Persistence of Discrete Dynamical Systems in Infinite Dimensional State Spaces

by

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A Dissertation Presented in Partial Fulfillment
of the Requirement for the Degree
Doctor of Philosophy

Approved April 2014 by the
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ARIZONA STATE UNIVERSITY

May 2014

ABSTRACT

Persistence theory plays an important part in population biology study. It provides a mathematically rigorous answer to the question of persistence by establishing an initial-condition-independent positive lower bound for the long-term value of a component of a dynamical system such as population size or disease prevalence. The notion of a semiflow is used to give the dynamics. A semiflow consists of a state space, X, a time-set, J, and a semiflow map ϕ. The state space X can be finite dimensional or infinite dimensional. Our work mainly focus on the case when X is an infinite dimensional space. Since our interests are concentrated on biological systems, X is required to have some positivity properties. Hence the concept of a cone is introduced as the positive state space X_+. Details of a cone will be found in chapter 2. Chapter 3 studies the year-to-year development of a population in infinite dimensional state space. For a semiflow map F, a linear or homogeneous map A is studied as an approximation of F at 0, the extinction state. The spectral radius $\mathbf{r}(A)$ is a threshold between extinction and persistence. Chapter 4 is a generalization study for a stage structured population model of a plant populations. Chapter 5 discusses a model for a spatially distributed population of male and female individuals that mate and reproduce only once in their life during a very short reproductive season. The model within one year dispersion is modeled by a system of partial differential equations. For this partial differential equation, both Neumann boundary condition and Dirichlet boundary condition are considered. Chapter 6 provides an application to the example of a rank-structured population model with mating.

ACKNOWLEDGEMENTS

I would like to express my deepest appreciation to my advisor Professor Horst Thieme, who provided insightful guidance and persistent help during my whole Ph.D. study. Without him, this dissertation would not have been possible.

I also would like to thank my committee members, Professor Hal Smith, Professor John Spielberg, Professor John Quigg and Professor Fabio Milner for their invaluable support. In addition, thanks to Professor Kuang Yang, who is one of my former committee member.

I also thanks the financial support from National Science Foundation grant DMS-0715451.

At last, I am very much grateful to my family and friends, their company granted me a happy and interesting graduate student life.

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Chapter 1

INTRODUCTION

Persistence theory studies the problem whether a given species in a mathematical model of interacting species survives over the long term or not. Mathematically, we use the concept of a semiflow to describe the dynamics of the species population. A semiflow consists of a state space, X, a time-set, J, and a map Φ.

The state space X comprehends all possible states of the system. It could be the amounts or densities of the system parts or structural distributions if there are one or several system structures. Mostly, we consider X as finite dimensional space if the system structures are finite, like patch models. However, when we study diffusion equations for spatial distributions, we will take X as an infinite dimensional space. We distinguish discrete and continuous time semiflow by the time set J: nonnegative integers and nonnegative real numbers.

The semiflow map is defined as $\Phi : J \times X \to X$. If $x \in X$ is the initial state of the system, then $\Phi(t, x)$ is the state at time t. Often Φ itself is called a semiflow. Semiflows are induced by differential equations of all kind (ordinary, partial, functional, and combinations of these).

Following the book Smith and Thieme (2011), we use a persistence function $\rho : X_+ \to \mathbb{R}_+$ to determine whether a system or a part of it persists or not. We say that a system is uniformly ρ-persistent if there exists some $\epsilon > 0$ such that

$$\liminf_{t\to\infty} \rho(\Phi(t, x)) \geq \epsilon.$$

for all $x \in X$ with $\rho(x) > 0$.

A system is called uniformly weakly ρ-persistent if there exists some $\epsilon > 0$ such that

$$\lim_{t \to \infty} \sup \rho(\Phi(t, x)) \geq \epsilon.$$

for all $x \in X$ with $\rho(x) > 0$.

In our work, we focus on those dynamical population systems which can be modeled as discrete semiflow in infinite dimensional space. By defining some appropriate persistence function, we give some general rules to determine a given population model's persistence property.

In the book (Smith and Thieme, 2011, Chapter 7), to study the year-to-year development of populations, a nonlinear matrix model is introduced:

$$x(n+1) = F(x(n)), n \in \mathbb{Z}_+,$$

where $F : \mathbb{R}^m_+ \to \mathbb{R}^m_+$. Chapter 3 generalizes the state space X from the finite dimensional case to the infinite dimensional case. To do that, we need to define a positive vector subset, and a positive operator to replace the state space $\mathbb{R}^n_+$ and semiflow map F; hence we introduce cone and homogeneous operator in the background material. Some specific positive operators F have properties such as the existence of forward invariant balls and compact attractors of bounded sets. For persistence property, we look into F with some special form and find a linearized operator A where A is an approximation of F at the extinction state, for example $F'(0) = A$ if F is differentiable. We will show that the spectral radius $\mathbf{r}(A)$ is a threshold between extinction and persistence.

Chapter 4 discusses the next generation model with the form $x_{n+1} = Ax_n + f(x_n)$. This model is generalized from Smith and Thieme (2013); the original model studies the dynamics of a stage structured population model for a plant populations $x_{n+1} = Ax_n + f(c^T x_n)b$, $n \geq 0$. We generalize the model step by step, first get rid of the linearity inside the function f, we have $x_{n+1} = Ax_n + \tilde{f}(x_n)b$ where $\tilde{f} : X_+ \to [0, \infty)$ and $b \in X_+$. Then we get rid of the b term, now the model has the form $x_{n+1} = Ax_n + \hat{f}(c^* x_n)$ where $\hat{f} : [0, \infty) \to X_+$ and c^* is a linear functional on X_+. We view these two models as special examples for our generalized model, they inherit all results from the model $x_{n+1} = Ax_n + f(x_n)$. Due to their own special property, they also have some extra results in stability and persistence study. Because these two examples have special linearized form, spectral radius can be calculated directly by the definition.

Chapter 5 studies a two-sex population model. Here we consider a model for a spatially distributed population of male and female individuals that mate and reproduce only once in their life during a very short reproductive season. The dispersion within one year is modeled by a partial differential equation and the next year offspring map is given by a mating and reproduction function ϕ. We consider both Neumann boundary and Dirichlet boundary conditions for the partial differential equation. For this particular model, we can still apply the theories we discussed in previous chapters. By introducing a reproduction number $\mathcal{R}_0$, we will prove that $\mathcal{R}_0$ can be used to determine whether the population persists or dies out. A part of chapter 5 (Neumann boundary condition) has been submitted as an article (Jin and Thieme, to appear).

Chapter 6 gives an example of a rank-structured population model with mating. For this specific example, we provide a way to find the spectral radius of the linearized map B. When the spectral radius is greater than 1, we prove a uniform ρ-persistence result.

Chapter 2

BACKGROUND MATERIAL ON HOMOGENEOUS MAPS ON CONES

2.1 Cones

Definition 2.1. A closed subset X_+ of a normed real vector space X is called a *wedge* if

(i) X_+ is convex.

(ii) $\alpha x \in X_+$ whenever $x \in X_+$ and $\alpha \in \mathbb{R}_+$.

Definition 2.2. A wedge $X+$ is called a *cone* if

(iii) $X_+ \cap (-X_+) = \{0\}$.

Nonzero points in a cone or wedge are called *positive*.

A wedge is called *solid* if it contains interior points.

A wedge is called *reproducing* (also called *generating*) if $X = X_+ - X_+$, and *total* if X is the closure of $X_+ - X_+$.

A cone X_+ is called *normal*, if there exists some $\delta > 0$ such that

$$\|x + z\| \geq \delta \text{ whenever } x \in X_+, z \in X_+, \|x\| = 1 = \|z\|. \tag{2.1}$$

In function spaces, typical cones are formed by the nonnegative functions. For more information about cones see Krasnoselskiĭ (1964); Krasnoselskiĭ *et al.* (1989); Lemmens and Nussbaum (2012); Thieme (2013).

A map B on X with $B(X_+) \subseteq X_+$ is called a *positive* map.

If X_+ is a cone in X, we introduce a *partial order* on X by $x \leq y$ if $y - x \in X_+$ for $x, y \in X$.

Definition 2.3. Let X and Z be normed ordered vector spaces with cones X_+ and Z_+ and $U \subseteq X$. A map $B : U \to Z$ is called *order preserving* (or *monotone* or *increasing*) if $Bx \le By$ whenever $x, y \in U$ and $x \le y$.

Positive linear maps are order-preserving.

2.2 Homogeneous Maps

In the following, X, Y and Z are ordered normed vector spaces with cones X_+, Y_+ and Z_+ respectively.

Definition 2.4. $B : X_+ \to Y$ is called *(positively) homogeneous (of degree one)*, if $B(\alpha x) = \alpha Bx$ for all $\alpha \in \mathbb{R}_+$, $x \in X_+$.

Since we do not consider maps that are homogeneous in other ways, we will simply call them homogeneous maps. It follows from the definition that

$$B(0) = 0.$$

Homogeneous maps are not Frechet differentiable at 0 unless $B(x+y) = B(x)+B(y)$ for all $x, y \in X_+$. For the following holds.

Proposition 2.5. *Let $B : X_+ \to Y$ be homogeneous. Then the directional derivatives of B exist at 0 in all directions of the cone and*

$$\partial B(0, x) = \lim_{t \to 0_+} \frac{B(tx) - B(0)}{t} = B(x), \qquad x \in X_+.$$

Theorem 2.6. *Let $F : X_+ \to Y$ and $u \in X$. Assume that the directional derivatives of F at u exist in all directions of the cone. Then the map $B : X_+ \to Y_+, B = \partial F(u, \cdot)$,*

$$B(x) = \partial F(u, x) = \lim_{t \to 0_+} \frac{F(u + tx) - F(u)}{t}, \qquad x \in X_+,$$

is homogeneous.

Proof. Let $\alpha \in \mathbb{R}_+$. Obviously, if $\alpha = 0$, $B(\alpha x) = 0 = \alpha B(x)$. So we assume $\alpha \in (0, \infty)$. Then

$$\frac{F(u + t[\alpha x]) - F(u)}{t} = \alpha \frac{F(u + [t\alpha]x) - F(u)}{t\alpha}.$$

As $t \to 0$, also $\alpha t \to 0$ and so the directional derivative in direction αx exists and

$$\partial F(u, \alpha x) = \alpha \partial F(u, x).$$

□

For a homogeneous map $B : X_+ \to X_+$, we define

$$\|B\|_+ = \sup\{\|Bx\|; x \in X_+, \|x\| \leq 1\} \tag{2.2}$$

and call B *bounded* if this supremum is a real number. Since B is homogeneous,

$$\|Bx\| \leq \|B\|_+ \|x\|, \qquad x \in X_+. \tag{2.3}$$

Let $H(X_+, Y)$ denote the set of bounded homogeneous maps $B : X_+ \to Y$ and $H(X_+, Y_+)$ denote the set of bounded homogeneous maps $B : X_+ \to Y_+$ and $HM(X_+, Y_+)$ the set of those maps in $H(X_+, Y_+)$ that are also order-preserving.

$H(X_+, Y)$ is a real vector space and $\| \cdot \|_+$ is a norm on $H(X_+, Y)$; $H(X_+, Y_+)$ and $HM(X_+, Y_+)$ are cones in $H(X_+, Y)$. We write $H(X_+) = H(X_+, X_+)$ and $HM(X_+) = HM(X_+, X_+)$.

It follows for $B \in H(X_+, Y_+)$ and $C \in H(Y_+, Z_+)$ that $CB \in H(X_+, Z_+)$ and

$$\|CB\|_+ \leq \|C\|_+ \|B\|_+.$$

2.3 Cone Spectral Radius and Orbital Spectral Radius

Definition 2.7. We define the *cone spectral radius* of B as

$$\mathbf{r}_+(B) := \inf_{n \in \mathbb{N}} \|B^n\|_+^{1/n} = \lim_{n \to \infty} \|B^n\|_+^{1/n}. \tag{2.4}$$

Theorem 2.8 (Mallet-Paret, Nussbaum, 2010). *Let X_+ be a reproducing cone in the ordered Banach space X. Then there exists some $c \geq 1$ such that, for all bounded linear positive maps B on X, $\|B\|_+ \leq \|B\| \leq |c|\|B\|_+$, $\mathbf{r}_+(B) = \mathbf{r}(B)$.*

Definition 2.9 (Förster, Nagy 1989). We define the local spectral radius of B at x by

$$\gamma_B(x) := \limsup_{n\to\infty} \|B^n(x)\|^{1/n}, \qquad x \in X_+.$$

We define the orbital spectral radius by

$$\mathbf{r}_o(B) = \sup_{x\in X_+} \gamma_B(x).$$

(Mallet-Paret, Nussbaum 2002).

