes he has, you find out how many square feet. Then you take the square root of that to find the length of each side, and then multiply by four because that's the perimeter.

Int.: Okay. I know Brian didn't ask you for a formula and all, but so that he doesn't have to call you and keep asking questions, do you think you could help him figure out a formula?

Bridget: If X equals number of giraffes, four times the square root of 10,000 times X.

Int.: Oh, okay. Can you break that down for me one more time?

Bridget: This is, F is the length of fence you need.

Int.: Okay.

Bridget: X is the number of giraffes, times the amount you need per giraffe.

Bridget: Then you find the square root of that to find this dimension (*indicating the length of one side of the pen*), then multiply it by four to find the perimeter.

After being prompted to consider a formula for this situation (lines 11 through 13), Bridget writes $F = 4\sqrt{1000g}$, and is able to give coherent explanations of the meaning of g, $1000g$, $\sqrt{1000g}$, and $4\sqrt{1000g}$. Taken together, this suggests robust quantification and quantitative reasoning, and a robust understanding of the use of variables to represent the values of those quantities.

Summary Characterization of Bridget

Bridget was sometimes able to understand and use function composition to build new functions in applied contexts. This is seen, for example, in Excerpt 46. However, in other contexts, such as the Circle Problem, she was unable to successfully build a function linking two quantities that were difficult to link directly (see Excerpt 43). Several factors were revealed in this study as either facilitating or complicating her ability to make progress on the function composition problems, as described in the following paragraphs.

Action view of function, in transition to process view.

Bridget's view of function could best be characterized as an action view, probably in transition to a process view (Dubinsky and Harel, 1992). She does not possess a robust understanding of the reversibility of functions, as was manifest in her solutions to the Circle Problem (see Excerpt 43) and the later parts of the Graphical Composition Problem (see Excerpt 39).

Bridget possesses a strong understanding of function notation and different function representations in most situations. For example, this strength is indicated in her solution to the first part of the Graphical Composition Problem, in which she evaluates functions, and coordinates input and output values with ease. However, she does display some fragility later in the same problem, as she is faced with a need to consider the inverse of the functions in the graphs; this fragility is likely compounded by (and at least in part the result of) possessing an action view of function. On numerous occasions, Bridget showed that she was very strong at using and understanding graphs. She described herself as a "graph person", and stated that she found graphs to be very helpful when solving many problems.

Consistent with the characterization of Bridget as having an action view of function, when she encounters a novel problem, Bridget tends to immediately search for one or more "correct" formulas or procedures to use. This can be seen, for example, in her solution to the Circle Problem (see Excerpts 41 and 42). In situations where her task requires linking two quantities that cannot be linked directly, such as unfamiliar function composition problems, Bridget's action view of function impedes her ability to do so.

Robust ability to conceive quantities in problem situations.

Bridget demonstrated strong quantitative reasoning in most situations. For example, when solving the Giraffe Pen problem, Bridget solved the two-giraffe case with no formulas (Excerpt 46), suggesting that she possessed a robust understanding of the quantitative relationships involved in this problem. Similar success can be seen in part of her solution to the Circle Problem, in Excerpt 43. However, she did display some fragility when solving the Salary Problem, as she found it difficult to conceive of $m + 12$ as a suitable input quantity for the functions being described (Excerpt 40).

Positive affective factors.

Bridget exhibited considerable self-confidence when solving many problems. She clearly thought of herself as someone who could "do math", and expected to succeed when attempting to solve problems. She tended to proceed confidently through her solutions to most problems, accepting setbacks as part of the problem solving process without becoming visibly distressed.

Throughout her interviews, Bridget demonstrated a strong ability to assess the "reasonability" of potential solutions. She assessed her results by comparing them to other values within the problem situation, and to what she knows about the "real world" when solving problems that appeared to her to be based on real world contexts. This was a key behavior that helped her successfully solve problems for which she would otherwise have obtained incorrect answers. This can be seen clearly, for example, in her solution to the Dinner Problem, where she rejected an intermediate result that would have required Matt to drive 180 miles per hour to reach the restaurant (Excerpt 31).

The next chapter will include further discussion of the results presented above, concerning Bridget's knowledge and use of function composition. For now, I turn to a discussion of the responses of my third subject, Karen, on select tasks.

Karen

This section presents findings from the clinical interviews that reveal insights about Karen's understandings, reasoning abilities, and problem solving behaviors, initially in the context of the clinical interview tasks. I follow this by discussion how these abilities influence her response to novel function composition tasks.

Karen's solution to the dinner problem.

To better understand Karen's quantitative reasoning and her approach to problem solving, I present selected episodes from her solution to the Dinner Problem (Table 13).

Table 13: Task 1 – The Dinner Problem
Two friends that live 42 miles apart decide to meet for dinner at a location half way between them. The first friend, Tom, leaves his house at 6:05 and drives an average speed of 34 miles per hour on his way to the restaurant. Task 1a: If the second friend, Matt, leaves at 6:10, what average speed will he need to travel to arrive at the same time as Tom?

Table 13: Task 1 – The Dinner Problem
Task 1b: If before leaving Matt knows that he averages driving 15 miles per hour to the restaurant, what time would he have to leave to arrive at the same time as Tom?

Karen demonstrates an understanding of some aspects of the problem in Excerpt 47, but the most interesting aspect of this excerpt is what is revealed about Karen's problem solving abilities. In this brief excerpt, she states, "I don't know" six times, and "I don't think this is right" three times. This weak confidence was exhibited repeatedly by Karen during the interviews.

Excerpt 47

Karen:	So Matt left five minutes after Tom, and they're meeting halfway, but Tom's going 34 miles an hour. Um, I don't know if I – I don't know. I don't know if I should do – 'cause what I, what I'm thinking about doing is doing a, doing what we did in class, but I don't know 'cause usually I write time on the bottom. But I don't think that, I don't think this is right 'cause that's just miles, and this is, like, the average, so I don't think I'm doing this right.
Int.:	Okay. Well, what are you thinking here?
Karen:	Um, well, this is gonna get me – I don't know. This doesn't – I don't know.

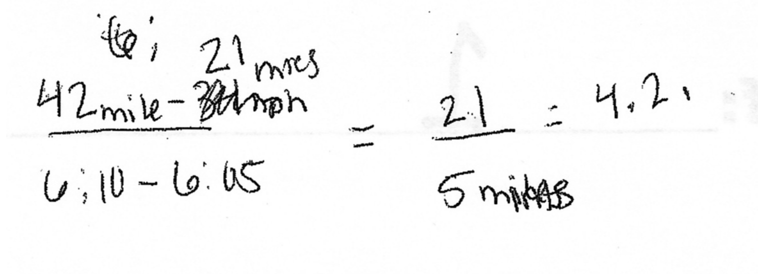
At this point, Karen was ready to give up, so I decided to probe her understandings of the problem, and to see if this would help her progress toward a solution. A portion of this conversation is shown in Excerpt 48, with comments to follow.

Excerpt 48

Int.:	What do you see the problem as asking you? What’s your goal here?
Karen:	Because, um, well, I guess I could do – I don’t know. ‘Cause this is, this is just the average speed that he’s going, so that doesn’t really have to do with – like it’s asking me what is average speed that Matt will need.
Karen:	So but they live 42 miles apart, and they’re meeting halfway, so I don’t know if I have to, if half, half of 42 is 21, so I don’t know – I don’t know if I do this since it’s, since they’re meeting halfway, but –
Karen:	Well, it’s still 21, so I won’t… [*Pauses*]
Int.:	Okay. Well, what does that 21 represent to you?
Karen:	Um, 21 is half the, half the distance since they’re meeting halfway. ‘Cause they both live 42 miles apart, but what this means to me is that in – oh, wait, this is minutes. I don’t know because it’s saying in five minutes he’ll go 21 miles. But if I divided that, it’s 4.2 so 4.2 is the average speed, but that doesn’t that doesn’t make sense to me, so I know I’m missing a part. [*Pauses*]

Note that in Excerpt 48, there is no evidence that Karen possesses an understanding of the relationships among distance, speed, and elapsed time. She recalls a formula – average speed equals change in distance divided by change in time, but isn’t sure what relevance that might have to the problem. She was given two times – 6:05 and 6:10 – so she decides to plug those into her formula, and since the problem states Tom and Matt are meeting "halfway", she decides to plug 42 miles and 21 miles into her formula as well, as shown in Figure 4.

Figure 4: Karen's Solution to the Dinner Problem



Karen's focus throughout Excerpt 48 is on calculations to perform. She feels a need to calculate *something*, even when she is unable to make sense of what she is calculating, what she wants to be calculating, or what she should be calculating. After performing her calculations, she realizes that she is stuck, and again is ready to stop working on the problem.

In Excerpt 49, I made a final attempt to probe Karen's understanding, and perhaps help her progress toward a solution. In lines 6 through 8, Karen is able to articulate what the problem is asking. However, in lines 15 through 18 she describes *average speed* as meaning *his speed, on average* – a circular definition, not a particularly useful definition, and a definition that evidences a lack of understanding of average speed.

Excerpt 49

Int.:	Okay. Um, can you tell me anything about the, the time that they want to arrive or, or the time they're going to arrive or…
Karen:	Oh. It doesn't – I don't know. It doesn't really say. Well, what is the average speed he will need to travel… at the same time as Tom, but I don't… is that what it's asking, what time he's gonna arrive or I

thought it was just saying what is the average speed that he will need to go to arrive at the same time as Tom since Tom left five minutes earlier. [*Pauses*]

Int.: Well, the second thing you said – exactly that. How fast will Matt have to go to get there at the same time that Tom gets there since Tom left five minutes earlier? So, for Tom, we do have an average speed.

Karen: Uh huh.

Int.: What does that mean to you, his, his average speed?

Karen: His average speed means that on, on average he’s going 34 miles an hour even though he could be slowing down or speeding up at a given time. But on average, from leaving his house to the restaurant, his average speed was 34.

Int.: Okay. And do you, do you know how long it takes him to get there? Or if not, can you figure out how long it takes him to get there?

Karen: I’m sure I can. I just don’t know what to do for his… [*Pauses*] Um, okay. So I don’t really know what to do.

Karen was ultimately unsuccessful in making much progress toward a solution of the Dinner Problem. Karen's responses reveal how problematic a calculationally oriented approach can be when attempting to respond to a problem that requires conceptualizing, relating and combining quantities. Her responses also allow insights about the mental pictures she constructed when attempting to construct a meaningful formula.

Karen's solution to the box problem.

To better understand Karen's reasoning and understandings, I present episodes from her solution to the Box Problem (Table 14). It is important to note that Karen had encountered the Box Problem previously, as the focus of an in-class discussion. This realization is essential when analyzing some of Karen's comments and behaviors when completing the tasks related to the Box Problem.

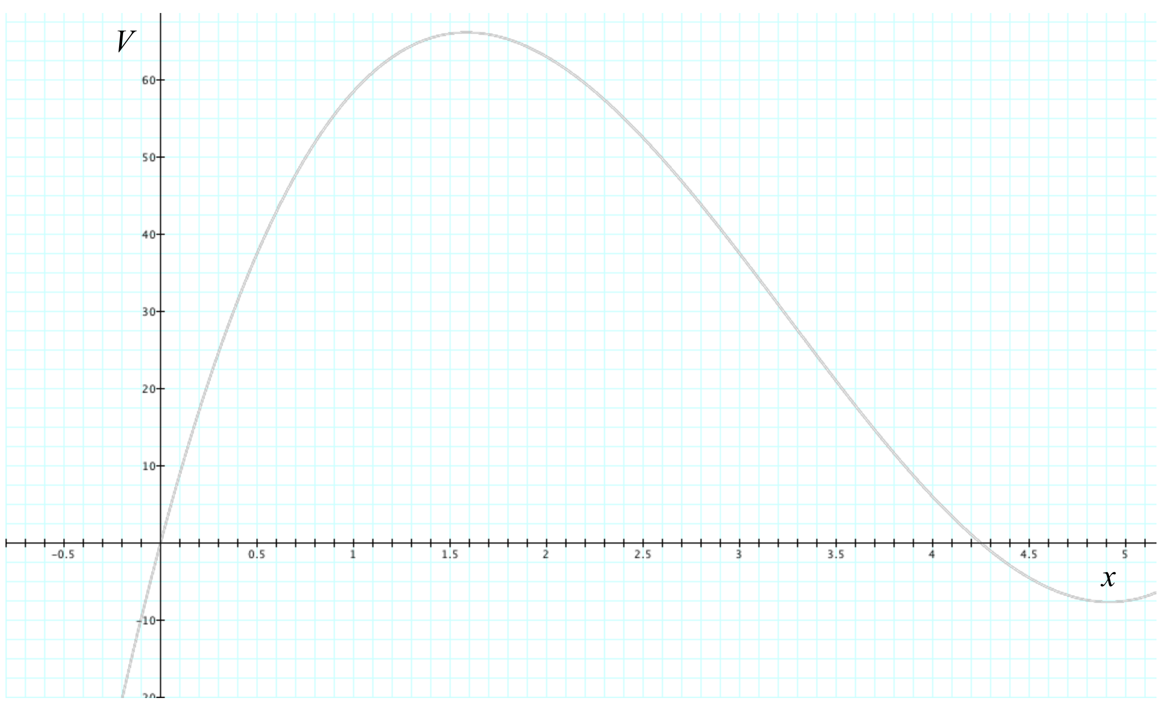
Table 14: Task 2 – The Box Problem
Starting with an 8.5" x 11" sheet of paper, a box is formed by cutting equal-sized squares from each corner of the paper and folding the sides up. Task 2a: Describe how the length of the side of the cutout and the volume of the box covary. Task 2b: Write a formula that predicts the volume of the box from the length of the side of the cutout. Task 2c: Given the graph below, how does the volume change as the length of the side of the cutout varies from 1.8 inches to 1.9 inches? 

Table 14: Task 2 – The Box Problem
Task 2d: Use a formula to determine how much the volume changes as the length of the side of the cutout varies from 1.8 inches to 1.9 inches.

In Excerpt 50, Karen describes how the length of the side of the cutout and the volume of the box covary. Making statements such as “the box is gonna get smaller, which makes the volume bigger” (lines 4-5), Karen seems to have failed to construct a helpful mental image of quantities in the box and how they change together. This is evident by her assertion in lines 5 and 6 that eventually the box will be “a flat piece of paper again”, rather than describing a situation in which the box gets taller and narrower, becoming a *folded* piece of paper in the end.

Excerpt 50

Karen:	Okay. Um, they co-vary because as you cut, um, squares out of the sides of the box, the volume’s gonna eventually get bigger, and then it’s gonna, it’s gonna increase and then it’s gonna decrease because as you’re cutting, the box is gonna get smaller, which makes the volume bigger. But eventually, you’re gonna cut to where it’s not a box anymore, and it’s gonna be, like, a flat piece of paper again.
Int.:	Okay. And what’s the volume gonna be at that point?
Karen:	Zero.

Few conclusions can be drawn from Excerpt 50 regarding Karen’s covariational reasoning. This is because her response to the task seems to be based on remembering words and phrases from the in-class discussion, rather than being supported by a useful mental image allowing meaningful quantitative reasoning.

The second part of this problem asked Karen to create a formula that predicts the volume of the box from the length of the side of the cutout. The formula she created is shown in Figure 5, and is correct.

Figure 5: Karen's Formula for the Box Problem

$$V = \underset{\text{length}}{(11-2x)}\,\underset{\text{width}}{(8.5-2x)}\,\underset{\text{height}}{x}$$

While the formula is correct, consider Karen's attempts to describe this formula, as illustrated in Excerpt 51. Her descriptions suggest a weak mental image of the problem, and an inability to engage in helpful quantitative reasoning using quantifiable attributes of the situation. In lines 2-4, she explains the reason she needs the "2" coefficients in her formula as being because "there's two lengths and two widths, and you have to compensate both of those in the formula". This suggests that she has likely written the formula she remembers from class as being "correct", without having an underlying understanding of why the formula is correct.

This lack of understanding is exhibited throughout the excerpt. In lines 21 and 22, she describes the three factors in her formula as being the length, width, and height "of the paper", rather than the dimensions of the box. Lastly, she describes her variable, *x*, as being "equal to the cutout", but is unable to articulate what if anything, about the cutout is represented by *x*.

Excerpt 51

Karen: A formula that predicts the volume of the box. [*Writes formula*] Oh, okay. Um, I did this because, um, you have to take – there's two lengths and two widths, and you have to compensate both of those in the formula in order to get the volume.

Int.: Okay. All right. And, so this represents the volume then?

Karen: Yeah.

Int.: Okay. And, um, so then can, can you explain to me what each of these three factors means –

Karen: Uh huh. This is the, this is the length, um, and there's two sides to the length, so when you, whatever, however I'm, I'll just say centimeters – if it's with, if it's by centimeters, you can plug however many, however many centimeters that you're cutting and that'll give you your length. This is the width, and it's the same thing 'cause there's two sides to the width, so whatever centimeters you'll be plugging in here will give you the centimeters that you're cutting out.

Karen: This is just, um, I guess, um, the centimeters, like, you're cutting, but, um, this is for the height, the height of the box that you will be cutting.

Int.: So the, and so then the length and width you're talking about here, is, are these the, the length of the paper, or the length of the box?

Karen: The length of the paper. So it's length times width times height. Of the paper.

Int.: Of the paper, okay. And, um –

Karen: And X is equal to the cut out.

Int.: Any certain thing about the cutout?

Karen: Um, um, amount? – I don’t know. Amount or centimeters or inches. I don’t know.

In Excerpt 52, Karen completes the third part of the problem, which asks her to use a given graph to describe how the volume of the box changes as the length of the cutout increases from 1.8 inches to 1.9 inches. Note that Karen is able to quickly and correctly complete this portion of the problem, despite the weak mental imagery and quantitative reasoning she demonstrated earlier in the problem. The reason she is able to complete this portion is because she remembers the correct procedure to follow. Her understanding of the meaning of the graph itself is explored in the next excerpt.

This notion of “remembering the correct procedure to follow” is important, and is a recurring theme throughout Karen’s interviews. When she believes she can follow a procedure that she already knows, she completes tasks quickly, confidently, and usually correctly. When faced with less procedural problems, such as many novel word problems, she is often reluctant to take time to think about the quantities in the problem, and unable to make much progress toward a solution.

Excerpt 52

Karen: Okay. So from 1.8, so from – basically it’s saying how does the

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volume change as the side changes from 1.8 inches to 1.9. Um, well, at 1.8 inches, um, the volume of the box is higher than at 1.9.

Int.: Okay. And how do you know that there’s higher volume?

Karen: Um, the 1.8 has an increased volume because the graph, the graph goes up and then down. Like right, like around, around 1.5 it looks like it’s max volume I think.

Int.: So if you wanted to know how much the volume was for a cutout of 1.8, how would, how do you use the graph to find it?

Karen: You would look, you would trace it this way, um, so I’m, it’s probably like over here somewhere.

Int.: Okay. So you just went over horizontally from a point on the graph.

Karen: Yes.

Curious about the reason for Karen’s success in completing the third part of this problem, I asked her to explain what a point on the graph represents. The results, shown in Excerpt 53, suggest that her understanding of graphs is strong, despite her not having conceptualized the quantities in the situation. She demonstrated how she would move vertically down from the point on the graph to determine the value of x that corresponded to that point, and horizontally to the left from the point on the graph to determine the value of V that corresponded to that point, explaining that “at 3.4 inches, the volume is 25 centimeters cubed” (lines 11-12). Note that she does not explain the meaning of these values in the context of the problem, but she does use the graph correctly.

Excerpt 53

1	Int.:	All right. So, um, yeah, if we have some point on the graph, I can just
2		make a dot out here somewhere. Um, like, what, what's a point on
3		that graph mean to you? What's that represent?
4	Karen:	It means that this is the, I guess it's not, oh, it is labeled, okay. Well,
5		this is the volume, which has a V here, um, this is the X, this is X,
6		which is cutout, which I was saying on the other.
7	Int.:	Uh huh.
8	Karen:	So this would – I'll just draw a line.
9	Int.:	Okay.
10	Karen:	Well, okay, I'll make it even, I'll say 3.4. So if it's 3.4 inch – like the
11		cutout is 3.4, you would follow it over to 25. So at 3.4 inches, the
12		volume is 25 centimeters cubed.

The final portion of the Box Problem asked Karen to repeat the previous task, but using her volume formula instead of the graph. Karen's solution to this portion of the problem is shown in Excerpt 54. She once again quickly and correctly completes this task, despite her weak mental image of the problem situation. Her success can again be attributed to knowing what procedure to follow. The task directs her to "use a formula", and she only has one formula on her paper, so she uses it. She does engage in one notable helpful problem solving behavior, as she compares her results using the formula and using the graph (lines 10-12).

Again note that few conclusions can be drawn regarding Karen's covariational reasoning from this excerpt; while her comments in lines 10-12 hint at Level 2 covariational reasoning, her weak mental imagery makes it unclear what quantities she could have mentally constructed to covary.

Excerpt 54

Karen:	Okay. Use a formula to determine how much the volume changes. So I would use the formula that I created. For both of them.
Int.:	And, um, how do you know that's the formula you wanna use?
Karen:	Um, because it's asking how much the volume is in terms of the length, and that's the volume that I use. I used the volume in terms of the length. [*Pauses to use calculator*]
Karen:	Okay. So it's 65.26 centimeters. So when the cutout is 1.8, this is the volume and then I would do the same thing for 1.9 to see what the volume is. [*Pauses to use calculator*] The volume is 64.29, when the cutout is 1.9, so after doing the formula, the, the graph and the, and the answers kind of correspond because, um, at 1.8 it's showing that it's higher than 1.9.
Int.:	Okay.
Karen:	Not by much, which is true.
Int.:	Okay. All right. So then to find the amount of change between 1.8 and 1.9, what would you do with –
Karen:	Um, to find the difference, or the amount of change, I would subtract

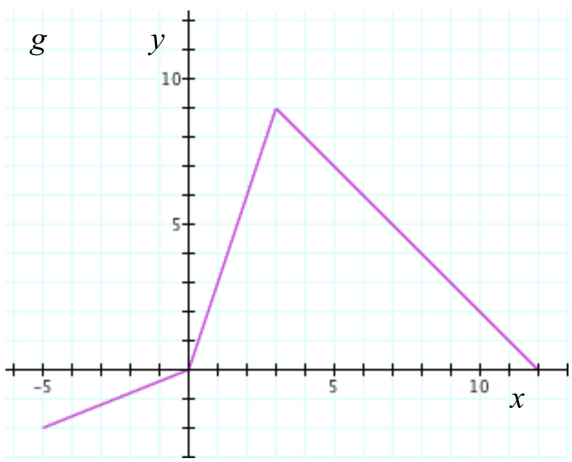
18	– then it's just .7, so the difference in volume is .7.

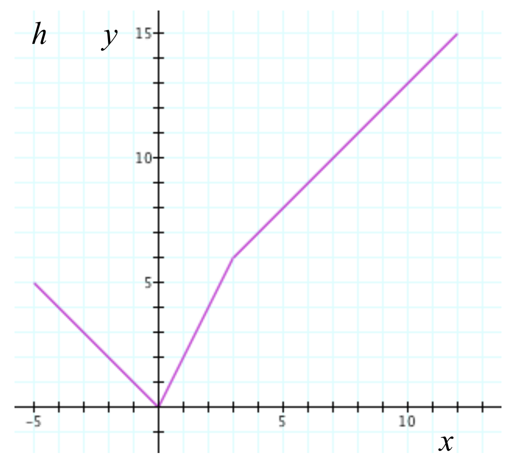
Karen's solution to the graphical composition problem.

To better understand Karen's covariational reasoning, view of function, and understanding of common function notation, I present select excerpts from her solution and interviews on the Graphical Composition Problem (Table 15).

Table 15: Task 3 – The Graphical Composition Problem

Functions g and h are defined by the graphs below.





Task 3a: Determine each of the following:

i) $h(g(1))$

ii) $g(h(5))$

iii) $h(h(2))$

Task 3b: How does the output $h(g(x))$ vary as x varies from 5 to 9?

Task 3c: How does x vary as $h(x)$ varies from 10 to 15?

Task 3d: How does x vary as $h(g(x))$ varies from 6 to 10?

The first question asks Karen to interpret commonly used function notation and evaluate two composed functions at three specific input values (rather than over an interval of input values). Her solution is described in Excerpt 55. Note that for each of the three cases she is asked to evaluate, Karen performs the evaluations quickly, correctly, and confidently (although she expresses a little uncertainty about the third case). Note that when solving this portion of the problem, Karen has a set procedure that she knows she must follow. The problem does not require her to conceptualize the quantities and their relationships; now are they expected to engage in quantitative reasoning, or covariational reasoning. It appears that she is a good memorizer and relies on the notation to help her know how to respond.