Theorem 2.10 (Mallet-Paret, Nussbaum, 2002). *Let B be bounded and homogeneous. Then, obviously, $\mathbf{r}_o(B) \leq \mathbf{r}_+(B)$. If X_+ is complete and normal and B is also continuous and order preserving, then $\mathbf{r}_+(B) = \mathbf{r}_o(B)$.*

2.4 Order Bounded Operators

Definition 2.11 (Adapted from Krasnosel'skii(1964) and coworkers(1989)). Let $B : X_+ \to X_+$, $u \in X_+$. B is called *pointwise u-bounded* if for any $x \in X_+$ there exists some $n \in \mathbb{N}$ and $\gamma > 0$ such that $B^n x \leq \gamma u$. B is called *uniformly u-bounded* if there exists some $c > 0$ such that $Bx \leq c\|x\|u$ for all $x \in X_+$.

Lemma 2.12. *Let X_+ be complete and $B : X_+ \to X_+$ be continuous, order preserving and homogeneous. Let $u \in X_+$ and B be pointwise u-bounded. Then some power of B is uniformly u-bounded.*

Proof. See (Thieme, 2013, Proposition 4.4). □

Theorem 2.13. *Let X_+ be normal and $B : X_+ \to X_+$ be homogeneous and order-preserving. Let $u \in X_+$.*

(a) If B is pointwise u-bounded, then

$$\mathbf{r}_o(B) = \lim_{n\to\infty} \|B^n u\|^{1/n} = \gamma_B(u).$$

(b) If B^m is uniformly u-bounded for some $m \in \mathbb{N}$, then

$$\mathbf{r}_+(B) = \lim_{n\to\infty} \|B^n u\|^{1/n} = \mathbf{r}_o(B).$$

Proof. See (Thieme, 2013, Theorem 4.5). □

2.5 Existence of Eigenvectors

Definition 2.14. We call B is *u-positive*, if for any $x \in X$, $x \neq 0$, there exists some $n \in \mathbb{N}$ and $\epsilon > 0$ such that $B^n x \geq \epsilon u$.

The following theorem has essentially been proved by Nussbaum in (Nussbaum, 1981, Theorem.2.1) but some finishing touches are contained in the introduction of Lemmens and Nussbaum (2013).

Theorem 2.15. *Let X_+ be the cone of a normed vector space and $B : X_+ \to X_+$ be homogeneous, continuous, order-preserving and compact. Assume that $r := \mathbf{r}_+(B) > 0$. Then there exists some $v \in X_+$, $v \neq 0$, such that $Bv = rv$.*

Chapter 3

PERSISTENCE AND STABILITY FOR DISCRETE TIME POPULATION MODELS WITH INFINITE DIMENSIONAL STRUCTURE

To extend the theories from finite dimensional space to infinite dimensional space, we replace the positive orthant of an Euclidean space by the positive cone of an ordered normed vector space, and also replace nonnegative matrices by positive linear bounded operators.

The year-to-year development of populations is often modeled by systems

$$x(n+1) = F(x(n)), \quad n \in \mathbb{Z}_+ \tag{3.1}$$

where $F : X_+ \to X_+$.

3.1 Forward Invariant Balls and Compact Attractors of Bounded Sets

Lemma 3.1. *Assume that $F : X_+ \to X_+$ is continuous and that there exists some $R > 0$ such that $\|F(x)\| \le \|x\|$ whenever $\|x\| \ge R$.*

(a) Then there exists some $\tilde{R} > 0$ such that

$$F(X_+ \cap \bar{B}_s) \subset \bar{B}_s, \qquad s \ge \tilde{R},$$

for all closed balls $\bar{B}_s$ with radius s and the origin as center.

(b) If $\|F(x)\| < \xi\|x\|$ whenever $\|x\| \ge R$, $\xi < 1$ then

$$\liminf_{n\to\infty} \|F^n(x)\| \le R, \qquad \limsup_{n\to\infty} \|F^n(x)\| \le \max\{\tilde{R}, R\}.$$

Proof. The proof is similar to the one for the finite dimensional case (Smith and Thieme, 2011, Lemma 7.1).

(a) Suppose that such an $\tilde{R} > 0$ does not exist. Then there exist sequences (s_n) in $\mathbb{R}_+$ and (x_n) in X_+ such that $s_n \to \infty$ and $\|x_n\| \leq s_n < \|F(x_n)\|$. Since $s_n > R$ for large n, this is a contradiction.

(b) Suppose that $x \in X_+$ and $\liminf_{n\to\infty} \|F^n(x)\| > R$. Then there exists some $N \in \mathbb{N}$ such that $\|F^n(x)\| > R$ for all $n \geq N$. By assumption, $\|F(F^n(x))\| < \xi\|F^n(x)\|$, $n \geq N$. By induction, $\|F^{N+n}(x)\| < \xi^n\|F(x(N))\|$ for all $n \in \mathbb{N}$ and $F^n(x) \to 0$ as $n \to \infty$, a contradiction.

In particular, for each $\epsilon > 0$, there exists some $n \in \mathbb{N}$ such that $\|F^n(x)\| \leq R + \epsilon$. Let $\hat{R} = \max\{R + \epsilon, \tilde{R}\}$. By part (a), $\|F^j(x)\| \leq \hat{R}$ for all $j \geq n$. Since $\epsilon > 0$ was arbitrary, the last assertion follows. □

Proposition 3.2. *Let X_+ be an ordered Banach space, assume the norm $\|\cdot\|$ defined on X_+ is monotone.*

If A is a positive bounded linear operator on X_+, then for every $r > \mathbf{r}(A)$, there exists a norm $\|\cdot\|'$ on X_+ such that

$$\|Ax\|' \leq r\|x\|', \qquad x \in X_+.$$

Further, $\|\cdot\|'$ is monotone, i.e., $\|x\|' \leq \|y\|'$ whenever $0 \leq x \leq y$.

Proof. The proof is similar to the one for the finite dimensional case (Smith and Thieme, 2011, Proposition A.24).

Let $r > \mathbf{r}(A)$, since

$$r > \mathbf{r}(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}},$$

we could find some $N \in \mathbb{N}$ such that

$$r > \|A^n\|^{\frac{1}{n}}$$

for all $n \geq N$. Therefore

$$r^n > \|A^n\| \text{ and } \|A^n x\| \leq \|A^n\|\|x\| < r^n\|x\|$$

for all $n \geq N$, $x \in X_+$.

Define $\|\cdot\|'$ as

$$\|x\|' = \|x\| + \sum_{j=1}^{k} \|A^j x\|/r^j$$

where $k + 1 = N$.

First prove that if the original norm is monotone, then the new norm is monotone.

Since $\|\cdot\|$ is monotone, if $x < y$ for $x, y \in X_+$, then $\|x\| < \|y\|$. A is positive, so $Ax < Ay$ if $x < y$. Therefore

$$\|x\|' = \|x\| + \sum_{j=1}^{k} \|A^j x\|/r^j < \|y\| + \sum_{j=1}^{k} \|A^j y\|/r^j = \|y\|'$$

if $x < y$. So $\|\cdot\|'$ is monotone.

Then prove $\|Ax\|' < r\|x\|'$ for all $x \in X_+$.

$$\begin{aligned} \|Ax\|' &= \|Ax\| + \sum_{j=1}^{k} \|A^{j+1}x\|/r^j \\ &= \|Ax\| + \sum_{j=1}^{k-1} \|A^{j+1}x\|/r^j + \|A^{k+1}x\|/r^k \\ &< \|Ax\| + \sum_{j=1}^{k-1} \|A^{j+1}x\|/r^j + r^{k+1}\|x\|/r^k \\ &= r(\|x\| + \sum_{j=1}^{k} \|A^j x\|/r^j) = r\|x\|'. \end{aligned}$$

So $\|Ax\|' < r\|x\|'$. □

Proposition 3.3. *Let X_+ be an ordered Banach space, assume the norm $\|\cdot\|$ defined on X_+ is monotone.*

Let $F : X_+ \to X_+$ be continuous. Assume that there exists some $R > 0$, some element $y \in X_+$ and some linear positive bounded operator D such that $\mathbf{r}(D) < 1$ and

$$F(x) \leq y + Dx, \qquad x \in X_+, \qquad \|x\| \geq R.$$

If $R = 0$*, then*

$$x(n) = F^n(x) \leq \sum_{i=0}^{n-1} D^i y + D^n x(0) \to (I - D)^{-1} y, \qquad n \to \infty \tag{3.2}$$

In the general case $R \geq 0$*, after introducing an equivalent norm, there exists some* $\tilde{R} > 0$ *such that*

$$F(X_+ \cap \bar{B}_s) \subset \bar{B}_s, \quad s \geq \tilde{R},$$

for all closed balls $\bar{B}_s$ *with radius* $s \geq \tilde{R}$ *and the origin as center. Further, there exists some* $\hat{R} > 0$ *such that*

$$\limsup_{n\to\infty} \|F^n(x)\| \leq \hat{R}, \quad x \in X_+.$$

Proof. The proof is similar to the one for the finite dimensional case (Smith and Thieme, 2011, Proposition 7.2).

First assume $R = 0$. We have $x(n+1) \leq y + Dx(n)$, $n \geq 0$ and Lemma 3.2 follows by iteration because D is positive. The limit results from $(\mathbb{I} - D)^{-1} = \sum_{i=0}^{\infty} D^i \geq 0$ and $D^n \to 0$ because $\mathbf{r}(D) < 1$.

Now consider the general case $R \geq 0$. By Proposition 3.2, there exists some $\zeta \in (\mathbf{r}(D), 1)$ and a norm $\|\cdot\|'$ on X_+ such that

$$\|Dx\|' \leq \zeta \|x\|', \qquad x \in X_+.$$

Since the new norm is equivalent to the original one, there exists $R' > 0$ such that

$$F(x) \leq y + Dx, \qquad \|x\|' \geq R'.$$

If $\|x\|' \geq R'$,

$$\|F(x)\|' \leq \|y\|' + \|Dx\|' \leq \|y\|' + \zeta\|x\|'.$$

Choose some $\xi \in (\zeta, 1)$. Then, if $\|x\|' \geq R'$,

$$\|F(x)\|' \leq \|y\|' + (\zeta - \xi)\|x\| + \xi\|x\|' \leq \|y\|' + (\zeta - \xi)R' + \xi\|x\|'.$$

Increasing R' as needed such that $\|y\|' + (\zeta - \xi)R' < 0$, then we have

$$\|F(x)\|' \leq \|y\|' + (\zeta - \xi)R' + \xi\|x\|' < \xi\|x\|' < \|x\|'.$$

Then the assumptions of Lemma 3.1 are satisfied and the assertion follow. □

Corollary 3.4. *Make the assumptions of Proposition 3.3, and also assume that F can be written as $F = L + K$, where K is a compact operator and L is linear bounded positive operator such that $\mathbf{r}(L) < 1$. Then the semiflow induced by F has a compact attractor of bounded sets.*

Proof. If F can be written as $F = L + K$ where K is compact and L is a linear positive operator such that $\mathbf{r}(L) < 1$, we claim that for all $n \in \mathbb{N}$, $F^n = L^n + K^{(n)}$ where $K^{(n)}$ is compact.

We prove this claim by induction: For $n = 1$, $F = L + K$ where K is a compact operator.

Now assume the claim is true for $n = k$, $F^k = L^k + K^{(k)}$ where $K^{(k)}$ is compact.

For $n = k + 1$,

$$\begin{aligned} F^{k+1} &= (L + K)(L^k + K^{(k)}) = LL^k + LK^{(k)} + K(L^k + K^{(k)}) \\ &= L^{k+1} + LK^{(k)} + K(L^k + K^{(k)}). \end{aligned}$$

$K^{(k+1)} = LK^{(k)} + K(L^k + K^{(k)})$ is compact. Therefore the claim is proved.

Apply (Smith and Thieme, 2011, Theorem 2.46), since for all $n \in \mathbb{N}$, $F^n = L^n + K^{(n)}$, $K^{(n)}$ is compact, therefore it satisfies the second condition of (Smith and Thieme, 2011, Theorem 2.46).

For the first condition, we need to prove that

$$\liminf_{k \to \infty} diam(L^k(C)) = 0$$

for bounded closed set C. Since

$$diamL^k(C) = \sup\{d(x,y), x, y \in L^k(C)\}$$

and C closed, let $x, y \in L^k(C)$. Then, there exist $x_0, y_0 \in C$ such that $L^k(x_0) = x, L^k(y_0) = y$.

$$\begin{aligned} d(x,y) =& d(L^k(x_0) - L^k(y_0)) = d(L^k(x_0 - y_0)) = \|L^k(x_0 - y_0)\| \\ \leq& \|L^k\|\|x_0 - y_0\| \leq \|L^k\| diam(C). \end{aligned}$$

So

$$diamL^k(C) \leq \|L^k\| diam(C)$$

Since C is bounded closed set, $diam(C)$ is finite. Since $\mathbf{r}(L) < 1$, we can find some s such that $\mathbf{r}(L) < s < 1$.

$$\mathbf{r}(L) = \lim_{n\to\infty} \|L^n\|^{\frac{1}{n}} < s < 1,$$

so for n large enough, $\|L^n\| < s^n \to 0$. Therefore

$$\liminf_{k\to\infty} diam(L^k(C)) \leq \|L^k\| diam(C) \to 0.$$

By Proposition 3.3, the semiflow induced by F is point-dissipative and eventually bounded on bounded sets. So the semiflow has a compact attractor of bounded sets by (Smith and Thieme, 2011, Theorem 2.33). □

3.2 Uniform Persistence and Persistence Attractors

Theorem 3.5. *Let F be of a positive operator. Assume:*

(a) We have a linear positive operator A_0 satisfy the following properties: For any $\eta \in (0,1)$, there exists some $\delta > 0$ such that $F(x) \geq \eta A_0 x$ whenever $\|x\| \leq \delta$.

(b) There exists some positive linear functional v^* (for each $x \in X_+ \setminus \{0\}$, $v^* x > 0$) and $r_0 > 1$ such that $v^* A_0 \geq r_0 v^*$.

(c) If $v^*x > 0$, then $v^*F(x) > 0$.

(d) There exists some $m \in \mathbb{N}$ and $c > 0$ such that $\|F^m(x)\| \leq c|v^*x|$ for all $x \in X_+$.

Then the semiflow induced by F is uniformly weakly $\rho-$persistent for $\rho(x) = v^*x$.

Assume in addition:

(e) The semiflow induced by F has a compact attractor K which attracts all points in X.

Then the semiflow induced by F is uniformly $\rho-$persistent for $\rho(x) = v^*x$.

Proof. Let $x \in X_+$, $v^*x > 0$. Set

$$x(n) = F^n(x), n \in \mathbb{Z}_+.$$

By assumption (c),

$$v^*x(n) > 0$$

for all $n \in \mathbb{Z}_+$.

Let $\epsilon > 0$, to be determined later. Suppose that the semiflow induced by F is not uniformly weakly $\rho-$persistent. Then there exists some $N \in \mathbb{N}$ such that when $n > N$,

$$\limsup_{n>N} v^*x(n) < \epsilon.$$

Then after a shift in time, rewrite $x = x(N)$, we will have

$$0 < v^*x(n) < \epsilon$$

for all $n \in \mathbb{Z}_+$. By assumption (d), after another shift in time, rewrite $x = x(m)$, then

$$\|x(n)\| \leq c\epsilon$$

for all $n \in \mathbb{Z}_+$.

Choose $\eta \in (0,1)$ such that $\eta r_0 > 1$. By assumption (a), there exists some $\delta > 0$ such that

$$F(x) \geq \eta A_0 x$$

whenever $x \in X_+$, $\|x\| \leq \delta$. Choose $\epsilon > 0$ such that $c\epsilon \leq \delta$. Then

$$\|x(n)\| \leq \delta$$

for all $n \in \mathbb{N}$, and

$$\rho(x(n+1)) = v^* F(x(n)) \geq v^* \eta A_0 x(n) = \eta v^* A_0 x(n) \geq \eta r_0 v^* x(n) = \eta r_0 \rho(x(n)) > 0.$$

Since $\eta r_0 > 1$, $\rho(x_n) \to \infty$ as $n \to \infty$, a contradiction.

Therefore the semiflow induced by F is uniformly weakly $\rho-$persistent for $\rho(x) = v^* x$. By (e) and (Smith and Thieme, 2011, Theorem 4.5), the semiflow is uniformly $\rho-$persistent. □

Theorem 3.6. *Let F be differentiable at 0. Suppose that:*

(a) $X_+ \setminus \{0\}$ is forward invariant under F.

(b) There exists $r_0 > 1$ and v^* is a positive bounded linear functional on X (for each $x \in X_+ \setminus \{0\}$, $v^* x > 0$) such that $v^* A_0 \geq r_0 v^*$ where $A_0 = F'(0)$.

(c) There exists some $\eta > 0$ such that $v^* x > \eta \|x\|$ for all $x \in X_+$.

Then the semiflow induced by F is uniformly weakly ρ-persistent for $\rho(x) = v^* x$. Assume in addition:

(d) The semiflow induced by F has a compact attractor K which attracts all points in X.

Then the semiflow induced by F is uniformly $\rho-$persistent for $\rho(x) = v^* x$.

Proof. We need to apply (Smith and Thieme, 2011, Proposition 3.16) to get the result. Hypothesis (a) of (Smith and Thieme, 2011, Proposition 3.16) holds by assumptions (a) and (b).

As for Hypothesis (b) of (Smith and Thieme, 2011, Proposition 3.16), let $\varepsilon > 0$, since F is differentiable at 0, $A_0 = F'(0)$, there exists some $\delta > 0$ such that

$$\|x\| < \delta \Rightarrow \|F(x) - F(0) - A_0(x)\| \leq \varepsilon \|x\|.$$

By assumption (c), we have $\eta > 0$ such that $\rho(x) > \eta\|x\|$ for all $x \in X_+$.

$$\begin{aligned}
\rho(F(x)) &= v^*F(x) = v^*(A_0(x) + F(0) + F(x) - F(0) - A_0(x)) \\
&\geq v^*(A_0(x)) + v^*(F(x) - F(0) - A_0(x)) \\
&\geq v^*A_0(x) - \|v^*\|\|F(x) - F(0) - A_0(x)\| \\
&\geq v^*A_0(x) - \varepsilon\|v^*\|\|x\| \\
&> v^*A_0(x) - \frac{\varepsilon}{\eta}\|v^*\|v^*x \\
&\geq (r - \frac{\varepsilon}{\eta}\|v^*\|)\rho(x) \\
&= \gamma\rho(x)
\end{aligned}$$

with $\gamma > 1$ if $\varepsilon > 0$ and $\delta > 0$ are chosen sufficiently small. Hence $\frac{\rho(F(x))}{\rho(x)} = \gamma > 1$, hypothesis (b) of (Smith and Thieme, 2011, Proposition 3.16) follows. □

Theorem 3.7. *Let F be of nonlinear operator form $F(x) = A(x)x$, where $A(x)$ is homogeneous, bounded and continuously depending on $x \in X_+$, and X_+ is solid. Let $v \in \breve{X}_+$, define*

$$\|x\|_v = \inf\{\beta \geq 0 : -\beta v \leq x \leq \beta v\}.$$

Assume:

(a) F is forward invariant on $X_+ \setminus \{0\}$.

(b) For any $x \in X_+ \setminus \{0\}$*, there exists some* $m \in \mathbb{N}$ *such that* $F^m(x) \in \breve{X}_+$*.*

(c) There exist $r_0 > 1$ *and* $v \in \breve{X}_+$ *such that* $A(0)v \geq r_0 v$*.*

Then the semiflow induced by F *is uniformly weakly* $\rho-$*persistent for* $\rho(x) = \|x\|_v$*. Assume in addition:*

(d) The semiflow induced by F *has a compact attractor* K *which attracts all points in* X*.*

Then the semiflow induced by F *is uniformly* $\rho-$*persistent for* $\rho(x) = \|x\|_v$*.*

Proof. First we prove that there is some $c > 0$ such that $\|x\|_v \leq c\|x\|$ for all $x \in X$.

Since $v \in \breve{X}_+$, there exist some $\varepsilon > 0$ such that $v \pm \varepsilon u \geq 0$ for any $u \in X_+$ with $\|u\| = 1$. This implies $v \geq -\varepsilon u$ and $v \geq \varepsilon u$. Choose $u = \frac{x}{\|x\|}$, we have

$$v \geq -\varepsilon \frac{x}{\|x\|} \ \text{ and } \ v \geq \varepsilon \frac{x}{\|x\|}.$$

So

$$-v \leq \varepsilon \frac{x}{\|x\|} \leq v.$$

By definition of $\|\cdot\|_v$,

$$\|\varepsilon \frac{x}{\|x\|}\|_v \leq 1.$$

$\varepsilon\|x\|_v \leq \|x\|$. Choose $c = 1/\varepsilon$, we get $\|x\|_v \leq c\|x\|$ for all $x \in X$.

We introduce another functional:

$$[x]_v = \sup\{\alpha \geq 0 : \alpha v \leq x\}.$$

We claim that there exists some δ and $\gamma > 1$ such that $[F(x)]_v \geq \gamma[x]_v$ when $||x|| < \delta$.

$$\begin{aligned} F(x) &= A(x)x \geq A(x)([x]_v v) = [x]_v A(x)v \\ &= [x]_v(A(0) + A(x) - A(0))v \\ &\geq [x]_v(A(0) - \|((A(x) - A(0))v\|_v)v \\ &\geq [x]_v(r_0 - c\|(A(x) - A(0))v\|)v \\ &\geq [x]_v(r_0 - c\|A(x) - A(0)\|\|v\|)v. \end{aligned}$$

Since $A(x)$ is bounded, continuously depending on $x \in X_+$, we can choose x close to 0 such that$\|A(x) - A(0)\|$ small enough. Hence there exists some $\delta > 0$ such that if $||x|| < \delta$, $r_0 - \|A(x) - A(0)\|\|v\| > 1$.

Let

$$\gamma = r_0 - \|A(x) - A(0\|\|v\| > 1.$$

From above discussion, we have $F(x) \geq \gamma[x]_v v$, then $[F(x)]_v \geq \gamma[x]_v$.

Let $x \in X_+ \setminus \{0\}$, set $x(n) = F^n(x), n \in \mathbb{Z}_+$. By assumption (a), $\|x(n)\|_v > 0$ for all $n \in \mathbb{Z}_+$. By assumption (b), there exists some $x(m)$ such that $[x(m)]_v > 0$.

Now suppose the semiflow induced by F is not uniformly weakly $\rho-$persistent. Let $\epsilon > 0$ to be determined later, after a shift in time,

$$0 < \|x(n)\|_v < \epsilon$$

for all $n \in \mathbb{N}$. There exists some $\delta > 0$ such that $[F(x)]_v \geq \gamma[x]_v$ with $\gamma > 1$ if $\|x\| < \delta$. Choose $\epsilon < \delta$, then $\|x(n)\| < \delta$ for all $n \in \mathbb{N}$, and by assumption (b),

$$[F(x(n+m))]_v \geq \gamma^m [x(n)]_v > 0$$

for some m. After another shift in time, since $\gamma > 1$,

$$[F(x(n))]_v = \gamma^n [x]_v \to \infty$$

as $n \to \infty$, which contradict with

$$0 < [x(n)]_v < \|x(n)\|_v < \epsilon$$

for all $n \in \mathbb{N}$.

So F is uniformly weakly ρ-persistent.

By (Smith and Thieme, 2011, Theorem 4.5) and assumption (d), the semiflow is uniformly $\rho-$persistent. □

Now we claim that the semiflow is uniformly $\tilde{\rho}-$persistent for $\tilde{\rho}(x) = [x]_v$.

Theorem 3.8. *Let assumptions of Theorem 3.7 hold, and further, let* $\Phi : J \times X \to X$ *be a state-continuous semiflow induced by* F*. We assume*

(i) Φ *is point-dissipative and asymptotically smooth.*

(ii) If $\phi : \hat{J} \to X$*,* $\hat{J} = J \cup (-J)$*, is a total* $\Phi-$*trajectory with pre-compact range and* $\inf_{t\in\hat{J}} \rho(\phi(t)) > 0$*, then* $\tilde{\rho}(\phi(0)) > 0$ *where* $\rho(x) = \|x\|_v$*,* $\tilde{\rho}(x) = [x]_v$*.*

Then (3.1) is uniformly $\tilde{\rho}-$*persistent for* $\tilde{\rho}(x) = [x]_v$*.*

Proof. By Theorem 3.7, we have F is uniformly $\rho-$persistent for $\rho(x) = \|x\|_v$.