Excerpt 55

Karen:	Um, how I look at these, well, I think how everyone is supposed to, but, um, the, the input is G of one, so you go to the G line, which is this graph 'cause it's labeled G, you find one in the corresponding point, which is three, so that's now your input for H. So you go three over here, one, two, three, and up and the corresponding line is six, so H of G of one is six.
Int.:	Okay.
Karen:	And that's how you do the next one and the following one the same way. So this is the opposite so you start with H of five, and that's five, six, seven, eight, it's eight. And then you take – that's now the

input for G, so you go to the G line and find the corresponding, which is four, so G of H of five is four.

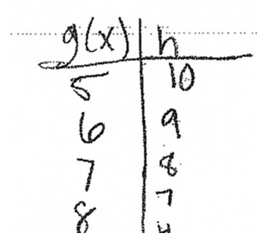
Int.: Okay.

Karen: Um, so H of two is four, and then I think you just, since it's the same line, then you do, it's not the inverse so I guess you just over four, which is seven. If I did that one right, I'm not sure, but I think so.

The second part of this problem asks Karen to describe how the output value, $h(g(x))$, varies in response to specified changes in the input value, x. Her response to this question is illustrated in Excerpt 56, and provides several insights into her view of function and covariational reasoning. Keep in mind that her ultimate "answer" is at least partly correct, despite the fragile understandings that she exhibits.

In line 2 of the excerpt, Karen identifies "G of X" as the input. This suggests that she is viewing h as "the function" in this portion of the problem. For Karen, evaluating $g(x)$ is a procedure that must be completed to determine what the input value is. This is reemphasized in lines 16-17, and in the table of values she constructs (Figure 6).

Figure 6: Karen's Table for Task 3b



Karen's inability to conceive of the inputs of two separate functions, and of two functions joined together in some way, is consistent with the action view of function she has demonstrated when solving other problems.

Note also that in lines 12-13 of Excerpt 56, Karen appears to demonstrate Level 2 covariational reasoning. Perhaps this should be called “pseudo-Level 2”, because although her statement in lines 12-13 is correct, her statements throughout the excerpt suggest that her understanding of what constitutes the input quantity is fragile.

Excerpt 56

Int.:	Okay. What about part two then?
Karen:	Okay. So input is G of X so five on G of X is seven, so then you’d go to H of seven, which is ten. So when H of G of X is five, wait. I could make a table - that would be helpful. G of X. I’m just making a table so I can see if it increases or decreases.
Karen:	And then whatever these numbers do, I’ll know if H of G of X increases or decreases by the table. Instead of just, like, writing it out, it’s just easier for me to look at. Um, so G of six, six, nine, so G of seven is five, five is eight, so G of eight is four, and seven, and G of nine is three, and that’s four… Yeah, so as, as X varies from – okay. So hold on. I’m just writing this sentence. Okay.
Karen:	Output of H of G of X, um, so, okay, so the output of H of G of X decreases as X varies from five to nine.
Int.:	Okay. Um, so in your table, the, the left column of your table, is it, is that G of X or is that, what, what is that column?
Karen:	Um, yeah, it’s G of X because G of X is the input and the inputs are given from five to nine.

In the third part of this problem, Karen is asked to describe how the *input* varies in tandem with specified variations in the *output* quantity. Her response is illustrated in Excerpt 57, and strongly supports the conclusion that she possesses an action view of function, with its resultant inability to conceive of the reversibility of functions. The problem statement indicates that $h(x)$ is what varies from 10 to 15; however, in Karen's understanding, varying the *input* is the procedure that is followed when working with functions. Even when this procedure is complicated by the need to evaluate the function at input values for which the function is undefined (lines 5-6), she is not deterred. In her understanding, functions only go one way – values are input, and results are calculated. This is emphasized even more by her comments in lines 12 through 15, which suggest that $h(x)$ is not a *quantity* for Karen, but is rather a statement about what to do – *this function is called h, and you plug values in for x*. Note also that once again in this excerpt that Karen appears capable of Level 2 covariational reasoning (line 9), although her description of the input and output quantities is reversed.

Excerpt 57

Int.:	Okay. All right. Well, how bout task three then?
Karen:	Okay. Okay. So this is just asking about H of X, so as H of X varies
	from 10 to 15, so 10 to 15 is your input.
Int.:	Okay.
Karen:	So H of 10 is 13… 11 is 14. I think just going up by one, so that's 15,
	H of 13. Is the line supposed to disappear? I don't know. [*Pauses*]
Int.:	Assume whatever you think you need to assume.

Karen: Okay. Well, since the line disappears, I'm just gonna say, um, H of X increases as X, um, varies from 10 to 15.

Int.: Okay. Um, can you tell me what's the difference between X and H of X? Writing those things, do those refer to the same thing?

Karen: Um, um, I guess I just look as X as just any number.

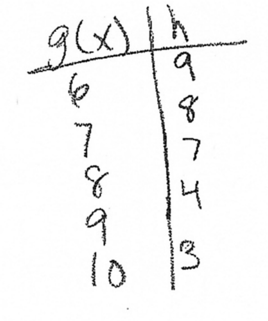
Karen: And when I see H of X, it's just saying that X is the number but then number has to be plugged into whatever H of X is.

Karen: So I guess it's just your input value.

The fourth and final part of this problem asks Karen to go one step further – to explain how the input value, x, varies in tandem with the output values $h\big(g(x)\big)$ varying from 6 to 10. Her response to this question is shown in Excerpt 58.

This excerpt gives further support to the assertion that Karen possesses an action view of function. In the first few lines, Karen notices that despite the different wording, the second and fourth parts of this problem seem (to her) to be asking the same thing. This supports the earlier assertion that she does not conceive of functions as being reversible. As a result, in lines 3-4 she decides to execute the same procedure she performed in the second part of the problem, producing the table shown in Figure 7.

Figure 7: Karen's Table for Task 3d



When solving this portion of the problem, Excerpt 58 suggests that $h\big(g(x)\big)$ is not a *quantity* for Karen, but is rather a statement about what to do – *Use the functions h and g, plug values into g, and then plug the results into h*. This is further support for her action view of function, and her inability to conceive of something denoted as $h\big(g(x)\big)$ as being a quantity.

Excerpt 58

Karen: Um, okay. So this is – well, now I'm getting confused 'cause task four is like two, but task two's asking the out, how does the output vary, task four's just saying how does X vary. Okay. I don't know. I'll just do it how I think. [*Laughs*]. Um, okay. So I'm gonna make a table. So… G of seven is five, G of eight is four, G of nine is three, and G of ten is two – is three, so X decreases as H of G of X varies from six to ten.

Int.: Okay. Um, you used a, a table for – basically you used tables for all of them. When you do that, how does, what does that help you with?

Karen: Um, I think it's 'cause I'm more of a visual, like, learner. Like, I, I can, I can look on the table and see and write it out, but to know if it's increasing or decreasing, I can look at a table and know that all the numbers are decreasing, or if the numbers got higher, I would know that it was increasing, or increasing then decreasing. It's just easier to look at it, and it's just kind of neater looking I think.

Karen's solution to the salary problem.

Continuing to examine Karen's view of function and quantitative reasoning, I consider a portion of her solution to the Salary Problem (Table 16).

Table 16: Task 4 – The Salary Problem
Using function notation, suppose $S(m)$ represents the monthly salary, in hundreds of dollars, of an employee after m months on the job. What would the function determined by $R(m) = S(m + 12)$ represent? a) the salary of an employee after $m + 12$ months on the job b) the salary of an employee after 12 months on the job c) \$12 more than the salary of someone who has worked for m months d) an employee who has worked for $m + 12$ months e) The salary after m months is the same as the salary after $m + 12$ months.

A portion of Karen's response to the Salary Problem is shown in Excerpt 59. Note that she ultimately chooses the correct answer. However, she is unable to give a coherent explanation of the reasoning that led her to this conclusion, and it is not clear what led her to discard choices b, d, and e. Also, in lines 9 through 18 she demonstrates that she has not conceptualized of the relationship between the function notation and the quantities described in the answer choices.

Excerpt 59

Karen: It's either A or, um, I guess C, but I think it's A because it's asking what would the function determined by – so, if SM represents the monthly salary, and M is the, um, is an employee after M months on the job, so whatever months M is, plus another 12 months is the salary in hundreds of dollars. So I think what it's saying is, um, the salary of an employee after however months, plus another 12 months on the job. So I think it's A.

Int.: What do you think C would look like?

Karen: Um, 12 more than the – $12.00 more than the salary of someone who has worked for M months. Twelve over – um, I think, um, I think it would be SM times 12, maybe, or S times 12 plus M because the salary – it's saying 12 more than the salary, so to me, I think, oh, I have to times 12 by the salary that I already had, or by the salary that I make or something. So you would have to times it by that, so that's what I think how it would look different.

Int.: So can you write for me what C would look like?

Karen: Um, maybe like that (*writes* “$= S(12) + M$)”. I don't really know, actually. I'm just kind of guessing.

Excerpt 59 demonstrates that Karen’s understanding of function notation is very weak. In problems where interpreting the notation as a command to calculate leads to a correct answer, she will often be successful. However, when attempting problems in which function notation is used to convey something more than a “command to calculate”, she is likely to be unsuccessful.

Karen's solution to the circle problem.

Turning to Karen's understanding and use of function composition when solving novel problems, I present portions of her solution to the Circle Problem (Table 17).

Table 17: Task 5 – The Circle Problem
Express the circumference of a circle as a function of the area of the circle.

Similar to the other students in this study, Karen immediately turns her attention to finding the appropriate formula or formulas, as shown in Excerpt 60. Unlike the other two students, however, Karen immediately recalls the correct formulas, writing $A = \pi r^2$ and $C = 2\pi r$. Immediately after this, she becomes stuck (lines 6-8), and takes a long pause.

It is interesting that all three students immediately wrote (or tried to write) formulas they knew that were related to circles. The significance of this is not immediately clear, but it probably warrants further investigation outside this study.

Excerpt 60

Karen: Okay. Um, express the circumference of a circle as a function of the area of a circle. Um – I'm trying to remember what the circumference is. I know area equals pi R squared. Circumference is 2 pi R I think, maybe.

Int.: What's, what's R here?

Karen: Radius. R is the radius. So express the circumference of a circle as a function of a circle. [*Pauses*] Um, um, so, um, I don't know. This is hard. [*Pauses*]

After a long pause, Karen writes $C(A) = 2\pi r^2$, giving the explanation shown in Excerpt 61. This excerpt illustrates that she believes she needs to create a single formula that includes both area and circumference. It also illustrates that she identifies area as the "input" to this formula, and that her intent is for the formula to give her a circumference. However, she does not conceive of the area formula as being "reversible", so she does not know how she might want to "combine" the two formulas (lines 5-7). In the end, she chose to write a formula that included all of the elements of her two original formulas – it included a 2, a π, an *r*, and a "squared".

Excerpt 61

Karen:	Um, okay. So the area is the input. Um, what I wanna do is just kinda combine them so it looks like this (*writes* $C(A) = 2\pi r^2$), but I don't know.
Karen:	I'm not really sure.
Int.:	You wrote C of A equals two pi R squared.
Karen:	Which means that the input is the area. Um, to the circumference, but not exactly sure what goes on this side, but I wrote this just 'cause I combined those two.

Recognizing that Karen's formula would not allow her to calculate circumference corresponding to a given area, I chose (Excerpt 62) to give her a particular value – in this case, 8. In lines 4-6, Karen realizes that 8 is not a value for the radius, so she substitutes it into the only other non-numerical part of the formula. At this point, she has an equation with one unknown, so she wants to solve for that unknown. However, she quickly realizes this will not give her circumference, and she again gets stuck.

Excerpt 62

Int.:	So if, if you have that formula there, now if I tell you that the area is, I don't know, eight – then can you use the formula to tell me what the circumference would be? Suppose the area is eight square feet.
Karen:	[*Pauses*] I don't – don't I have to know what the radius is? I don't know… I guess I would set it equal to this (*substitutes 8 for* $C(A)$*, writing* $8 = 2\pi r^2$), that's what I'm thinking.
Karen:	And then, um, I wanna say I divide by two. Oh man, I don't know. Um, now I don't know what to do. 'Cause I'm just, now I just have a variable that I don't know what to do with. And this is the radius, which isn't gonna give me the circumference.
Karen:	I don't, I don't really know what to do.

Finally, in Excerpt 63 I again probe Karen's reasoning for combining the area and circumference formulas in the way that she did. This time, she explicitly states that she knew she had to use both formulas, so she "just combined them". Since she did not conceive of the area function as being invertible, she combined the formulas in the only way that was apparent to her, although her final comments in line 8 indicate that she did not really believe this was the correct thing to do.

Excerpt 63

Int.: When you combined those two formulas and came up with this, why did you decide to combine them in the way you did, to have, C of A as two pi R squared?

Karen: Um, because – I don’t know. I used area as input, but I combined them because I, I know that there would have to be a function of another, and I know that I had to use both, um, both formulas, I just didn’t know how to put them, so I just combined them. But – I don’t, I don’t know. I don’t really know how to do it.

Karen’s ability to complete the Circle Problem was hampered by her inability to recognize that she needed to determine the inverse function by rewriting the area formula (in terms of the radius) as a process that determines radius when given the area. The ability to flexibility move from conceptualizing a function and its inverse has been closely linked to students possessing a process view of function (Dubinsky; Carlson, 1998), which illustrates the importance of a process view of function to the student’s successful completion of a function composition word problem, like the Circle Problem.

Karen’s solution to the giraffe pen problem.

To conclude my investigation of Karen’s understanding and use of function composition when solving novel problems, consider her solution to the Giraffe Pen Problem (Table 18).

Table 18: Task 6 – The Giraffe Pen Problem
Bryan has decided to open a wildlife park, and he would like to build a square pen for some giraffes. Bryan hasn’t decided how many giraffes to acquire, but he has been told

Table 18: Task 6 – The Giraffe Pen Problem
he should allow 10,000 square feet for each giraffe to graze. Help Bryan design his park – find a way to determine how many feet of fencing he must buy to enclose his square pen, based on the number of giraffes that will live in the pen.

After Karen first read the problem aloud, she was asked to describe the situation in her own words, and to explain what she believed she was being asked to do. The ensuing discussion is illustrated in Excerpt 64.

Excerpt 64

Karen: Um, well, Brian needs to build a fence for his giraffes and someone told him that he should have at least 10,000 square feet for each giraffe. Um, but it doesn't say how many giraffes he's gonna have. Um, find a way to determine how many feet of fencing he must buy to enclose his pen. Um, well, the only value they give us is 10,000 square feet. I don't – I don't have a value for square feet, but okay. Um… how many feet of fencing he must buy to enclose his pen, based on the number of giraffes. Um, okay. So – based on the number of giraffes, um – well, I want to do just however many giraffes there are times 10,000.

Int.: Okay.

Karen: Because, um – because for each – for each giraffe, they're saying you need 10,000 square feet. So however many giraffes there are, um, you would times that by 10,000, since it's for one.

Int.: Okay, and what – what are you trying to find by doing that?

Karen: Um, how many feet of fencing. So if this is feet, uh, however many giraffes there were would give you how many feet of fencing for however many giraffes.

Karen's comment in line 6 of Excerpt 64 is important, and hints at something that will prove crucial to her solution of this problem – "I don't have a value for square feet". Karen realizes that she needs 10,000 square feet for each giraffe, and she notes that this is "the only value they give us" (line 5). As has been demonstrated in earlier problems, Karen has a tendency to want to perform calculations, so she performs the only calculation that seems appropriate to her, multiplying 10,000 by the number of giraffes to be acquired. As far it goes, this is a correct step to perform. However, Karen's next action shows that she failed to conceptualize what a square foot is a measurement of, and provides no evidence that she sees a foot as being distinct from a square foot. She appears to view both a square foot and foot as linear measurements, concluding that the multiplication she performed would tell her "how many feet of fencing for however many giraffes" (lines 17-18). In an effort to determine whether this was a misreading of the problem on her part, or evidence of a misunderstanding, I probed the distinction between feet and square feet in Excerpt 65.

Excerpt 65

Int.: Okay, except that the 10,000 wasn't feet, right? It was square feet.

Karen: Oh, square feet. I don't know – Square feet, hm. Well, if I knew how many feet were in a square foot or the opposite. I don't really – I can't really tell you 'cause I don't really know. [*Pauses*]

Int.: So it's square feet, so what – what would that 10,000 square feet be a measurement of? Is that an, um – like a length of something, or a circumference, or an area, or a volume, or –

Karen: An area. I think it's an – it's an area – like how you describe how big someone's house is. You say it's – something square feet. But – I don't really… [*Pauses*]

Karen's response in Excerpt 65 to this probing revealed information about her conception of square feet. In lines 2-3, she comments that she doesn't know "how many feet were in a square foot". This suggests that she is looking for a way to *convert* square feet into feet, which in turn suggests that feet and square feet do not measure the same quantity. In the remainder of Excerpt 65, I try to determine how she is conceptualizing the idea of "square feet". Karen correctly notes that "square feet" is used to describe an area – "how big someone's house is" (lines 8-9). However, note that this does *not* mean Karen possesses a useful understanding of the idea of area.

Karen was next asked to consider situations involving more than one giraffe. Excerpt 66 provides her description of the situation in which a total of two giraffes were to be acquired. Notice that again, despite previous attempts to prompt Karen to discern between feet and square feet as being non-equivalent units, she continues to use feet and square feet as being commensurable units. In lines 8-10, Karen again laments her lack of a "conversion", and with her comments in lines 11-12 – "just so I answer, kind of, your questions" – her growing frustration with the problem (and with the interviewer, perhaps) begins to show. Note that an ability to conceptualize what a unit (in this case, a foot, or a square foot) is measuring is essential for conceptualizing the quantities and how they are related in this situation.

Excerpt 66

Int.: Well, what total area would he need for two giraffes, if he had two giraffes?

Karen: Um, well, if it was just in feet, he would need 20,000 feet of fencing.

Int.: Tell me what you mean, “if it was just in feet”.

Karen: Um, so this would be 20,000 feet – feet. It’s because I don’t know what to do with square foot – square feet.

Int.: Oh, okay.

Karen: So I’m just saying, if it was in feet – because I – I feel like that you – that I would need, like, a conversion – something to turn square feet into feet, and I don’t know – I don’t know how many feet are in a square foot or whatever, so I’m just saying – just so I answer, kind of, your questions – that if it was just in feet, then if I had two giraffes, I would need 20,000 square feet for two.

At this point, I realized I needed to probe Karen’s understanding further, and in Excerpt 67 Karen was asked to explain exactly *what* about a square is its area. Note that her first attempt (line 3) is procedural in nature – she gives the formula she has memorized for the area of a square. After this, she describes the area as being the distance between two opposite sides of the square (lines 4 through 7), suggesting she recognizes area as being *something* about the inside of the square – the part that lies between the two sides. After persistent probing she fails to accurately describe the meaning of a square inch as the area enclosed in a square that is 1" on each side. Her description of her calculation in lines 8 through 10 that the area is calculated by multiplying the distances obtained by measuring from one corner to both of the adjacent corners does not appear to be grounded in meaning.

Excerpt 67

Int.:	If we've got a square, like this square here –what about that square is the area?
Karen:	Isn't area like width times height, I thought?
Karen:	I – I don't know – area… Area – area – I don't know. Um, I guess the distance from – from one – from one side of the fence – wait – from, like – from this side to this side (*indicating two opposite sides of a square she has sketched*) would be the area.
Karen:	From this point to this point and this one to this one (*indicating the distances from one corner to each of the adjacent corners*) – if you times, then that would give you the area.

Eventually, with quite a bit of assistance, Karen constructed a square pen that measured 100 feet along each side. Asked about two giraffes, Karen placed two of these pens side by side, constructing a 100 by 200 rectangular pen. Exploring this situation, I asked Karen to consider that the problem asked her to create a *square* pen (Excerpt 68). She was able to create a mental image that supported conceiving of a pen that was shortened and widened (lines 5 through 9), but still didn't possess a robust quantitative structure that would allow her to calculate the length of one side of the new, square pen (lines 10-11).

Excerpt 68

1	Int.:	If Bryan wanted to build a *square* pen, could he build a square pen
2		that gives him 20,000 square feet of area?
3	Karen:	Um – square, um –
4	Int.:	‘Cause we’ve built him a *rectangle* that has 20,000 square feet.
5	Karen:	Uh, um, yeah. I guess if you were to shorten the sides – these sides
6		(*indicating the 200-foot sides*) – and make this higher (*indicating the*
7		*100-foot sides*), but it would – and it would still give you 20,000
8		square feet. But if you shorten these, made these longer, then they
9		just wouldn’t – it would just be taller this way and shorter this way.
10	Int.:	Yeah, okay. So you don’t know what the dimensions would be?
11	Karen:	Yeah.

After all of this discussion with Karen, she seemed to making progress toward conceptualizing area as some measurement of amount of space, even though she never expressed the meaning of a square inch. Her underdeveloped understanding of area was further revealed (Excerpt 69) when asked about a pen for a single giraffe.

Excerpt 69

1	Int.:	So, um, how could you figure out how many feet of fencing he needs,
2		if he’s got one giraffe?

Karen: I think – I don't – I don't really know 'cause I think I'm still confused on the whole square feet and feet thing. Like, I understand that this is square feet, but I don't really know if you would need 10,000 feet of fence. That just sounds, uh, like a lot.

As a final attempt to prompt Karen to make some progress on this problem, I sketched a 100-foot by 100-foot square. Karen was able to calculate the area of this square as "10,000", and began to make a comment about "100 feet of fence", but then stopped herself. Excerpt 70 illustrates the discussion that followed.

Excerpt 70

Int.: What could he do with 100 feet of fence? Tell me more.

Karen: Um, well, if he had – if this was 100, so that's one side, so, um – I don't know. Would he just need – but this is – you need four sides of this, so – if this is just one side and – uh. I'm not sure. [*Long pause*]

Karen: I really don't like these, at all. I'd rather have, like, problems.

In lines 2 through 4 of this excerpt, Karen appeared to be on the verge of a "breakthrough" that might allow her to build a mental model of the problem that would allow her to complete it successfully. However, her weak understanding of area, and her under developed mental model of the context, were not robust enough to support her formation of a coherent and accurate response to the question.

Line 5 of this excerpt where she indicated that she wanted "problems" reveals her strong dislike for applied problems and that she doesn't believe that these are "problems."

Summary Characterization of Karen

Karen was unable to understand or use function composition to build new functions in applied contexts. This is seen, for example, in Excerpts 61, 63, and 65 through 67. Her inability to make progress on the function composition problems was complicated by several factors revealed in this study, and described in the following paragraphs.

Action view of function.

Karen did not have a process view of function, so she was unable to imagine function composition as a stringing together of two function processes. In fact, she never expressed this as a goal when attempting Tasks 5 and 6 that required her to do so. Karen's view of function can be characterized as an *action* view, with little evidence that she was in transition to a process view of function.

Karen possessed a very strong "procedural" inclination consistent with an action view of function. When attempting problems in this study, her first action was typically to find a formula to use or a procedure to follow for the purpose of calculating *something*. When no formula or procedure was apparent that she considered "correct", she made little progress in advancing her solutions. This lack of progress was associated with an inability to conceptualize the quantities in the situation.

Consistent with this characterization, in Karen's understanding, function notation tells her what to do. For Karen, function notation is a signal that she needs to plug in a value and calculate. For example, Karen demonstrated an immediate focus on formulas when solving the Circle Problem (see Excerpt 60). It is worthwhile to note that even though she was able to recall the correct formulas with no difficulty (unlike the other two students in this study), her lack of a process view of function left her unable to successfully complete the problem.

Further evidence of Karen's weak understanding of function notation could be seen, for example, in her solution to the Salary Problem (see Excerpt 59). This problem reveals an inability to conceptualize the output from evaluating an expression as an input to a function. For Karen, function notation signals a command to calculate, and the subtle differences among the choices in the Salary Problem held little meaning for her.

Karen was unable to conceive of the reversibility of functions. In her understanding, functions only went in one direction; a "function" meant she was to plug in values, and calculate based on those values. For example, Karen was unable to conceive of the reversibility of the area formula, when attempting to respond to the Circle Problem (see Excerpt 62). Similar inability to conceive of the reversibility of a function could be seen in her solution to the Graphical Composition Problem (see Excerpts 57 and 58).

Finally, it is very interesting to note that Karen did not consider problems like the Giraffe Pen Problem to be math problems. In her view of mathematics, math problems are tasks that are completed with memorized formulas and procedures. This is not surprising, and is consistent with her approach to solving the problems in this study, her greater success (relative to the other two students in the study) in remembering formulas related to attributes of a circle, and her tendency to get frustrated and give up when the correct formula or procedure was not apparent to her.