First we prove that $\tilde{\rho}(x) = [x]_v$ is continuous. Let $\|y - x\|_v \le \epsilon$, then

$$-\epsilon v \le y - x \le \epsilon v,$$

$$y = x + y - x \ge [x]_v v - \epsilon v = ([x]_v - \epsilon)v.$$

So $[y]_v \ge [x]_v - \epsilon$. We can switch x and y to get $[x]_v \ge [y]_v - \epsilon$. Thus $|[x]_v - [y]_v| < \epsilon$, so $[x]_v$ is continuous. Therefore it is semicontinuous. Apply (Smith and Thieme, 2011, Corollary 4.22), we get that the semiflow is uniformly $\tilde{\rho}$-persistence. □

Theorem 3.9. *Let Y be a metric space, $F : Y \to Y$ be a continuous map generating the discrete dynamical system $y(n+1) = F(y(n))$, $y(0) \in Y$, and let $\rho : Y \to \mathbb{R}_+$ be continuous.*

Recall

$$Y_0 = \{y \in Y; \forall n \in \mathbb{N} : \rho(F^n(y)) = 0\}. \tag{3.3}$$

Assume that $Y_0 \neq \emptyset$, also assume that there exists a set $B \subseteq Y$ and some $c > 0$ such that $F^n(x) \to B$ as $n \to \infty$ for all $x \in Y$ and $B \cap \{\rho \leq c\}$ has compact closure in Y.

Further assume that, for any $y \in Y$ with $\rho(y) > 0$, $\rho(F^n(y)) > 0$ for infinitely many $n \in \mathbb{N}$.

Let $Y_0 \neq \emptyset$ and

$$\Omega = \bigcup_{y \in Y_0} \omega(y) \subseteq \bigcup_{i=1}^{k} M_i, \qquad M_i \cap M_j = \emptyset, \quad i \neq j,$$

where each M_i is a compact, forward invariant subset of Y_0 that is isolated in Y_0. Assume that $\{M_1, \ldots, M_k\}$ is acyclic.

Finally assume that, for $i = 1, \ldots, k$, there exists an open set U_i with $M_i \subseteq U_i \subseteq Y$.

(i) $\rho(F(y)) \geq \rho(y)$ *for all $y \in U_i$;*

(ii) *There is no sequence $(y_n)_{n \in Z}$ in U_i such that $y_{n+1} = F(y_n)$ and $\rho(y_n) = \rho(y_0) > 0$ for all $n \in \mathbb{Z}$.*

Then the semiflow induced by F is uniformly weakly ρ-persistent.

Compare the conditions (i) and (ii) for more in (Smith and Thieme, 2011, Proposition 3.16).

Proof. Guided by (Smith and Thieme, 2011, Theorem 8.17), we only need to show that each M_i is isolated in Y if M_i is isolated in Y_0 and each M_i is weakly ρ-repelling.

First we prove that each M_i is isolated in Y if M_i is isolated in Y_0. Since M_i is isolated in Y_0, there exists some neighborhood W_i of M_i such that W_i isolates M_i in Y_0: If N_i is compact, invariant and contained in $Y_0 \cap W_i$, then $N_i \subset M_i$. By assumption, we have a neighborhood U_i of M_i satisfying (i) and (ii), we show that $W_i \cap U_i$ isolates M_i in Y.

Let $\tilde{N}_i$ be a compact invariant subset of $W_i \cap U_i$. Suppose that $\tilde{N}_i$ contains a point in $y \in Y \setminus Y_0$. By (Smith and Thieme, 2011, Theorem 1.40), since $\tilde{N}_i$ is invariant, there exists a sequence $(y_n)_{n\in\mathbb{Z}}$ such that $y_0 = y$, $y_{n+1} = F(y_n)$ for all $n \in \mathbb{Z}$ and $y_n \in \tilde{N}_i$ for all $n \in \mathbb{Z}$. Since $y \in Y \setminus Y_0$, then by (i), $\rho(y_n) \geq \rho(y_{n-1}) \geq \rho(y_{n-2}) \geq \cdots \geq \rho(y) > 0$. So the sequence $(\rho(F^n(y)))$ is increasing and bounded, it has a limit $\alpha = \lim_{n\to\infty} \rho(y_n) > 0$.

The $\omega-$set of $(y_n)_{n\in\mathbb{Z}}$ is

$$\{\lim y_{n_j} | y_{n_j} \text{ is the subsequence of } (y_n)\}.$$

Since $(\rho(F^n(y)))$ has the limit α, $\rho(z) = \alpha$ for all $z \in \omega((y_n)_{n\in\mathbb{Z}})$. Since $\omega((y_n)_{n\in\mathbb{Z}})$ is an invariant set, apply (Smith and Thieme, 2011, Theorem 1.40) again, we could find a sequence $(z_n)_{n\in\mathbb{Z}}$ such that $z_{n+1} = F(z_n)$ for all $n \in \mathbb{Z}$ and $z_n \in \omega((y_n)_{n\in\mathbb{Z}})$ for all $n \in \mathbb{Z}$. And

$$\rho(z_1) = \rho(z_2) = \rho(z_3) = \cdots = \rho(z_n) = \alpha > 0$$

for all $n \in \mathbb{Z}$, contradicting (ii). Therefore $\tilde{N}_i \subset M_i$ in Y_0, M_i is isolated in Y.

Now we need to prove that M_i is weakly $\rho-$repelling. Suppose not, there exists a $y \in Y$ such that $\rho(y) > 0$, $F^n(y) \to M_i$. Then there exists some $N \in \mathbb{N}$ such that $\rho(F^n(y)) \in U_i$ for all $n > N$. By the assumption that $\rho(F^n(y)) > 0$ for infinitely many $n \in \mathbb{N}$ when $\rho(y) > 0$, we can find a subsequence of (y_n), which we denote as

(y_n) again such that $\rho(y_n) > 0$ and $\rho(y_n) \in U_i$ for all $n \in \mathbb{N}$. By (i), $\rho(y_n)$ is increasing for all $n \in \mathbb{N}$ and bounded, so it has a limit point $y_0 \in M_i$ such that $\rho(y_0) > 0$. But ρ is zero on $M_i \subseteq Y_0$, a contradiction.

If $\Phi(t, y) \to M_i$, $\rho(y_0) = \lim_{n\to\infty} \rho(y_n) \to 0$ since ρ is continuous. Therefore $\rho(y_0) > 0$ contradict with $\rho(y_0) = 0$. So each M_i is weakly $\rho-$repelling.

Now by (Smith and Thieme, 2011, Theorem 8.17), Φ is uniformly weakly ρ-persistent. □

Chapter 4

GENERALIZATION STUDY FOR AN ABSTRACT DENSITY DEPENDENT INTEGRAL PROJECTION MODEL

Let A be a positive linear bounded operator on an ordered Banach space X with positive cone X_+, that is, $AX_+ \subset X_+$. Assume that $\mathbf{r}(A) < 1$.

Let $f : X_+ \to X_+$ be continuous and compact, $f(0) = 0$ where $0 \in X_+$. Consider the semilinear map $F : X_+ \to X_+$ defined by

$$F(x) = Ax + f(x), \qquad x \in X_+.$$

Our next generation model is given by

$$x_{n+1} = F(x_n),\ n \geq 0,\ x_0 \in X_+.$$

Hence we have

$$x_{n+1} = Ax_n + f(x_n), \qquad n \geq 0. \tag{4.1}$$

To help us understanding this general model, we introduce two specialized models Example 4.1, Example 4.2. By studying the persistence and stability properties for these two examples, we get a better idea for how to derive some general results for our general model 4.1.

Example 4.1.

$$x_{n+1} = Ax_n + \tilde{f}(x_n)b, \qquad n \geq 0$$

where $\tilde{f} : X_+ \to [0, \infty)$ is continuous and compact, $b \in X_+$.

Example 4.2.

$$x_{n+1} = Ax_n + \hat{f}(c^* x_n), \qquad n \geq 0$$

where $\hat{f} : [0, \infty) \to X_+$ is continuous and compact, c^* denotes a positive bounded linear functional on X.

4.1 Fundamentals of the Dynamics

By (4.1), we have

$$x_n = A^n x_0 + \sum_{k=1}^{n} A^{k-1} f(x_{n-k}), \quad n \in \mathbb{N}.$$

By the triangle inequality,

$$\|x_n\| \leq \|A^n x_0\| + \sum_{k=1}^{n} \|A^{k-1}\| \|f(x_{n-k})\|, \quad n \in \mathbb{N}. \tag{4.2}$$

Proposition 4.3. *The discrete dynamical system F has the following properties:*

(a) F is asymptotically smooth.

(b) If $\{x_n\}_{n\geq 0}$ is bounded, then the orbit $O(x_0) = \{x_n = F^n(x_0); n \in \mathbb{N}\}$ has compact closure. Its omega limit set $\omega(x_0)$ is nonempty, compact, invariant, and $x_n \to \omega(x_0)$.

(c) If F is point dissipative,

$$\Omega = \bigcup \{\omega(x_0); x_0 \in X_+\}$$

has compact closure in X_+, attracts every point of X_+.

(d) If F is point dissipative and eventually bounded on bounded sets, the semiflow induced by F has a compact attractor, $\mathcal{A}$, of bounded sets.

Proof. (a) Notice that $F = A + f$ with a compact map f. By induction, for each $n \in \mathbb{N}$, $F^n = A^n + f_n$ with a compact map $f_n : X_+ \to X_+$. By (Smith and Thieme, 2011, Theorem 2.46), F is asymptotically smooth.

(b) Since $\{x_n\}_{n\geq 0}$ is bounded and f is compact, the closure of $\{f(x_n)\}_{n\geq 0}$ is a compact set, which is bounded. Then there exists some $C > 0$ such that $\|f(x_n)\| \leq C$ for all $n \in \mathbb{Z}_+$.

By (4.2),

$$\|x_n\| \leq \|A^n x_0\| + C(\sum_{k=0}^{n-1} \|A^k\|), \quad n \in \mathbb{N}.$$

The series converges because $\mathbf{r}(A) = \lim_{k\to\infty} \|A^k\|^{1/k} < 1$. Compactness of $O(x_0)$ follows from (a) and (Smith and Thieme, 2011, Proposition 2.27). The properties of $\omega(x_0)$ follow in the usual way ((Smith and Thieme, 2011, Theorem.2.11)).

(c) (Smith and Thieme, 2011, Theorem.2.28.).

(d) See Corollary 3.4. □

Let's consider the special case Example 4.1, (4.2) has the form

$$\|x_n\| \leq \|A^n x_0\| + \sum_{k=1}^{n} \tilde{f}(x_{n-k})\|A^{k-1}b\|, \qquad n \in \mathbb{N}.$$

We define positive vector e by

$$e = (I - A)^{-1}b = \sum_{j\geq 0} A^j b. \tag{4.3}$$

Lemma 4.4. *In Example 4.1, let $S \subset X_+$ be invariant and bounded,i.e. $F(S) = S$. Assume $\tilde{f}$ maps bounded set to bounded set, then there exist some $C > 0$ such that $S \subset [0, Ce]$.*

Proof. Let $z \in S$. By (Smith and Thieme, 2011, Theorem.1.40), there exists a sequence $(z_n)_{n\in\mathbb{Z}}$ in S with

$$z_{n+1} = F(z_n) = Az_n + \tilde{f}(z_n)b, \quad n \in \mathbb{Z}, \qquad z_0 = z.$$

Since (z_n) is bounded, there exists some $C > 0$ such that $\tilde{f}(z_n) \leq C$ for all $n \in \mathbb{Z}$ with C not depending on z and n. Hence

$$z_n \leq Az_{n-1} + Cb, \; n \in \mathbb{Z}.$$

By induction,

$$z_n \leq A^k z_{n-k} + C \sum_{j=0}^{k-1} A^j b, \quad n \in \mathbb{Z}, \quad k \in \mathbb{N}.$$

The right hand side converges to Ce as $k \to \infty$. Since X_+ is closed, $z_n \leq Ce$ for all $n \in \mathbb{Z}$. In particular $z \leq Ce$. □

Remark 4.5. Proposition 4.3 apply to Example 4.1 and Example 4.2, and for Example 4.1, the omega limit set in part (b) also have the property $\omega(x_0) \subset [0, Ce]$ where C comes from Lemma 4.4.

Now we look at the linearization of F at $x = 0$. If F is Frechet differentiable at 0 with respect to the cone, there exists a positive bounded linear operator $L : X_+ \to X_+$ such that

$$\frac{\|f(x) - f(0) - L(x)\|}{\|x\|} \to 0, \qquad x \to 0_+$$

We denote L as $f'(0)$, then the linearization of F at 0 is of the form $Bx = Ax + f'(0)(x)$, $x \in X$, where $f'(0)$ is an additive and homogeneous map on X_+.