Inability to conceptualize quantities.

Karen's ability to construct useful mental imagery of problem situations was very weak. This resulted in Karen's inability to conceptualize quantities from a problem statement. In Tasks 5 and 6, for example, conceptualizing the quantities to be related was essential for constructing a composed function to relate two quantities through a "middle" quantity of the kind described by Engelke (2007). As revealed by Engelke, students and mathematicians who were successful read and re-read problems until they had identified the quantities to be related, and then the "middle quantity" that would be useful for relating the desired quantities. Excerpts 61, 63, and 64 through 70 provide evidence that Karen had not conceptualized function composition as a necessary concept to help her relate two quantities that could not be directly related by a single formula.

Karen had a weak conception of units that complicated her efforts to conceptualize measurements of area. This highlights the important role of units in conceptualizing a quantity, as demonstrated by Karen in Excerpts 65, 66, and 69. Karen did not possess a useful understanding of the concept of area, in the Giraffe Pen Problem. Instead, she expressed a desire to know the "conversion" to get from square feet to feet. She described the area of the square pen as being a "distance" between two opposite sides of the pen; this may indicate that she understood area is referring to *something* about the space in the middle of the pen, but she did not show that she understood what that something might be.

CHAPTER 6

DISCUSSION AND CONCLUSIONS

In this section, I discuss the results of this study relative to the research questions presented in Chapter 1.

1. What is the nature of precalculus students' understanding of function composition?
2. What reasoning abilities and understandings support precalculus students in understanding and using function composition?
3. **3.** What factors are important facilitators of, or obstacles to, students' possessing a robust conception of function composition?

Research Question 1: What is the nature of precalculus students' understanding of function composition?

The students in this study varied in their understanding of function composition. The degree to which students in this study were able to understand and use function composition to build new functions in applied contexts varied from student to student, and from problem to problem. A student's understanding of function composition and success at solving the applied function composition problems notably reflected the student's view of function, quantitative reasoning, and covariational reasoning, as described in the following paragraphs.

View of function.

A key finding is the importance of a student's view of function, relative to the student's success in linking processes to relate quantities that could not be directly related. Students possessing a strong procedural orientation were unable to link processes to relate quantities that could not be directly related. "Procedural orientation" is intended to mean a procedural understanding of functions (and composed functions) as representing a series of commands to either perform calculations or use graphs. This view of function has been called an action view (Dubinsky and Harel, 1992) and the findings are consistent with findings reported by Dubinsky and Harel and by Carlson (1998).

For example, consider Karen's procedural approach to the Circle Problem (excerpt 60), and subsequent inability to link the formulas she recalled (excerpts 61 through 63). Karen immediately focused on writing down the formulas that she believed applied to this problem, and was successful in doing so (excerpt 60). After this, she becomes stuck, and was unsure what to do. In excerpt 61 she described area as being "the input… to the circumference", a statement that seems to hold little meaning to her, and is likely the result of having used the terms, input and output, in class. When prompted with a specific given value for the area ("Suppose the area is eight square feet", excerpt 62), Karen is still unsuccessful at inverting one of her formulas.

By contrast, students who were in transition to a process view of function were more successful at linking processes to relate quantities that could not be directly related, and in constructing and evaluating composed functions. For example, consider Patricia's solution to the Circle Problem (excerpts 18 through 21). In lines 7 through 12 of excerpt 18, Patricia reasoned about her desired input (area) and output (circumference), using the terms input and output, and writing $f(A) = C$ (line 2 of excerpt 19). Although Patricia initially had trouble inverting one of her functions to create a composite function (indicating that she does not fully possess a process view of function), she ultimately created a correct function formula to express the area of a circle as a function of its circumference and explained that the key to her solution is that “you have to be very comfortable with flipping area and circumference back around with each other”. This comment suggested that she was thinking about function inverse as reversing the process of the original function.

Students’ understanding of representations of functions also proved to be significant to students’ understanding and use of function composition. Seeing function formulas, graphs and tables as tools or ways of representing how one quantity changes with another is a critical way of thinking that is foundational for composing two functions. When students had this way of thinking they were often successful in solving function composition problems. In this study, for example, Bridget was adept at working with functions defined by graphs (see excerpts 38 and 39), and as a result was able to successfully evaluate composed functions given a single input value for the first function. In contrast, Bridget was less adept at working with functions defined by formulas (see excerpt 43), which complicated her efforts to solve such problems, such as the Circle Problem (excerpts 41 through 43).

Quantitative and covariational reasoning.

This study suggests that students’ quantitative reasoning is critical to how students approach novel function composition problems. Recall that both Patricia and Bridget sometimes exhibited strong and robust quantitative reasoning. By contrast, Karen showed weak quantitative reasoning abilities throughout the semester. Correspondingly, Patricia and Bridget were more successful at creating and linking functions that related appropriate pairs of quantities, and to recognize when the output of one function was “suitable” as an input to a second function.

Students' understanding of function composition also appeared to be linked to the student's covariational reasoning abilities. Students who engaged in higher-level covariational reasoning were able to coordinate input and output quantities of composed functions in much the same as they could with a single function. Students with more limited covariational reasoning were unable to coordinate the changing values of the three different relevant pairs of quantities involved in function composition problems.

As a demonstration of the importance of a student's covariational reasoning, consider Bridget's solution to the Graphical Composition Problem, described in excerpts 38 and 39 in the previous chapter. Her actions when completing the subtasks of this problem reveal that she was able to reason covariationally about pairs of changing quantities at lower levels of the framework presented by Carlson et al. (2002). Bridget's covariational reasoning abilities were evident in her solution, and in her partial success at building and refining mental images of problems that supported constructing quantitative relationships that were then linked to relate two quantities that the student could not relate directly.

Research Question 2: What reasoning abilities and understandings support precalculus students in understanding and using function composition?

Constructing a mental picture of the situation described in an applied problem is essential for constructing meaningful formulas to relate quantities in the situation. Constructing a mental picture of the context of the problem involves such things as imagining two friends driving to a restaurant (the Dinner Problem, Task 1), a box being folded up (the Box Problem, Task 2), or a pasture being enclosed by fencing (the Giraffe

Problem, Task 6). When students are unable to do this, they are unable to conceptualize the quantities in a situation.

This study demonstrated that a student's conception of the relevant quantities in a problem statement and conception of how they are related is emergent by the student's engaging in repeated acts to make sense of the problem statement (see excerpts 18, 19, and 22 through 26). The manner in which students constructed an image of the quantitative relationships in the problem, and came to conceptualize the linking together of two function processes for the purpose of linking two quantities that could not be related by a single formula, varied from problem to problem. However, an analysis of the collection of findings supports that, in instances where the student persisted in making meaning of the quantities, building quantitative structures, and conceptualizing how to relate the desired quantities, she was eventually able to identify a linking quantity needed to concatenate two processes that would relate the desired quantities.

The results from analyzing Patricia's explanations revealed that her persistent efforts to develop a detailed mental picture of problem contexts were critical to her eventually conceptualizing and relating the quantities in the situation. For example, consider her solutions to the Circle Problem (excerpts 18 through 21) and the Giraffe Pen Problem (excerpts 22 through 26).

Analysis of Patricia's solution to the Circle Problem, described in Excerpts 18 through 21 in the previous chapter, reveals that she engaged in repeated acts of sense-making. In Excerpt 18, she explored the meaning of "express" and "a function of". Later in her solution, in Excerpt 19, she asked "Now what am I doing here?" and then engaged in an additional attempt at sense-making, and decided that her solution should be of the form $f(A) = C$. As she progressed in her solution to the problem, in Excerpt 20 she once again paused to consider the problem and her understanding of it.

While attempting to solve the Giraffe Pen Problem, illustrated and discussed in the previous chapter using Excerpts 22 through 26, Patricia repeatedly returned to the problem statement, building and refining mental images that supported her progress toward a solution. This reflects a pattern of persistence that was seen throughout her interviews; time and again she returned to the problem statement, tenaciously examining her understanding of the words in the problem statement and the mathematical relationships implied by those words (to her). The result of this persistence was her development and refinement of rich quantitative relationships between pairs of quantities, and ultimately success at solving problems. This success seemed to surprise her. On several occasions during her interviews, she seemed to have come to a dead end when trying to solve a problem, before persisting in making changes in her approach until she found a new approach that allowed her to continue.

In contrast, when Karen was responding to the Dinner Problem (excerpts 47 through 49), her inability to create a coherent mental picture made it impossible for her to conceptualize the relevant quantities. By the end of excerpt 47, Karen was ready to give up, and only persisted because she was asked additional questions about the problem. She lacked the desire to persist in developing a mental picture of the situation, the desire that was seen so strongly in Patricia's approach to problems described earlier.

This finding suggests that some students would benefit by curriculum and instruction placing greater emphasis on developing students' persistence in developing a mental picture of the context described in a problem statement, prior to attempting to define variables or write a formula.

Research Question 3: What factors are important facilitators of, or obstacles to, students' possessing a robust conception of function composition?

In addition to the factors already discussed – view of function, covariational reasoning, and quantitative reasoning – this study revealed that students' understanding of area, perimeter, and circumference complicated students' success in solving applied function composition problems.

Word problem statements designed to prompt the student to use function composition in the solution often use terms such as area (for example, Tasks 5 and 6 in this study), perimeter (Task 6), and circumference (Task 5). The results of this study indicate that some college precalculus students possess weak understandings of these terms and the concepts they represent, hindering students' ability to make sense of problem statements and conceptualize quantitative relationships.

As an example, contrast Bridget's solution to the Giraffe Pen Problem (described using Excerpts 44 through 46 in the previous chapter) with Karen's solution to the same problem (described using Excerpts 64 through 70). Bridget's explanations revealed that she understood that a square must have sides of equal length, that the area (in square feet) equals the product of the length and width of the square (in feet), that the perimeter of a square is four times the length of one side of the square, and that the perimeter of the square (in feet) is the quantity whose value she is being asked to determine (lines 8 through 11 of Excerpt 44, as clarified by lines 1 through 4 of Excerpt 43). The concepts of area and perimeter are not problematic for her. Her explanations and correct solution to the problem supports the assertion that the quantities in these formulas were meaningful, and she understood the relationships expressed by the formulas.

In contrast, consider Karen's solution to the same problem. In lines 2 through 4 of Excerpt 65, her response revealed a weak understanding of what is measured using a square foot as a unit, as opposed to what is measured using a foot as a unit. This lack of understanding of the distinction between feet and square feet is demonstrated again in lines 9 and 10 of Excerpt 66, where she describes a need for a "conversion" between the two. Finally, in Excerpt 67, Karen gave a description that confirmed that she possessed a very weak understanding of the concept of area. Ultimately, Karen's weak understanding of the meanings of, and relationships among, the area, length, width, and perimeter of a square left her unable to make any significant progress toward a correct solution to the problem.

This finding suggests that some students would benefit by curriculum and instruction placing greater emphasis on developing students' understanding of area, perimeter, and circumference, and the units commonly used to measure these quantities.

Re-examination of Initial Conjectures

Having examined students' understanding of function composition, and the reasoning and understandings that support students' understanding and approach to function composition problems, I turn to a re-examination of the conjectured mental actions from Chapter 3.

The first major change to my conjectures concerns sequencing. In Chapter 3, I presented a conjectured "sequence" of mental actions, based on a conceptual analysis of my own solution to the "ripple problem". The data collected and analyzed under this study suggest that this sequencing does not align with what was revealed when examining the subjects' thinking in this study.

For example, Patricia's solution to the Circle Problem was not as orderly and sequential as the conjectured framework might suggest. She began by recalling memorized formulas for area and circumference and then explained what each of the variables "stood for". However, her creation of a mental picture of the situation, and quantifying attributes of the situation, were not actions performed once, at the start of her solution. Rather, she repeatedly attempted to make sense of the problem statement, each time refining her mental picture of the event and understanding of the relevant quantities.

Much like Patricia's solution, Bridget's solution to the Circle Problem did not "step through" the mental actions in the conjectured framework. First, note that Bridget began by recalling memorized formulas for area and circumference and explaining what each of the variables "stood for", much like Patricia had done. Bridget created a mental picture of the situation, and quantified attributes of the situation, as a result of repeated acts of sense-making. This was similar to Patricia's approach to the circle problem.

Turning to Karen's solution to the circle problem, it seems at first glance to be very poorly described by the conjectured series of mental actions in the conjectured "sequencing". Note that Karen began by recalling memorized formulas for area and circumference, while this was the fourth mental action in the original framework.

As a result of these considerations, the framework presented later in this chapter does not include sequencing. The student solutions examined in this study do not warrant the inclusion of such sequencing since no pattern emerged regarding the order in which students exhibited the mental actions and behaviors described in the conjectured framework (Table 19).

The results of this study also provided information for refining the mental action described as, “Conceptualize the output of the first function as a suitable input to the second function”. The mental actions performed by students in this study were more varied than what was described in Chapter 3. For example, as Patricia solved the Circle Problem, she was able to describe (see Excerpt 18 in the previous chapter) what she understood to be the inputs and outputs of the function formulas she had created, and she wanted to “combine” these formulas (Excerpt 19). She was even able to articulate that something wasn’t quite right when she attempted to do so, but it was not clear from Patricia’s actions that she was consciously trying to “conceptualize the output of the first function as a suitable input for the second”. She simply knew that she wanted to combine the formulas, and get circumference and area into the same formula. She was unable to conceive of the quantity “radius” as the key to making this connection. In this example, radius is the “connecting” quantity that Engelke (2007) labeled the “middle man”. Consistent with Engelke’s results, it was evident that determining this connecting quantity was key to solving the problem successfully.

Similarly, Bridget attempted to combine the function formulas she had created for area and circumference, and was able to articulate a perceived need to do so (Excerpt 42). However, just like Patricia, she was unable to do so. Again, it is not clear from Bridget’s actions that she was trying to “conceptualize the output of the first function as a suitable input for the second”. Just as in Patricia’s solution to this problem, Bridget acted out of a desire to “combine” the formulas she had created, and get circumference and area into the same formula. Much like Patricia, she was unable to conceive of the quantity “radius” as the key to making this connection.

Finally, even Karen attempted to combine the function formulas she had created, and was able to articulate a perceived need to do so (Excerpt 61). However, she was also unable to do so. Just as in the other two students’ solution to this problem, Karen acted out of a desire to “combine” the formulas she had created. She had formulas for both circumference and area, and she felt she should combine them into a single formula. Again, much like both Patricia and Bridget, she was unable to conceive of the quantity “radius” as the key to making this connection.

The actions of all three students suggest that “conceptualize the output of the first function as a suitable input for the second” is an incomplete description of the reasoning that students perform when trying to combine formulas in their solutions to function composition problems. Rather, the conceptualization of the connecting quantity was key to both students’ advancement toward a formula to relate two quantities through function composition, and their inability to advance their solution. This notion is explored further in the following paragraphs that discuss modifications to another of the conjectured mental actions.

A third modification to the conjectured framework is regarding the mental action described in Chapter 3 as, “Conceiving of the “reversibility” of one or both functions (as required) to yield the desired input and output quantities”. In some cases, students were successful at the problem solution step that was suggested in this description, even though their behaviors did not support the characterization of the students’ mental actions suggested by the description of this mental action.

For example, after being prompted with a specific value for "area" with which she could perform calculations (Excerpt 21), Patricia realized that she could solve her area formula for radius (creating the formula for the inverse function, whether she realized it or not). However, it is not clear that she was conceiving of the reversibility of a function; she may have simply been solving an equation with one variable (r). Regardless of the mental actions underlying her algebraic manipulations, with the function formula $r = \sqrt{\frac{A}{\pi}}$ in hand, Patricia was now able to describe "shoving" the output of one function (radius) into the second function (line 8 of Excerpt 21), to give circumference, concluding by confidently writing the function formula, $C = f(A) = 2\pi\left(\sqrt{\frac{A}{\pi}}\right)$.

Considering Bridget's solution to the same problem, note that after being prompted with a specific value for "area" with which she could perform calculations, Bridget realized that she could use her area formula to determine the corresponding radius (Excerpt 43). However, she appeared to have been solving an equation with one variable (r), rather than conceiving of the reversibility of the function. In addition, Bridget was unable to generalize her calculations to produce a formula for the case in which the value of area is not explicitly known, but is represented by the variable, A. It does not appear that Bridget conceptualized function reversibility; while she was performing the correct algebraic manipulations, the mental operations necessary to qualify as conceiving of function reversibility might not have been present. Instead, her actions might be described as pseudo-analytical behaviors.

In both of these examples, the conceptualization of a "connecting" quantity was key to students' advancement toward a formula to relate two quantities through function composition, or their inability to advance their solution.

As a result of the modifications described above, a framework that describes function composition mental actions is given in Table 19.

Table 19: Function Composition Mental Actions

Mental Action	Model Student Behavior	Facilitated by
Creating a static or dynamic mental picture, which forms a basis for subsequent quantification	Student is able to describe a coherent mental picture consistent with the given problem.	Conceptualizing the events described in the problem statement Persistence – Engaging in repeated acts to make sense of the problem statement.
Quantifying attributes of the problem situation	Student is able to describe varying and nonvarying measureable attributes of the problem, and note the units used to measure these attributes.	Mental imagery – conceiving of relationships among attributes Quantitative reasoning – conceiving of measurability of attributes of relevant objects in the mental picture constructed when reading (and re-reading) the problem statement. Persistence – Engaging in repeated acts to make sense of the problem statement.
Conceiving of the possibility of using variables to represent the values of the created quantities	Student assigns variables to represent the values of quantities in the problem context.	Understanding of variable – ability to conceive of using symbol to represent value of quantities

Mental Action	Model Student Behavior	Facilitated by
Relating the quantities using formulas	Student is able to create formulas to express relationships between values of different quantities.	View of function – student possesses an action or process view of function Covariational reasoning – student is able to conceive of the relationship between changing values of quantities Quantitative reasoning – student is able to conceive of the relationships between quantities and the quantitative structures that arise
Conceptualizing the relationships as functions, accepting inputs and producing outputs	Student is able to identify input and output quantities, to solve formulas to express the output quantity in terms of the input quantity.	View of function – student possesses a process view of function that supports reasoning about inputs and outputs of functions Covariational reasoning – student is able to conceive of the relationship between changing values of quantities Quantitative reasoning – student is able to conceive of the relationships between quantities and the quantitative structures that arise
Conceiving of a need to combine the functions previously constructed	Student describes wanting to "combine" two or more functions previously created.	View of function – student possesses a process view of function that supports reasoning about inputs and outputs of functions

Mental Action	Model Student Behavior	Facilitated by
Conceiving of the quantity that allows the two functions to be combined	Student is able to identify the quantity that the two functions share – the “connecting” quantity	View of function – student possesses a process view of function that supports reasoning about reversibility of functions, and function inverses Persistence – Engaging in repeated acts to make sense of the problem statement.
Conceiving of the “reversibility” of one or both functions (as required) to yield the desired input and output quantities	Student is able to correctly determine the inverse of one function, when necessary to provide a suitable input to the second function.	View of function – student possesses a process view of function Covariational reasoning – student is able to coordinate changing values of three pairs of quantities: input and output of first function; input and output of second function; and input of first function and output of second function
Conceptualizing the composed function as a single function	Student is able to correctly describe the input and output of the composed function. Student is able to create a formula, table, or graph defining the composite function.	Covariational reasoning – student is able to coordinate changing values of the input of first function and the output of second function

Contributions to the Literature

Function composition is a useful mathematical concept, but is often treated as an afterthought in current precalculus texts. Little prior research has focused on student knowledge or use of function composition, except to note that many students have difficulty with the concept, and to relate this difficulty to students’ understanding of the concept of function. This study contributes to the sparse body of research related to understanding function composition, and applying function composition in solving novel word problems. More specifically, this study highlights the importance of quantitative reasoning and students' willingness to engage in acts of meaning making until they accurately conceptualized the quantities in the study. The results from analyzing student thinking has provided an initial characterization of the mental actions students perform when solving novel word problems involving function composition, and highlights important understandings that facilitate these mental actions. In addition, this study provides insights into student learning that should contribute to improved teaching of function composition, and curricular refinements for teaching the concept meaningfully.

Limitations of the Study

Some of the tasks used in this study included questions that required students to determine the inverse function, in addition to composing two functions. As a result, the obstacles that students experienced when determining and defining the inverse of a function may impact some of my findings about function composition.

The students who participated in this study were chosen with the intent that they represent typical precalculus students. Students' PCA scores were used in an effort to choose students of different mathematical ability levels. However, there is no guarantee that these students were indeed typical, or that the results obtained by studying these students are applicable to broader populations of students. In particular, it was not possible to choose students randomly for this study, as the students engaged in a certain degree of self-selection by volunteering to participate (that is, students who didn't want to participate in this study couldn't be forced to participate). In addition, the relatively small number of students studied makes it difficult to make any useful statistical claims.

The students who participated in this study were all female. This has the important consequence that any gender-related factors are not visible in this study. This issue was considered when choosing students to participate in this study, but I decided to proceed with an all-female group for three reasons. First, the small total number of students being studied (three) would make it difficult to isolate factors related to gender even if the group of subjects had been a mixed-gender group. Second, the pool of students willing to participate included very few males. Third, gender issues were not a focus of this study.

Another limitation arises from the fact that the study was situated in a class that used a specialized curriculum that emphasized quantitative reasoning, covariation, and problem solving. As a result, students in this study may have experienced a precalculus class that was significantly different from that experienced by the "typical" student.

The theoretical perspective described in earlier chapters points to another limitation of this study. I performed conceptual analyses of students' actions when they performed tasks in their clinical interviews, with the intent of characterizing students' knowledge and thinking. However, conceptual analysis, as rooted in radical constructivism, holds that students' mathematical realities are fundamentally unknowable. This means that while I believe my analyses explain students' actions, it is not possible to guarantee that students were really thinking what I believe they were thinking.

A related limitation of this study is due to the use of videotape to analyze of student thinking that occurred during the interviews. The objects of analysis were not the students' actions themselves; rather, the analysis was performed on videotapes of student actions, photocopies of the students' written work, and my own handwritten notes taken during the interviews. Thus, the analysis was unavoidably influenced by my own prior decisions about what was important to capture, and by my own real-time interpretations about what was noteworthy during student interviews. I attempted to mitigate this weakness as much as possible by having an external observer watching and listening remotely, and comparing notes after each interview. However, while I believe this helped minimize this limitation of the study, it is not possible to completely eliminate this limitation.

This study begins to characterize student knowledge and use of function composition. However, it does not fully explain all aspects of learning, knowing, and teaching function composition. Much additional research is needed before such a claim can be made.

Future Research

Several lines of possible future research are suggested by this study. For example, additional research is needed to understand why some students persist in repeated acts to make sense of problem statements, and conceptualize quantities and quantitative relationships, while other students do not.

In light of the insights gained from analyzing your data, I offer some specific suggestions for areas where future studies should probe more deeply. In particular, more probing/focus is needed in future studies to gain insights about:

1. The mechanisms by which students acquire a process view of function, and what mental operations are entailed in making this acquisition, and to what degree this conceptualization is necessary for understanding and using function composition;
2. How students conceptualize variables in the context of attempting to build a composed function (e.g., were students seeing the variables as static or as representing varying values that a quantity could assume?); and
3. How students see the product of composing two functions.

Further research is needed to understand how students' understanding of function composition with polynomial functions (as articulated in the above framework) relates to student use of arguments in exponential and trigonometric functions (e.g., $f(\theta) = \sin\left(3\theta + \frac{\pi}{4}\right)$, $g(x) = 14(1.2)^{\frac{x}{3}}$). Understanding how the mental actions for composing polynomial functions relates to composing functions in more complex exponential and trigonometry functions could be useful in supporting greater student learning of these concepts.

Future research is also needed to understand how students conceptualize the composed function, and its relationship to concatenating two function processes. Because students had so much difficulty in conceptualizing and concatenating two function processes for the purpose of relating two quantities that could not be directly related, this study did not explore students' views of the composed function and how it relates to the two function processes.

Implications for Curriculum and Instruction

There is a tremendous need for curriculum to place increased focus on developing the reasoning patterns and understandings articulated in the function composition framework. As noted in Chapter 1, most precalculus curriculum has one or two sections focused on function composition of polynomial functions, with little opportunities for students to solidify their understanding of this important concept or to see how it is related to the use of arguments in exponential and trigonometric functions. Students need more opportunities to develop their confidence in and ability to conceptualize the words conveyed in a problem context. They also need greater support in conceptualizing quantities and how they are related in specific contexts. This will require a major shift in the nature of the problems and the presentations that appear in many widely used precalculus textbooks. The function composition framework presented earlier in this chapter should be useful for curriculum developers, and in developing more meaningful professional development to strengthen teachers' understanding of the processes by which students acquire the ability to understand and use function composition to solve novel problems.