There is a relationship between spectral radius $\mathbf{r}(B)$ of the linearized operator B and $f'(0)$.

Theorem 4.6. *Assume that X_+ is a reproducing cone with ordered Banach space X, $\mathbf{r}(A) < 1$, we denote $f'(0)$ as the derivative of f at 0. Since f is compact, we have $f'(0)$ is compact (Deimling, 1985, Proposition 8.2),(Smoller, 1983, Theorem 13.2), (Krasnoselskiĭ and Zabreĭko, 1984, Section 17.5) then $\mathbf{r}(A + f'(0)) - 1$ and $\mathbf{r}((\mathbb{I} - A)^{-1} f'(0)) - 1$ have the same sign. Moreover, if $\mathbf{r}(B) > \mathbf{r}(A)$, then $\mathbf{r}(B)$ is an eigenvalue of B with a positive eigenvector.*

This result is an infinite-dimensional generalization of Cushing and Zhou (1994)

To prove this theorem we need the following fact:

Let $\lambda > \mathbf{r}(A)$,

$$\lambda - A - f'(0) = (\lambda - A)(\mathbb{I} - (\lambda - A)^{-1}f'(0)) = (\mathbb{I} - f'(0)(\lambda - A)^{-1})(\lambda - A).$$

Hence the following are equivalent

(a) $\lambda \in \rho(A + f'(0))$

(b) $1 \in \rho((\lambda - A)^{-1}f'(0))$

(c) $1 \in \rho(f'(0)(\lambda - A)^{-1})$ and in either case

$$\begin{aligned}(\lambda - A - f'(0))^{-1} &= (\mathbb{I} - (\lambda - A)^{-1}f'(0))^{-1}(\lambda - A)^{-1} \\ &= (\lambda - A)^{-1}(\mathbb{I} - f'(0)(\lambda - A)^{-1})^{-1}.\end{aligned}$$

The compactness of $f'(0)$ can be dropped if X_+ is normal and generating (Thieme, 2009, Theorem 3.10).

Proof of Theorem 4.6. The compact map $f'(0) : X_+ \to X_+$ can be extended to a linear compact map from X to X because X_+ is generating and X an ordered Banach space. Assume that $\mathbf{r}((\mathbb{I} - A)^{-1}f'(0)) > 1$. Since $f'(0)$ is compact, $\mathbf{r}((\lambda - A)^{-1}f'(0))$ is a continuous function of $\lambda \geq 1$. By the intermediate value theorem, there exists some $\lambda > 1$ such that $\mathbf{r}((\lambda - A)^{-1}f'(0)) = 1$. Then 1 is in the spectrum of $(\lambda - A)^{-1}f'(0)$ and λ is in the spectrum of $A + f'(0)$. Hence $\mathbf{r}(A + f'(0)) > 1$.

In the same way it follows that $\mathbf{r}((\mathbb{I} - A)^{-1}f'(0)) \geq 1$ implies $\mathbf{r}(A + f'(0)) \geq 1$.

Let $\mathbf{r}(A + f'(0)) = r \geq 1$. Since $\mathbf{r}(A) < 1$, the spectral radius is larger than the essential spectral radius and $\mathbf{r}(A + f'(0))$ is an eigenvalue of $A + f'(0)$ Nussbaum (1981). So 1 is in the spectrum of $(r - A)^{-1}f'(0)$ and

$$s := \mathbf{r}((r - A)^{-1}f'(0)) \geq 1.$$

By the Krein-Rutman theorem, there exists some $v \in X_+$, $v \neq 0$, such that

$$(r - A)^{-1}f'(0)v = sv.$$

Since $(\lambda - A)^{-1} f'(0) v$ is a decreasing function of $\lambda \geq \mathbf{r}(A)$, we have

$$(\mathbb{I} - A)^{-1} f'(0) \geq v.$$

This implies that

$$\mathbf{r}((\mathbb{I} - A)^{-1} f'(0)) \geq 1.$$

By contraposition, $\mathbf{r}((\mathbb{I} - A)^{-1}) < 1$ implies $\mathbf{r}(A + f'(0)) < 1$.

Let $\mathbf{r}(A + f'(0)) = r > 1$. Since $\mathbf{r}(A) f'(0) < 1$ there exists some $v \in X_+$, $v \neq 0$ such that

$$(A + f'(0))v = rv.$$

Choose some $\xi \in (0, 1)$ such that $\xi r \geq 1$. Then

$$(A + \xi f'(0))v \geq \xi(A + f'(0))v = \xi rv \geq v.$$

So $(\mathbb{I} - A)v \leq \xi f'(0) v$ and $v \leq \xi(\mathbb{I} - A)^{-1} f'(0) v$ By Krasnoselskiĭ (1964),

$$1 \leq \mathbf{r}(\xi((\mathbb{I} - A)^{-1} f'(0)) = \xi \mathbf{r}((\mathbb{I} - A)^{-1} f'(0)).$$

Thus $\mathbf{r}((\mathbb{I} - A)^{-1} f'(0)) \geq 1/\xi > 1$.

The last statement follows by Nussbaum (1981) since $f'(0)$ is compact. □

Apply Theorem 4.6 to special cases Example 4.1, Example 4.2, we have following results:

For Example 4.1, the linearized form for F is represented by $B = A + b\tilde{f}'(0)$ where $\tilde{f}'(0)$ is the derivative of $\tilde{f}'(0)$, $\tilde{f}'(0)$ is a linear functional on X.

Corollary 4.7. *Let $\diamond$ denotes one of the relations $<, >, =$, then $\mathbf{r}(B) \diamond 1$ if and only if $\mathbf{r}(e\tilde{f}'(0)) \diamond 1$ where e comes from (4.3). Moreover, if $\mathbf{r}(B) > \mathbf{r}(A)$, then $\mathbf{r}(B)$ is an eigenvalue of B with a positive eigenvector.*

Here $e\tilde{f}'(0) = (\mathbb{I} - A)^{-1}b\tilde{f}'(0)$ represents $(\mathbb{I} - A)^{-1}f'(0)$ and B is $A + f'(0)$ in Theorem 4.6.

For Example 4.2, assume $\hat{f}$ be differentiable at 0, we denote the derivative as $\hat{f}'(0)$, $\hat{f}'(0) \in X_+$. We define

$$\hat{e} = (\mathbb{I} - A)^{-1}\hat{f}'(0) = \sum_{j\geq 0} A^j \hat{f}'(0). \tag{4.4}$$

The linearizations of F, if they exist, as

$$Bx = Ax + \hat{f}'(0)(c^*x), \qquad x \in X.$$

Apply Theorem 4.6 to this example, B is the linearized form $A + f'(0)$, $\hat{e}c^*$ is $(\mathbb{I} - A)^{-1}f'(0)$.

Corollary 4.8. *Let $\diamond$ denotes one of the relations $<, >, =$, then $\mathbf{r}(B) \diamond 1$ if and only if $\mathbf{r}(\hat{e}c^*) \diamond 1$. Moreover, if $\mathbf{r}(B) > \mathbf{r}(A)$, then $\mathbf{r}(B)$ is an eigenvalue of B with a positive eigenvector.*

Apply the definition of spectral radius, we can get the value of $\mathbf{r}(\mathbb{I} - A)^{-1}f'(0)$ in Example 4.1 and 4.2. Namely $\mathbf{r}((\mathbb{I} - A)^{-1}b\tilde{f}'(0)) = \tilde{f}'(0)e$ in Example 4.1 and $\mathbf{r}((\mathbb{I} - A)^{-1}\hat{f}'(0)c^*) = c^*\hat{e}$ in Example 4.2.

Since in both Example 4.1 and 4.2, $(\mathbb{I} - A)^{-1}f'(0)$ can be written as the form $(\mathbb{I} - A)^{-1}f'(0)x = (x^*x)a$ where $x^* \in X_+^*$, $a \in X_+$. Denote $D = (\mathbb{I} - A)^{-1}f'(0)$, since by Definition 2.7: $\mathbf{r}(D) = \lim_{n\to\infty} \|D^n\|_+^{1/n}$, we have

$$Dx = (x^*x)a,$$

$$\begin{aligned} D^2x &= x^*(x^*xa)a \\ &= (x^*x)(x^*a)a, \end{aligned}$$

$$
\begin{aligned}
D^3x &= x^*((x^*x)(x^*a)a)a \\
&= (x^*x)(x^*a)^2a.
\end{aligned}
$$

Therefore

$$D^n x = (x^*x)(x^*a)^{n-1}a,$$

$$\|D^n\| = \|x^*x\|(x^*a)^{n-1}\|a\|,$$

$$\mathbf{r}(D) = \lim_{n\to\infty} \|D^n\|^{1/n} = x^*a.$$

For Example 4.1, $(\mathbb{I} - A)^{-1}b\tilde{f}'(0)x = (\tilde{f}'(0)x)e$, so x^* is $\tilde{f}'(0)$, a is e. $\mathbf{r}((\mathbb{I} - A)^{-1}b\tilde{f}'(0)) = \tilde{f}'(0)e$.

For Example 4.2, $(\mathbb{I}-A)^{-1}\hat{f}'(0)c^*x = (c^*x)\hat{e}$, so x^* is c^*, a is $\hat{e}$. $\mathbf{r}((\mathbb{I}-A)^{-1}\hat{f}'(0)c^*) = c^*\hat{e}$.

4.2 Extinction and Stability of the Extinction State

We expect that the zero fixed point is stable when $\mathbf{r}(B) = \mathbf{r}(A + f'(0)) < 1$, equivalently, by Theorem 4.6, when $\mathbf{r}((\mathbb{I} - A)^{-1}f'(0)) < 1$, and unstable if the reverse strict inequality holds.

4.2.1 General Results

Theorem 4.9. *If* $\mathbf{r}(A+f'(0)) < 1$, *then 0 is asymptotically stable with respect to* $\|\cdot\|$ *for map* B.

Theorem 4.10. *If* $\mathbf{r}(A + f'(0)) < 1$ *and* $Fx \le Bx$, *then 0 is globally stable with respect to* $\|\cdot\|$ *for map* F.

Theorem 4.11. *If* $\mathbf{r}(A + f'(0)) < 1$, *then 0 is locally stable with respect to* $\|\cdot\|$ *for map* F.

Proof. After switching to an equivalent norm, we can assume that $\|A + f'(0)\| < 1$ (Krasnoselskiĭ, 1964, Theorem.2.5.2). Since f is Frechet differentiable at 0, let $\epsilon > 0$, there exist some δ such that for $\|x\| < \delta$:

$$\frac{\|f(x) - f'(0)(x)\|}{\|x\|} < \epsilon.$$

Therefore $\|f(x) - f'(0)(x)\| < \epsilon\|x\|$. We choose ϵ small enough such that $\epsilon + \|A + f'(0)\| < 1$. Now we claim that if $\|x_1\| < \delta$, $\|x_n\| < \delta$ for all $n \in \mathbb{N}$.

For $\|x\| < \delta$, $Fx = Ax + f(x) = Ax + f'(0)(x) + (f(x) - f'(0)(x))$. So

$$\begin{aligned} \|Fx\| &\leq \|A + f'(0)\|\|x\| + \|f(x) - f'(0)(x)\| \\ &< \|A + f'(0)\|\|x\| + \epsilon\|x\| \\ &= (\|A + f'(0)\| + \epsilon)\|x\| \\ &< \|x\| < \delta. \end{aligned}$$

So $\|x_n\| \leq \delta$ which implies $\|x_{n+1}\| \leq \alpha\|x_n\|$ for all $m \in \mathbb{N}$ with some $\alpha < 1$ and the statement follows. □

4.2.2 Special Cases for Example 1 and Example 2

For Example 4.1, we introduce a useful linear functional

$$\tilde{g} = \tilde{f}'(0)(I - A)^{-1}. \tag{4.5}$$

Then

$$\tilde{g} = \tilde{g}A + \tilde{f}'(0) \geq \tilde{f}'(0),$$

$$\tilde{g}b = \tilde{f}'(0)(I - A)^{-1}b = \tilde{f}'(0)e.$$

We have following two extinction results.

Theorem 4.12. *Let X_+ be normal. If $\tilde{f}'(0)(e) = 0$, then for all $n \in \mathbb{N}$, $\tilde{f}'(0)(x_n) = \tilde{f}'(0)(A^n x_0)$ and $\tilde{f}'(0)(x_n) \to 0$ as $n \to \infty$. If $\tilde{f}'(0)(e) > 0$, assume there exists some $\eta \in (0, \infty)$ such that $\tilde{f}(x)\tilde{f}'(0)(e) \leq \tilde{f}'(0)(x)$ for all $x \in X_+$ with $\tilde{g}(x) \leq \eta$.*

(a) *Then 0 is stable with respect to* $\tilde{g}$*: For any* $\delta \in (0,\eta)$, $\tilde{g}(x_n) \le \delta$ *for all* $n \in \mathbb{N}$ *whenever* $\tilde{g}(x_0) \le \delta$.

(b) *If there is some* $\tilde{\eta} \in (0,\infty]$ *such that* $\tilde{f}(x)\tilde{f}'(0)(e) < \tilde{f}'(0)(x)$ *for all* $\tilde{g}(x) < \tilde{\eta}$, *then* $x_n \to 0$ *whenever* $\tilde{g}(x_0) < \tilde{\eta}$, *and* 0 *is locally asymptotically stable.*

Proof. (a) If $\tilde{f}'(0)(e) = 0$:

$\tilde{f}'(0)(e) = 0$ implies that $\tilde{f}'(0)(A^n b) = 0$ for all $n \in \mathbb{Z}_+$. Since

$$x_n = A^n x_0 + \sum_{k=1}^{n} \tilde{f}(x_{n-k})A^{k-1}b,$$

we have

$$\tilde{f}'(0)(x_n) = \tilde{f}'(0)(A^n x_0) + \sum_{k=1}^{n} \tilde{f}(x_{n-k})\tilde{f}'(0)(A^{k-1}b).$$

Therefore we have $\tilde{f}'(0)(x_n) = \tilde{f}'(0)(A^n x_0)$ for all $n \in \mathbb{N}$.

Since $\mathbf{r}(A) < 1$, $A^n x_0 \to 0$ as $n \to \infty$. $\tilde{f}'(0)(x_n) = \tilde{f}'(0)(A^n x_0) \to 0$ as $n \to \infty$.

If $\tilde{f}'(0)(e) > 0$:

Let $x \in X_+$ and $\tilde{g}x \le \eta$. Then $\tilde{f}'(0)(x) \le \eta$, and

$$\tilde{g}(F(x)) = \tilde{g}Ax + \tilde{f}(x)\tilde{g}(b) = \tilde{g}(x) - \tilde{f}'(0)(x) + \tilde{f}(x)\tilde{f}'(0)(e) \le \tilde{g}(x) - \tilde{f}'(0)(x) + \tilde{f}'(0)(x) = \tilde{g}(x).$$

Let $x_n = F^n(x)$. Then the term $\tilde{g}(x_n)$ form a decreasing sequence and (a) follows.

(b) Let α be the limit of $\{\tilde{g}(x_n)\}$. If $\tilde{g}(x_0) < \tilde{\eta}$, $\alpha < \tilde{\eta}$, then

$$\alpha = \lim_{n\to\infty} \tilde{g}(x_{n+1}) = \lim_{n\to\infty} (\tilde{g}(x_n) - \tilde{f}'(0)(x_n) + \tilde{f}(x_n)\tilde{f}'(0)(e)).$$

This implies that $-\tilde{f}'(0)(x)_n + \tilde{f}(x_n)\tilde{f}'(0)(e) \to 0$ as $n \to \infty$. If $\tilde{f}'(0)(e) = 0$, by (a), we have our conclusion.

Now we consider $\tilde{f}'(0)(e) > 0$, since $\tilde{f}'(0)(x_n) \le \tilde{g}(x_n)$, $\{\tilde{g}(x_n)\}_{n\ge 0}$ bounded implies $\{\tilde{f}'(0)(x_n)\}_{n\ge 0}$ bounded. And we have $-\tilde{f}'(0)(x_n) + \tilde{f}(x_n)\tilde{f}'(0)(e) \to 0$, so $\{\tilde{f}(x_n)\}_{n\ge 0}$ is also bounded.

Suppose that

$$\beta := \limsup_{n\to\infty} \tilde{f}'(0)(x)_n > 0.$$

There exists a strictly increasing sequence (n_k) of natural numbers such that $\tilde{f}'(0)(x)_{n_k} \to \beta$. And since $\{\tilde{f}(x_n)\}_{n\geq 0}$ is bounded, by Remark 4.5 (b), we can find a subsequence of (x_{n_k}), here we denote as (x_{n_k}) again, such that $x_{n_k} \to \tilde{x}_0$, where $\tilde{x}_0 \in \omega(x_0)$, clearly $\tilde{g}(\tilde{x}_0) \leq \eta$. Since $\tilde{f}$ is continuous and $\beta > 0$,

$$\beta = \lim_{k\to\infty} \tilde{f}(x_{n_k})\tilde{f}'(0)(e) = f(\tilde{x}_0)\tilde{f}'(0)(e) < \tilde{f}'(0)(\tilde{x}_0) = \lim_{k\to\infty} \tilde{f}'(0)(x)_{n_k} = \beta,$$

a contradiction.

So $\beta = 0$ and $\tilde{f}'(0)(x)_n \to 0$ and $\tilde{f}(x_n) \to 0$ as $n \to \infty$. Since $\sum_{n=1}^{\infty} \|A^n b\|$ converges, we can apply Lebesgue's dominated convergence theorem (with the counting measure) to

$$\|x_n\| \leq \|A^n x_0\| + \sum_{k=1}^{n} \tilde{f}(x_{n-k})\|A^{k-1}b\|, \quad n \in \mathbb{N}.$$

and obtain $\|x_n\| \to 0$ as $n \to \infty$. □

Corollary 4.13. *(a) If $\tilde{f}$ is differentiable at 0 and $\tilde{f}'(0)(e) < 1$, then 0 is locally asymptotically stable.*

(b) If $\tilde{f}(x)\tilde{f}'(0)(e) < \tilde{f}'(0)(x)$ for all $x \in X_+$, then $F^n(x) \to 0$ as $n \to \infty$ for all $x \in X_+$, and 0 is globally asymptotically stable.

For Example 4.2, again, we introduce a useful linear functional

$$\hat{g} = c^*(I - A)^{-1}. \tag{4.6}$$

Then we have

$$\hat{g} = \hat{g}A + c^* \geq c^*,$$

$$\hat{g}\hat{f}'(0) = c^*(I - A)^{-1}\hat{f}'(0) = c^*\hat{e}.$$

Theorem 4.14. *Let X_+ be normal, assume that there exists some $\eta \in (0, \infty]$ be such that $\hat{g}\hat{f}(c^*x) \le c^*x$ for all $x \in X_+$ with $\hat{g}x \le \eta$.*

(a) Then 0 is stable with respect to $\hat{g}$: For any $\delta \in (0, \eta)$, $\hat{g}x_n \le \delta$ for all $n \in \mathbb{N}$ whenever $\hat{g}x_0 \le \delta$.

(b) 0 is locally stable with respect to $\| \cdot \|$.

Proof. (a) Let $x \in X^+$ and $\hat{g}x \le \eta$. Then $c^*x \le \eta$, and

$$\hat{g}F(x) = \hat{g}Ax + \hat{g}\hat{f}(c^*x) = \hat{g}x - c^*x + \hat{g}\hat{f}(c^*x) \le \hat{g}x - c^*x + c^*x = \hat{g}x.$$

Let $x_n = F^n(x)$. Then the term $\hat{g}x_n$ form a decreasing sequence, so the statement follows.

(b) By (a) and $\hat{g} \ge c^*$, we have $c^*x_n \le \delta$ for all $n \in \mathbb{N}$ whenever $\delta \in (0, \eta)$ and $\hat{g}x_0 \le \delta$. By (4.2), Example 4.2 has the form:

$$\|x_n\| \le \|A^n x_0\| + \sup \|\hat{f}[0, \delta]\| \sum_{k=1}^{n} \|A^{k-1}\|.$$

After switching to an equivalent norm, we can assume that $\|A\| < 1$ (Krasnoselskiĭ, 1964, Theorem 2.5.2) Let $\epsilon > 0$ and $\|x_0\| \le \epsilon$. Then, for $n \in \mathbb{N}$,

$$\|x_n\| \le \|A\|\epsilon + \sup \|\hat{f}([0, \delta])\| \sum_{k=1}^{\infty} \|A^{k-1}\|.$$

Choosing $\delta \in (0, \epsilon)$ small enough, we can achieve, $\|x_n\| \le \epsilon$ for all $n \in \mathbb{N}$ provided $\|x_0\| \le \epsilon$ and $\|x_0\| \le \epsilon$ and $\hat{g}x_0 \le \delta$. Now choose $\tilde{\delta} = \min\{\epsilon, \delta / \|\hat{g}\|\}$. Then $\|x_n\| \le \epsilon$ whenever $\|x_0\| \le \tilde{\delta}$. □

Corollary 4.15. *If $\hat{f}$ is differentiable at 0 and $c^*\hat{e} < 1$, where $\hat{e}$ comes from (4.4), then 0 is locally stable.*

4.3 Persistence

4.3.1 General Model

Theorem 4.16. *For $F : X_+ \to X_+$, $F(x) = A(x) + f(x)$, we assume*

(i) There exists $v^* \in X^*_+$, $v^* \neq 0$, $r > 1$ such that $v^*(\mathbb{I} - A)^{-1} f'(0) = rv^*$.

(ii) Define $w^* = v^*(\mathbb{I} - A)^{-1}$ and assume $v^*x > 0$ implies $w^*f(x) > 0$, $w^*x = 0 = w^*f(x)$ implies $f(x) = 0$.

(iii) Let $\eta \in (0, 1)$, there exists some $\delta > 0$ such that $w^*f(x) \geq (1 - \eta)w^*(f'(0)x)$ if $\|x\| < \delta$.

Define $\rho = w^* = v^*(\mathbb{I} - A)^{-1}$, then the semiflow induced by F is uniformly weakly $\rho-$persistent:
There exists some $\epsilon > 0$ such that $\limsup_{n\to\infty} w^*x_n \geq \epsilon$ if $w^*x_0 > 0$.

The semiflow is uniformly ρ-persistent if, in addition, the norm is monotone and there exist some $R > 0$, some linear bounded positive operator D and some $y \in X_+$ such that $f(x) \leq y + Dx$ for all $x \in X_+$ with $\|x\| \geq R$ and $\mathbf{r}(A + D) < 1$.

This proof uses some unpublished ideas of Hal Smith. Existence of v^* satisfying (i) follows from Krein-Rutman theorem if X_+ is generating.

Proof. By $w^* = v^*(\mathbb{I} - A)^{-1}$ and $w^* = w^*A + v^* \geq v^*$,

$$w^*x_{n+1} = w^*Ax_n + w^*f(x_n) = (w^* - v^*)x_n + w^*f(x_n). \tag{4.7}$$

Apply Proposition 3.9, here $\rho(x) = w^*x$, suppose that $w^*x_0 > 0$. If $v^*x_0 > 0$, then by (ii), $w^*x_1 \geq w^*f(x_0) > 0$. If $v^*x_0 = 0$, then $w^*x_1 \geq w^*x_0 > 0$.

It follows that if $w^*x_0 > 0$, then $w^*x_n > 0$ for all $n \geq 0$.

We want to apply Theorem 3.9, define

$$Y_0 = \{y \in Y; w^*(F^n(y)) = 0 \text{ for all } n \in \mathbb{N}\}.$$

First we show that solutions starting in Y_0 converge to 0: Let $y_0 \in Y$, we have $w^*(y_0) = 0$, by assumption (ii), $f(y_0) = 0$. Therefore

$$w^*(y_1) = w^*(F(y_0)) = w^*Ay_0 + w^*f(y_0) = (w^* - v^*)y_0 + w^*f(y_0).$$

Since $w^*(y_1) = w^*(F(y_0)) = 0$ and $f(y_0) = 0$, we have $v^*(y_0) = w^*(y_0) = 0$ and $y_1 = A(y_0) + f(y_0) = A(y_0)$.

Assume $y_n = A^n(y_0) \in Y_0$ for some $n \in \mathbb{N}$, we have $w^*(y_n) = 0$, by assumption (ii), $f(y_n) = 0$. Therefore

$$w^*(y_{n+1}) = w^*(F(y_n)) = w^*Ay_n + w^*f(y_n) = (w^* - v^*)y_n + w^*f(y_n).$$

Since $w^*(y_{n+1}) = w^*(F(y_n)) = 0$ and $f(y_n) = 0$, we have $v^*(y_n) = w^*(y_n) = 0$ and $y_{n+1} = A(y_n) + f(y_n) = A(y_n) = A^{n+1}(y_0)$.