This study also leads me to question the decision made by the writers of the Common Core State Standards in Mathematics (CCSSM) to make the ideas of function composition and function inverse *plus standards*. The ideas of function composition and function inverse are important and complex, and I call on curriculum developers to consider ways to support their longitudinal development from Algebra I. Such longitudinal development should continue from what it means to generalize the process of solving for the output variable in Algebra I, to applying function composition reasoning when modeling more complex exponential and trigonometric functions in precalculus).

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APPENDIX A

EXPLORATORY STUDY PRE-INTERVIEW PROTOCOL

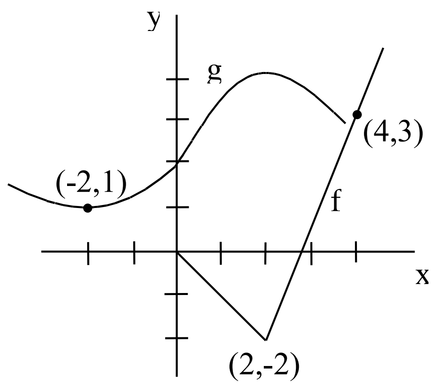
Problems:

Find the area of a square as a function of its perimeter. (Carlson *et al*., 2010)

Given the function $f(x) = 3x + 1$ and $g(x) = x^2$, evaluate $f\big(g(3)\big)$.

Express the diameter of a circle as a function of its area (Carlson, 1998).

Use the graphs of f and g to evaluate $g(f(2))$. (Carlson *et al*., 2010)



Using the data provided in the table below, find the following values:

x	1	2	3	4	5	6	7
$f(x)$	-5	-1	2	-1	0	4	-10
$g(x)$	2	1	4	7	-1	3	5

$f\big(g(1)\big)$
$g\big(f(4)\big)$
$g\big(g(7)\big)$

A therapeutic drug has the side affect of raising a patient's heart rate. The following table shows a patient's heart rate, r, as a function of the level of the drug Q, or $r = f(Q)$.

Q, drug level (mg)	0	50	100	150	200	250
$f(Q)$, heart rate (beats per minute)	60	70	80	90	100	110

Find $f(100)$ and describe the meaning of this value in this situation.

The level of the drug in a patient's bloodstream falls over time. The following table gives the drug level Q as a function of time, t, since the drug was administered, or $Q = g(t)$.

t, time (hours)	0	1	3	4	7	12
$g(t)$, drug level (mg)	250	200	150	100	50	0

Consider $f(g(t))$. Using the tables above, describe the following: What quantity represents the input for this function? What quantity represents the output for this function?

Input:
Output:

Find $f(g(12))$ and describe the meaning of this value in this situation.

APPENDIX B

EXPLORATORY STUDY POWERPOINT

Functions

… linking functions together

ASU
Arizona State University

Growing Circles

A pebble is thrown into a lake and the ripple travels outward at 0.7 meters/second. How can you determine the area inside the ripple at any time?

- *What* quantities are varying (changing) in the situation?
- *How* are these quantities changing?
- In your groups,
  - Draw a picture of the situation
  - Label quantities that are unknown and quantities that are known
  - Note the quantities that you've been asked to relate.
  - How would you determine the area inside the ripple if you are only able to measure the time since the pebble hit the water? Describe this in words, without using any algebraic equations…

ASU
Arizona State University

Growing Circles

A pebble is thrown into a lake and the ripple travels outward at 0.7 meters/second. How can you determine the area inside the ripple at any time?

- Write a formula for the following function relationships:
  - Radius as a function of time
  - Area as a function of radius

- Graph each function:
  - Radius as a function of time
  - Area as a function of radius

ASU
Arizona State University

Growing Circles

A pebble is thrown into a lake and the ripple travels outward at 0.7 meters/second.

What is the area of the circle 6 seconds after the pebble hit the water? In what ways can we determine this?

- Verify your answer using the graphs of $f(t)$ and $g(r)$.
- Verify your answer using the algebraic functions $f(t)$ and $g(r)$.
- Using input and output language, describe the processes of $f(t)$ and $g(r)$.

ASU
Arizona State University

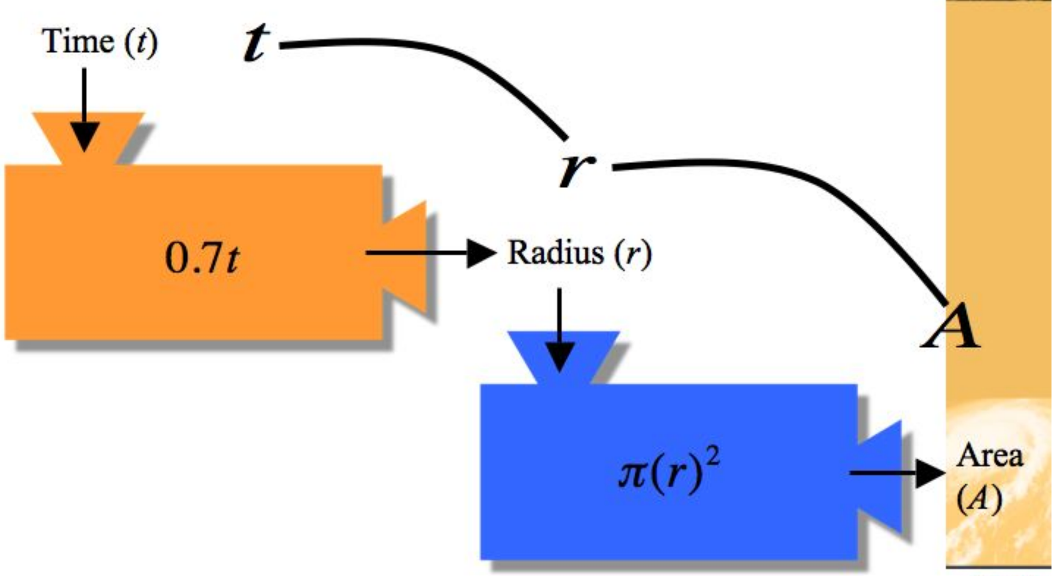
Growing Circles

- Area as a function of time…we need to know that:

  $r = f(t) = 0.7t$ and $A = g(r) = \pi r^2$

- What might $g(f(t))$ mean?
  - We have used the output (the length of the radius) $f(t)$ as the input (the length of the radius) to $g(r)$. This is referred to as *function composition*. $g(f(t))$ represents the area A at time t.

Growing Circles



Growing Circles

What about area as a function of time - that is, the composed function $g(f(t))$?

Let's call the new composed function, *h*, such that

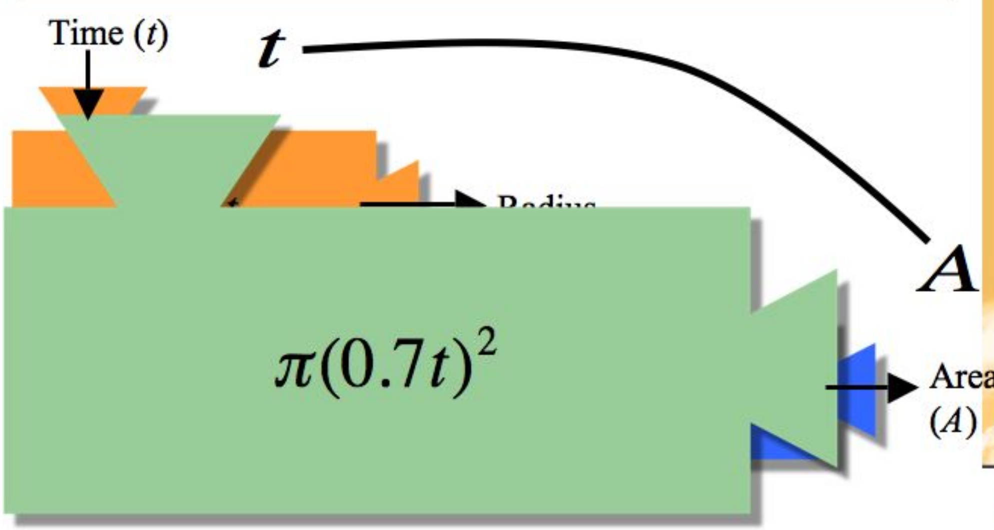
$$h(t) = g(f(t))$$

Since the final output of *h(t)* is area *A*, we can say that

$$A = h(t) = g(f(t)) = \pi(0.7t)^2$$

Growing Circles

What about area as a function of time - that is, the composed function $g(f(t))$?



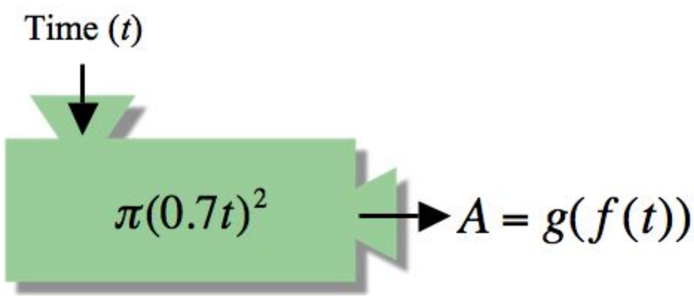
Growing Circles

- The composed function $g(f(t))$ is…

$$A = h(t) = g(f(t)) = \pi(0.7t)^2$$

- We also denote the composed function with the following notation:

$$A = (g \circ f)(t) = g(f(t)) = \pi(0.7t)^2$$



Composition: Using a table

Use the following table to determine

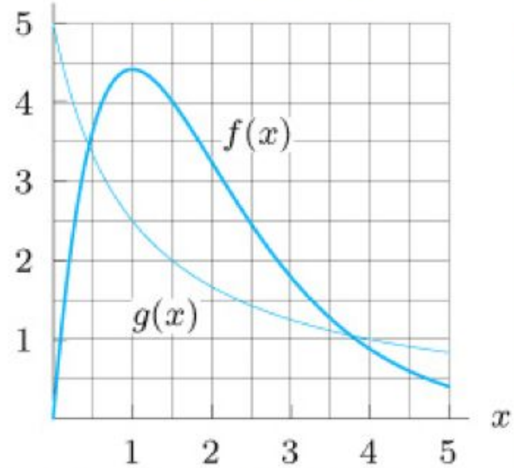
1. $g(f(-1))$
2. $f(f(3))$
3. $g(g(0))$
4. $f(g(3))$
5. $f^{-1}(3)$
6. $g^{-1}(f(3))$
7. $f^{-1}(f^{-1}(3))$

x	$f(x)$	$g(x)$
-2	0	5
-1	3	3
0	4	2
1	-1	1
2	6	-1
3	-2	0

Composition: Using a graph

Find the following:

1. $f(g(1))$
2. $g(f(1.5))$
3. $g(g(3))$
4. $f(f(0.5))$
5. $f(g(0))$



As x increases from 0 to 5, how does $g(x)$ change?

As x increases from 0 to 5, how does $f(g(x))$ change?

As x increases from 0 to 5, how does $f(x)$ change?

As x increases from 0 to 5, how does $g(f(x))$ change?

ARIZONA STATE UNIVERSITY

APPENDIX C

EXPLORATORY STUDY POST-INTERVIEW PROTOCOL

Task 1

A local Wisconsin ice cream shop tries to track the amount of customers and wait time based on the outside temperature. The following table gives the expected wait time W as a function of the number of customers, c, or $W = f(c)$.

c, number of customers	30	45	70	90	110	155
$W = f(c)$, average wait time per customer (in minutes)	5	15	30	40	50	70

The number of customers increases as the outside temperature increases. The following table shows the number of customers, c, as a function of the outside temperature t, or $c = g(t)$.

t, outside temperature (Fahrenheit)	40	50	60	70	80	90
$c = g(t)$, Number of Customers	30	45	70	90	110	155

Consider $f(g(t))$. Using the tables above, describe the following: What quantity represents the input for this composite function? What quantity represents the output for this composite function?

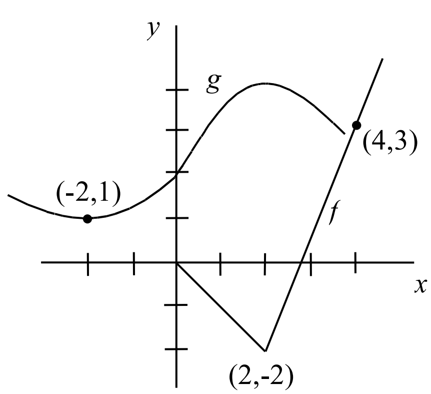
Find $f(g(50))$ and describe the meaning of this value in this situation.

Task 2

Given the function $h(x) = 3x - 1$ and $g(x) = x^2$, evaluate $g(h(2))$.

Task 3

Use the graphs of f and g to evaluate $g(f(2))$.



Task 4

Using the data provided in the table below, find the following values:

x	1	2	3	4	5	6	7
$f(x)$	-5	-1	2	-1	0	4	-10
$g(x)$	2	1	4	7	-1	3	5

a. $f(g(1))$ b. $g(f(3))$ c. $g(g(7))$

Task 5

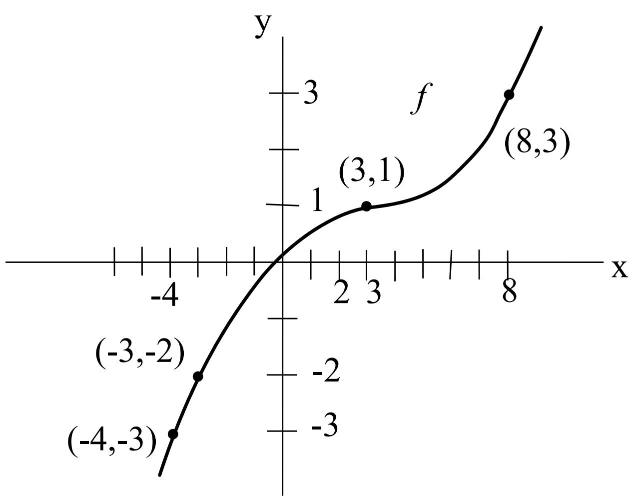
Using the functions g and h defined by the formulas, $g(x) = \dfrac{2}{3x-1}$ and $h(x) = x^3 + 6$, find $g(h(x))$.

Task 6

Using the graph of f, determine the following:

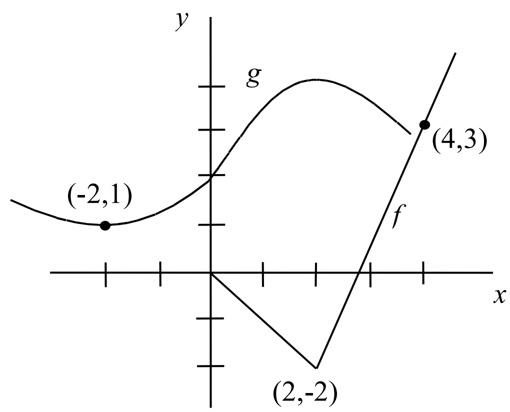
$f(f(-4))=$

$f^{-1}(f(8))=$



Task 7

Using the graphs of *f* and *g*, explain how $g(f(x))$ changes as x increases from 0 to 2.



Task 8

A pebble is thrown into a lake and the ripple travels outward at 3.1 inches per second.

Determine a function that describes the area of the circle, *A*, in terms of time, *t*.

Find the area inside the ripple 6.4 seconds after the pebble hits the water.

How long does it take for the circle to reach an area of 45 square inches?

Task 9

Consider the computer applet that depicts two functions, $f(x)$ and $g(x)$.

Experiment with moving the point on the *x*-axis of the graph of *f*. When you move this point a small amount, what does this mean with regard to the value of the input (*x*) to function *f*?

How does the value of the output of *f* change in response to small changes in the input to *f*? How do you know this?

How is the composed function $g(f(x))$ depicted in this computer applet? Please explain.

How could you use this computer applet to find the value of $g(f(x))$ for a particular value of *x*?

Using this computer applet, explain how the output of the composite function $g(f(x))$ varies as the value of x varies.

Task 10

Now consider the second applet, which shows two functions, s and u, graphed on the same set of axes.

Could you use this computer applet to find $s(u(x))$ for a particular value of x? If so, how? If not, why not?

Can you show me how you might use this computer applet to describe how the output of $s(u(x))$ varies as x increases from 4.1 to 4.4?

What role, if any, does the line $y = x$ play in the composite function $s(u(x))$?

Task 11

Find the area of a square as a function of its perimeter. Use function notation, and describe the quantities that are the inputs and outputs of any functions you create.

Task 12

Express the diameter of a circle as a function of its area. Use function notation, and describe the quantities that are the inputs and outputs of any functions you create.

APPENDIX D

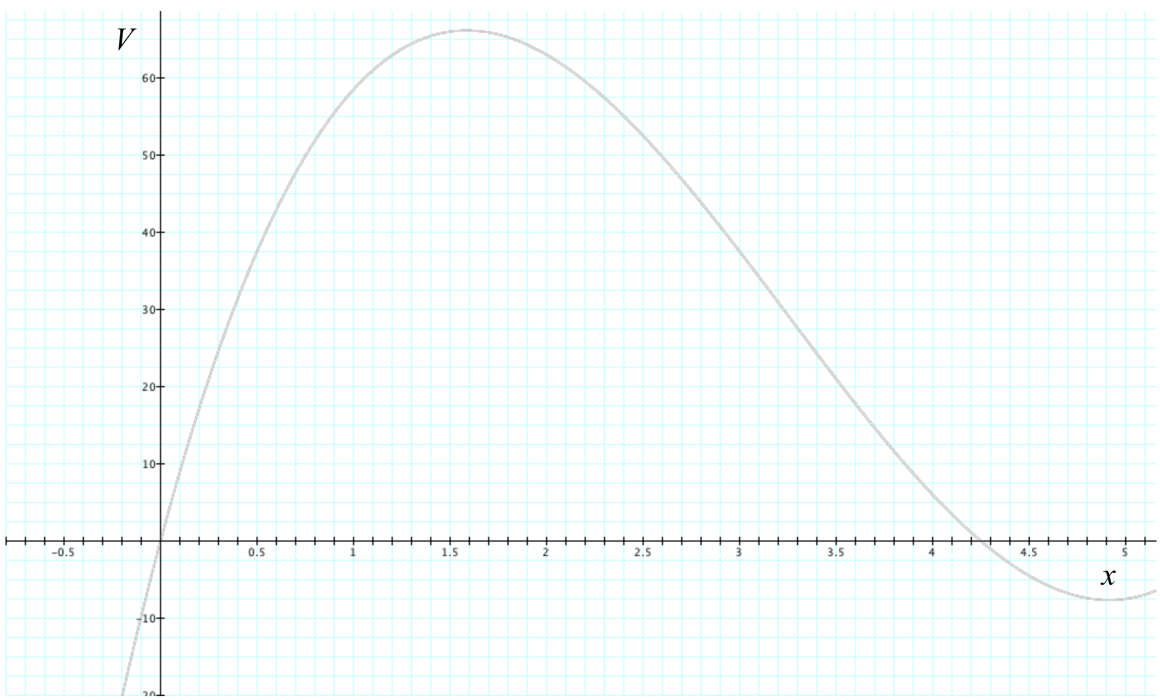
DETAILED INTERVIEW TASKS

Task 1 – The Dinner Problem:
Two friends that live 42 miles apart decide to meet for dinner at a location half way between them. The first friend, Tom, leaves his house at 6:05 and drives an average speed of 34 miles per hour on his way to the restaurant. Task 1a: If the second friend, Matt, leaves at 6:10, what average speed will he need to travel to arrive at the same time as Tom? Task 1b: If before leaving Matt knows that he averages driving 15 miles per hour to the restaurant, what time would he have to leave to arrive at the same time as Tom?

Task 2 – The Box Problem:

Starting with an 8.5” x 11” sheet of paper, a box is formed by cutting equal-sized squares from each corner of the paper and folding the sides up.

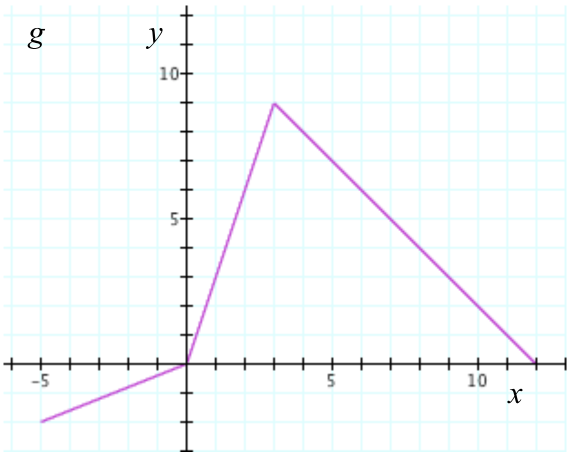
Task 2a: Describe how the length of the side of the cutout and the volume of the box covary.
Task 2b: Write a formula that predicts the volume of the box from the length of the side of the cutout.
Task 2c: Given the graph below, how does the volume change as the length of the side of the cutout varies from 1.8 inches to 1.9 inches?

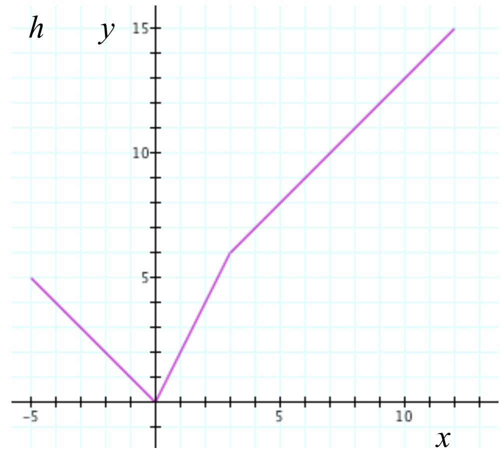


Task 2d: Use a formula to determine how much the volume changes as the length of the side of the cutout varies from 1.8 inches to 1.9 inches.

Task 3 – The Graphical Composition Problem:

Functions g and h are defined by the graphs below.





Task 3a: Determine each of the following:

i) $h(g(1))$

ii) $g(h(5))$

iii) $h(h(2))$

Task 3b: How does the output $h(g(x))$ vary as x varies from 5 to 9?

Task 3c: How does x vary as $h(x)$ varies from 10 to 15?

Task 3d: How does x vary as $h(g(x))$ varies from 6 to 10?

Task 4 – The Salary Problem:
Using function notation, suppose S(m) represents the monthly salary, in hundreds of dollars, of an employee after m months on the job. What would the function determined by R(m) = S(m + 12) represent? a) the salary of an employee after m + 12 months on the job b) the salary of an employee after 12 months on the job c) $12 more than the salary of someone who has worked for m months d) an employee who has worked for m + 12 months e) The salary after m months is the same as the salary after m + 12 months.

Task 5 – The Circle Problem:
Express the circumference of a circle as a function of the area of the circle.

Task 6 – The Giraffe Pen Problem:
Bryan has decided to open a wildlife park, and he would like to build a square pen for some giraffes. Bryan hasn't decided how many giraffes to acquire, but he has been told he should allow 10,000 square feet for each giraffe to graze. Help Bryan design his park – find a way to determine how many feet of fencing he must buy to enclose his square pen, based on the number of giraffes that will live in the pen.

APPENDIX E

CONSENT FORM

Project Pathways: Bowling Research Project
Letter of Consent

Dear Student:

I am a graduate student in the College of Liberal Arts and Sciences at Arizona State University working under the direction of Dr. Marilyn Carlson, Principal Investigator for ***Project Pathways***. I am conducting a research study to investigate the learning of important concepts in algebra and the impact of new curricular materials being developed by the Project Pathways research team.

I am requesting your participation, which may involve some or all of the following: (1) taking brief assessments; (2) participating in videotaped interviews, surveys, and observations as agreed upon; (3) teaching experiment sessions; and (4) allowing all of your written work to be duplicated for research purposes. The results of this research study may be published or presented at research conferences, but your name will not be used. You will be compensated at the rate of $20/hr for each hour you participate in this research. This funding will be paid by the NSF Project Pathways grant.

Your participation in this study is voluntary. If you choose not to participate, there will be no penalty, and it will not affect your grade in any ASU class. You may choose to withdraw from the study at any time.

If you have any questions concerning the research study, please call me at (480) 220-8935 or email me at stacey.bowling@asu.edu.

Sincerely,
Stacey A. Bowling

I am at least 18 years of age:

By signing below you are giving consent to participate in the above study.

Signature Printed Name Date

If you have any questions about your rights as a subject/participant in this research, or if you feel you have been placed at risk, you can contact the Chair of the Human Subjects Institutional Review Board, through the ASU Research Compliance Office, at (480) 965-6788.