By induction, we get that $y_n = A^n(y_0)$ and $v^*(y_n) = 0$ for all $n \in \mathbb{N}$. Since $\mathbf{r}(A) < 1$, the sequence $(y_n) \to 0$ as $n \to \infty$.

Then we claim that $\{0\}$ is isolated in Y_0.

Let W be a neighborhood of $\{0\}$, K is a compact invariant subset of W and $K \subset Y_0$. If there exists $z \neq 0$ such that $z \in K$, since K is invariant, we could find a sequence $\{(z_{-n})\} \subset N$ such that $z = F^n(z_{-n}) = A^n(z_{-n}) \to 0$ as $n \to \infty$. This contradicts $z \neq 0$. Then $K = \{0\}$. Hence we get $\{0\}$ is isolated in Y_0.

We also have Y_0 is acyclic. Suppose not, there exists $x_0 \neq 0$, the sequence $\{x_n\}$ have the relation $x_{n+1} = F(x_n)$ for all $n \in \mathbb{N}$ and $x_n \to 0$ as $n \to \pm\infty$.

By previous proof, we have $x_n \to 0$ as $n \to \infty$. Also $x_0 = A^n x_{-n} \to 0$ as $n \to \infty$, which contradicts $x_0 \neq 0$. Therefore Y_0 is acyclic.

Let U be the neighborhood of $\{0\}$ such that if $x \in U$, $\|x\| < \delta$.

First we show that if $\|x\| \in U$, then $w^* f(x) \geq \xi v^* x$.

By assumption (iii), let $\eta > 0$, there exists some $\delta > 0$ such that $w^* f(x) \geq (1-\eta) w^*(f'(0)x)$, so $w^* f(x) - w^*(f'(0)x) \geq -\eta w^*(f'(0)x)$.

$$
\begin{aligned}
w^* f(x) &= w^*(f(x) - f'(0)x + f'(0)x) \\
&= w^* f'(0)x + w^*(f(x) - f'(0)x) \\
&\geq r v^* x - \eta w^*(f'(0)x) \\
&= (r - r\eta) v^* x.
\end{aligned}
$$

We set $\xi = r(1-\eta)$, and let η small enough such that $\xi > 1$.

Then the assumptions of Theorem 3.9 are easy to check.

For (i), $w^* x_{n+1} = (w^* - v^*)x_n + w^* f(x_n) > w^* x_n + (\xi - 1) v^* x_n \geq w^* x_n$ for all x_n such that $w^* x_n < \delta$.

For (ii), if $w^*(x_{n+1}) = w^*(x_n) > 0$ for all $n \in \mathbb{Z}$, then by (4.7), we have $w^* f(x_n) = v^*(x_n)$ for all $n \in \mathbb{Z}$. By assumption (i) and (iii), we have $w^* f(x) \geq \xi v^* x$ for $\xi > 1$ when $\|x\| < \delta$. Hence $w^* f(x_n) = v^*(x_n) \geq \xi v^*(x_n)$ for all $n \in \mathbb{Z}$, this happens only if $v^* x_n = 0$ for all $n \in \mathbb{Z}$. Therefore $w^* f(x_n) = 0$ and $w^*(x_n) = w^* A^n x_0 \to 0$ for all $n \in \mathbb{Z}$. Since $\mathbf{r}(A) < 1$, $w^*(x_n) \to 0$ as $n \to \infty$, then $w^*(x_n) = 0$ for all $n \in \mathbb{Z}$.

For uniformly ρ-persistent, by Proposition 3.3 and Corollary 3.4, the semiflow induced by F has compact attractor of bounded sets. Hence apply (Smith and Thieme, 2011, Theorem 4.5), we have uniformly weakly ρ-persistent implies uniformly ρ-persistent result. □

4.3.2 Special Cases for Example 1 and Example 2

Apply Theorem 4.16 to Example 4.1, Example 4.2, we have some persistence results.

For Example 4.1, $\tilde{g}$ and $\tilde{f}'(0)$ is w^* and v^* respectively in Theorem 4.16, then we have:

Theorem 4.17. *For $F : X_+ \to X_+, F(x) = A(x) + \tilde{f}(x)b$, we assume that $\tilde{f}(0) = 0$ and $\tilde{f}$ is differentiable at 0. Further we assume*

(i) $\tilde{f}'(0)(e) > 1$.

(ii) $\tilde{f}'(0)(x) > 0$ *implies* $\tilde{g}\tilde{f}(x) > 0$.

(iii) Let $\eta \in (0,1)$, there exists some $\delta > 0$ such that $\tilde{f}(x) \geq (1-\eta)(f'(0)x)$ if $\|x\| < \delta$.

Then the semiflow induced by F is uniformly weakly persistent in the following sense:

There exists some $\epsilon > 0$ such that $\limsup_{n\to\infty} \tilde{f}(x_n) \geq \epsilon$ for any solution (x_n) with $\tilde{g}(x_0) > 0$.

Proof. Since $\tilde{f}'(0)(e) > 1$, let $r = \tilde{f}'(0)(e)$, then $\tilde{f}'(0)(\mathbb{I} - A)^{-1}b\tilde{f}'(0) = r\tilde{f}'(0)$. We view $\tilde{f}'(0)(\mathbb{I} - A)^{-1}$ as w^*, $b\tilde{f}'(0)$ as $f'(0)$, then assumption of Theorem 4.16: $v^*(\mathbb{I} - A)^{-1}f'(0) = rv^*$ and $r > 1$ is satisfied. Assumptions (ii) and (iii) are the same as Theorem 4.16. Therefore all assumptions of Theorem 4.16 are satisfied, the conclusion follows. □

For Example 4.2, similarly, we view $\hat{g}$ and c^* as w^* and v^* respectively in Theorem 4.16, then we have:

Theorem 4.18. *For $F : X_+ \to X_+$, $F(x) = A(x) + \hat{f}(c^*x)$, we assume that $\hat{f}(0) = 0$ and $\hat{f}$ is differentiable at 0. Further we assume*

(i) $c^*\hat{e} > 1$ *where* $\hat{e} = (\mathbb{I} - A)^{-1}\hat{f}'(0)$,

(ii) $\hat{g}\hat{f}(s) > 0$ *for all* $s \in (0, \infty)$.

Then the semiflow is uniformly weakly persistent in the following sense:

There exists some $\epsilon > 0$ *such that* $\limsup_{n\to\infty} \hat{g}x_n \geq \epsilon$ *for any solution* (x_n) *with* $\hat{g}x_0 > 0$.

Proof. Let $c^*\hat{e} = r$, we have $c^*(\mathbb{I} - A)^{-1}\hat{f}'(0)c^* = rc^*$, here we view c^* as v^*, $\hat{f}'(0)c^*$ as $f'(0)$, then we have $v^*(\mathbb{I} - A)^{-1}f'(0) = rv^*$ for $r > 1$ as assumption (i) in Theorem 4.16. Assumption (ii) is the same as Theorem 4.16.

For Assumption (iii), since $\hat{f}$ is differentiable at 0, $\hat{g}$ is linear, we have $\hat{g}\hat{f}$ is differentiable at 0. And

$$\frac{\hat{g}\hat{f}(s)}{s} \to \hat{g}\hat{f}'(0) = c^*\hat{e} > 1$$

as $s \to 0$ where $s \in \mathbb{R}_+$.

Let $\eta > 0$, there exists some $\delta > 0$ such that $\frac{\hat{g}\hat{f}(s)}{s} > (1-\eta)\hat{g}\hat{f}'(0)$ for all $|s| < \delta$. Therefore there exists $\delta_1 > 0$ such that if $\|x\| < \delta_1$, $s = c^*x < \delta$. Then $\hat{g}\hat{f}(s)s > (1-\eta)\hat{g}(\hat{f}'(0))(s)$ for $s = c^*x$ and $\|x\| < \delta_1$. So assumption (iii) of Theorem 4.16 is satisfied.

Since all assumptions of Theorem 4.16 are satisfied, the conclusion follows. □

Remark 4.19. If the semi-dynamical system generated by F is point dissipative, then for Example 4.1, F is uniformly $\tilde{g}$-persistence under the assumptions of Theorem 4.17. For Example 4.2, F is uniformly $\hat{g}$-persistence under the assumptions of Theorem 4.18.

Proof. Apply (Smith and Thieme, 2011, Corollary 4.8). □

Chapter 5

PERSISTENCE AND EXTINCTION OF DIFFUSING POPULATIONS WITH TWO SEXES AND SHORT REPRODUCTIVE SEASON

5.1 Introduction

In discrete infinite-dimensional dynamical system, the next generation operator maps the spatial offspring distribution of one reproductive season to the distribution of the next season. Here we consider a model for a spatially distributed population of male and female individuals that mate and reproduce only once in their life during a very short reproductive season. There is only one reproductive season in a year, and individuals reproduce during the reproductive season that follows the one during which they have been born. Between birth and mating, females and males move by diffusion on a bounded domain Ω under Neumann boundary conditions and are subject to density-dependent per capita mortality rates. Mating and reproduction is described by a (positively) homogeneous function (of degree one). The model is a special case of a periodic impulsive reaction diffusion system, but similarly as in Lewis and Li (2012) we treat it as an abstract difference equation or a discrete semiflow on an infinite dimensional state space Smith and Thieme (2011). The semiflow is induced by a map that relates this year's offspring density to next year's offspring density.

We identify a basic reproduction number $\mathcal{R}_0$ that acts as a threshold between extinction and persistence. If $\mathcal{R}_0 < 1$, the population dies out while it persists (uniformly weakly) if $\mathcal{R}_0 > 1$. Because of the homogeneous mating function, the map that induces the discrete semiflow is not differentiable at the origin, and so $\mathcal{R}_0$ is not the spectral radius of a bounded positive linear operator Diekmann *et al.*

(1990, 2012); Smith and Thieme (2011); Zhao (2003) but the cone spectral radius of a continuous, order-preserving, homogeneous map that approximates the semiflow map at the origin. More generally, the threshold separating extinction and persistence is not determined from a linear problem Cantrell and Cosner (2004); Cantrell *et al.* (2007) but from a homogeneous (i.e.,non-additive) one.

To a certain degree, our model is a two-sex version of the model in Lewis and Li (2012) for bounded domains. There do not seem to be many models that combine spatial and sexual structure. Diffusive spread and traveling waves are studied on the real line in Ashih and Wilson (2001); the mating is not modeled by a homogeneous function but one that is of Michaelis-Menten (alias Monod alias Holling II) form, and the mating is not seasonal. A system of two integro-difference equations is used in Miller *et al.* (2011), and the speed of two-sex invasions on the real line is investigated; the mating is modeled by a homogeneous function that is multiplied by a density-dependent reproductive success factor. In our model, it is the per capita mortality rate that is density-dependent.

5.2 The Model

We assume that mating and reproduction only occurs once a year during a very short season. We let this season mark the transition from one year to the next. During the year, females and males move by diffusion on a bounded domain Ω in some Euclidean space $\mathbb{R}^2$.

The density of offspring at location $x \in \Omega$ at the end of the reproductive season is denoted by $f(x)$ with a continuous function $f : \bar{\Omega} \to \mathbb{R}_+$. Our aim is to construct a map $F : C_+(\bar{\Omega}) \to C_+(\bar{\Omega})$ on the nonnegative continuous functions that maps this year's density of offspring to next year's density of offspring and thus gives rise to a discrete semiflow $(F^n)_{n\in\mathbb{Z}_+}$ and a difference equation

$$f_n = F(f_{n-1}), \qquad n \in \mathbb{N}. \tag{5.1}$$

5.2.1 The Model for within One Year Dispersion

The dispersion within one year is modeled by a system of partial differential equations. Let $u_1(t,x)$ and $u_2(t,x)$ represent the density of females and males at location $x \in \Omega$ at time $t \in [0,1]$ where the number 1 marks the end of the year,

$$\begin{aligned} (\partial_t - d_i\Delta)u_i &= -\mu_i(t,x,u)u_i, \qquad 0 < t \leq 1, x \in \Omega, i = 1,2 \\ u &= (u_1,u_2) \end{aligned} \tag{5.2}$$

with Neumann boundary conditions

$$\partial_\nu u_i(t,x) = 0, \qquad x \in \partial\Omega, \tag{5.3}$$

and initial conditions

$$u_i(0,x) = p_i(x)f(x), \qquad i = 1,2. \tag{5.4}$$

Here $\triangle$ is the Laplace operator in x, d_i are diffusion constants of females and males respectively, μ_i are density dependent per capita mortality rates, $p_i(x)$ are the probabilities that the offspring at x is female/male.

We also consider the linear partial differential equations (5.2),(5.3), and (5.4)with $\mu_i(t,x,u)$ being replaced by $\mu_i(t,x,0)$.

Assumption 5.1. We assume that $\mu_i : [0,\infty) \times \Omega \times [0,\infty) \to \mathbb{R}$ are nonnegative and differentiable and that the derivatives and μ_i themselves are bounded on all sets $[0,a] \times \Omega \times [0,a]$. We also assume that the boundary $\partial\Omega$ is C^4. The functions $p_i : \bar{\Omega} \to [0,1]$ are assumed to be continuous and to be strictly positive on Ω.

Under these assumptions, unique continuous mild solutions u_1, u_2 exist on $[0,1]\times\bar{\Omega}$ as can be seen by a standard fixed point argument. These solutions are classical in the sense that they have partial derivatives $\partial_t u_i$ and first and second partial derivatives with respect to x on $(0,1]\times\Omega$. This can be seen by a bootstrap argument similar to the one in the proof of (Smith, 1995, Theorem.3.1)

5.2.2 The Next Year Offspring Map

The next year offspring map $F : C_+(\bar{\Omega}) \to C_+(\bar{\Omega})$ is given by

$$F(f)(x) = \phi(x, u_1(1,x), u_2(1,x)), \qquad f \in C_+(\bar{\Omega}), x \in \bar{\Omega} \tag{5.5}$$

where u_1 and u_2 are the solutions of (5.2), (5.3) and (5.4), and $\phi : \Omega\times\mathbb{R}^2_+ \to \mathbb{R}_+$ is the mating and reproduction function. If there are u_1 females and u_2 males at location x in Ω, $\phi(x, u_1, u_2)$ is the amount of offspring produced at x.

5.2.3 Properties of Mating and Reproduction Function

- ϕ is continuous on $\bar{\Omega}\times\mathbb{R}^2_+$.
- $\phi(x,\cdot)$ is order preserving on $\mathbb{R}^2_+$ for each $x \in \Omega$.
- $\phi(x,\cdot)$ is (positively) homogeneous for each $x \in \Omega$,

$$\phi(x, \alpha u) = \alpha\phi(x, u), \quad \alpha \geq 0, x \in \Omega, u \in \mathbb{R}^2_+.$$

- $\phi(x, 1, 1) > 0$ for all $x \in \bar{\Omega}$.

Example 5.2. Harmonic mean $\phi(x, u_1, u_2) = \beta(x)\frac{u_1 u_2}{u_1+u_2}$ with strictly positive β.

Example 5.3. Minimum function $\phi(x, u) = \min\{\beta_1(x)u_1, \beta_2(x)u_2\}$ with strictly positive β_1 and β_2.

5.3 Main Result

We introduce the map $B : C_+(\bar{\Omega}) \to C_+(\bar{\Omega})$ as an approximation of F at zero, which means B is F with $\mu_i(t,x,u)$ being replaced by $\mu_i(t,x,0)$ in (5.2). B is the next year offspring map under optimal conditions(when the per capita death rate is not increased by overcrowding effects).

We will show that the map B is homogeneous, order-preserving, continuous and compact. We can therefore define the cone spectral radius $\mathbf{r}_+(B)$ of B which is the threshold parameter separating extinction from persistence. If no distinction were made between males and females, the map B would be linear and bounded and $\mathbf{r}_+(B)$ the usual spectral radius of B. Since individuals are assumed to reproduce only once in their life, in the next reproduction season after their birth, B can be interpreted as the next generation map and its spectral radius as the basic reproduction number of the population, i.e., the mean number of offspring one average newborn will have produced at the end of its life under optimal conditionsDiekmann *et al.* (2012, 1990). So we introduce the notation $\mathcal{R}_0 = \mathbf{r}_+(B)$. As in the linear case, no closed formula is available in general, but estimates and approximations can be found in a similar way. For instances, if f is strictly positive continuous function on $\bar{\Omega}$, then $\mathbf{r}_+(B) = \lim_{n\to\infty} \|B^n f\|^{1/n}$.

Theorem 5.4. *There exists a threshold parameter $\mathcal{R}_0 = r_+(B)$ that separates population extinction and persistence as follows:*

- *If $\mathcal{R}_0 < 1$, the extinction state 0 is locally asymptotically stable in the following sense: There exist $\delta_0 > 0$, $\alpha \in (\mathbf{r}_+(B), 1)$ and $M \geq 1$ such that $\|F^n(f)\| \leq M\alpha^n \|f\|$ for all $f \in C_+(\bar{\Omega})$ with $\|f\| \leq \delta_0$.*

- *If $\mathcal{R}_0 < 1$ and $\mu_i(t,x,u) \geq \mu_i(t,x,0)$ for all $t \geq 0, x \in \Omega, u \geq 0$, then the pop-*

ulation always dies out: $F^n(f) \to 0$ *as* $n \to \infty$*, uniformly on* Ω *and uniformly for* f *in bounded subsets of* $C_+(\bar{\Omega})$*.*

- *If* $\mathcal{R}_0 > 1$*, the population persists uniformly weakly: There exists some* $\epsilon > 0$ *such that, for any* $f \in C_+(\bar{\Omega})$*,* $f \not\equiv 0$*,*

$$\limsup_{n\to\infty} \sup_{x\in\Omega} F^n(f)(x) \geq \epsilon$$

Weak persistence implies instability of the origin.

5.3.1 Stability and Persistence Result for Maps

In this section, we generalize the principle of linearized stability to the case that the approximation at the origin is not linear but only homogeneous.

A cone X_+ is normal (see (2.1)) if and only if its norm is equivalent to a norm $\|\cdot\|$ which is monotone Krasnoselskiĭ (1964); Krasnoselskiĭ *et al.* (1989): $\|x\| \leq \|y\|$ for all $x, y \in X_+$ with $x \leq y$.

Theorem 5.5. *Let* X_+ *be the normal cone of an ordered normed vector space. Let* $F, B : X_+ \to X_+$ *and let* B *be homogeneous, bounded and order preserving,* $r = r_+(B) < 1$*. Assume that for each* $\eta > 0$ *there exists some* $\delta > 0$ *such that* $F(x) \leq (1+\eta)B(x)$ *for all* $x \in X_+$ *with* $\|x\| \leq \delta$*. Then* F *is locally asymptotically stable in the following sense:*

There exists some $\delta_0 > 0$*,* $\alpha \in (r, 1)$ *and* $M \geq 1$ *such that* $\|F^n(x)\| \leq M\alpha^n\|x\|$ *for all* $n \in \mathbb{N}$ *and all* $x \in X_+$ *with* $\|x\| \leq \delta_0$*.*

Proof. Since X_+ is normal, we can replace the original norm by an equivalent monotone norm. This does not affect $r_+(B)$ given by (2.4)

Choose some s such that $r_+(B) < s < 1$. Then there exists some $m \in \mathbb{N}$ such that $\|B^n\|_+ < s^n$ for all $n \in \mathbb{N}$, $n \geq m$. Further there exists some $c \geq 1$ such that $\|B^n\|_+ \leq c$ for $n = 1, \ldots, m$.

Choose $\eta > 0$ such that $\alpha := (1+\eta)s < 1$ and then choose $\delta > 0$ such that $F(x) \le (1+\eta)B(x)$ for all $x \in X_+$ with $\|x\| \le \delta$. (Since we switched to an equivalent norm, the δ may not be the same as in the assumption of the theorem.) Choose $\delta_0 \in (0, \delta)$ such that

$$(1+\eta)^m c\delta_0 \le \delta. \tag{5.6}$$

Let $x \in X_+$ and $\|x\| \le \delta_0$. Then $F(x) \le (1+\eta)B(x)$. Since the norm is monotone, by (5.6) and (2.3),

$$\|F(x)\| \le (1+\eta)\|B\|_+\|x\| \le (1+\eta)c\delta_0 \le \delta.$$

Since B is order-preserving and homogeneous,

$$F^2(x) \le (1+\eta)B(F(x)) \le (1+\eta)^2 B^2(x).$$

Since the norm is monotone,

$$\|F^2(x)\| \le (1+\eta)^2\|B^2\|_+\|x\| \le (1+\eta)^2 c\delta_0 \le \delta.$$

Proceeding this way we obtain that

$$F^n(x) \le (1+\eta)^n B^n(x), \qquad n = 1, \dots, m,$$

and

$$\|F^n(x)\| \le (1+\eta)^m c\|x\| \le (1+\eta)^n c\delta_0 \le \delta, \qquad n = 1, \dots, m. \tag{5.7}$$

For $n = m+1$,

$$F^n(x) \le (1+\eta)B(F^m(x)) \le (1+\eta)^n B^n(x).$$

Since the norm is monotone,

$$\|F^n(x)\| \le (1+\eta)^n\|B^n\|_+\|x\| \le (1+\eta)^n s^n\|x\| \le \alpha^n\|x\| < \delta.$$

Now, by induction,

$$F^n(x) \leq (1+\eta)^n B^n(x), \qquad n \geq m,$$

and

$$\|F^n(x)\| \leq \alpha^n \|x\|, \qquad n \geq m.$$

We combine this estimate with the one in (5.7) and obtain the assertion. □

This is a global version of the previous result.

Theorem 5.6. *Let X_+ be the normal cone of an ordered normed vector space. Let $F, B : X_+ \to X_+$ and let B be homogeneous, bounded and order preserving, $r = \mathbf{r}_+(B) < 1$. Assume that $F(x) \leq B(x)$ for all $x \in X_+$. Then $F^n(f) \to 0$ as $n \to \infty$ uniformly for f bounded subsets of X_+.*

Proof. Since B is order-preserving, $F^n(x) \leq B^n(x)$ for all $n \in \mathbb{N}$ and $x \in X_+$. Since X_+ is normal, we can replace the original norm by an equivalent monotone norm. This does not affect $\mathbf{r}_+(B)$. Then $\|F^n(x)\| \leq \|B^n\|_+ \|x\|$ and $\|B^n\|_+ \to 0$ and $n \to \infty$. □

In the next result, we assume that the cone X_+ has nonempty interior, $\breve{X}_+$. For $v \in X_+$, we recall the functional on X_+

$$[x]_v = \sup\{\alpha \geq 0, x \geq \alpha v\}, \qquad x \in X_+. \tag{5.8}$$

If $v \in \breve{X}_+$, the functional $[\cdot]_v$ is continuous. It is also homogeneous and concave. Since X_+ is closed,

$$x \geq [x]_v v, \quad x \in X_+. \tag{5.9}$$

Theorem 5.7. *Let $F : X_+ \to X_+$ and $B : X_+ \to X_+$, and let B be homogeneous, order preserving and continuous, $r = \mathbf{r}_+(B) > 1$. Assume*

- *$F^n(x) \in \breve{X}_+$ for infinitely many $n \in \mathbb{N}$ if $x \in X_+$, $x \neq 0$.*

- *For any $\eta \in (0,1)$ there exists some $\delta > 0$ such that $F(x) \geq (1-\eta)B(x)$ for all $x \in X_+$ with $\|x\| \leq \delta$.*

- *There is some $r > 1$ and some $v \in \breve{X}_+$ such that $Bv \geq rv$*

Then the semiflow induced by F is uniformly weakly norm-persistent: There exists some $\epsilon > 0$ such that $\limsup_{n\to\infty} \|F^n(x)\| \geq \epsilon$ for any $x \in X_+$ with $x \neq 0$.

Proof by contradiction. According to assumption (a), we choose $r > 0$ and $v \in \breve{X}_+$ such that $Bv \geq rv$.

Choose $\eta \in (0,1)$ such that $r(1-\eta) > 1$. According to assumption (b), choose some $\delta > 0$ such that $F(x) \geq (1-\eta)B(x)$ for all $x \in X_+$ with $\|x\| \leq \delta$.

Suppose the semiflow is not uniformly weakly norm-persistent. Then, for $\epsilon \in (0,\delta)$, there exists some $x \in X_+$ with $x \neq 0$ and some $m \in \mathbb{N}$ such that $\|F^n(x)\| \leq \epsilon$ for all $n \geq m$.

By assumption (a), we can find some $M > m$ such that $F^M(x) \in \breve{X}_+$. Let $y = F^M(x)$. Since $\|y\| \leq \epsilon < \delta$, $F(y) \geq (1-\eta)B(y)$. By (5.9), since B is order-preserving and homogeneous,

$$B(y) \geq B([y]_v v) = [y]_v B(v) \geq r[y]_v v.$$

We combine the inequalities,

$$F(y) \geq (1-\eta)r[y]_v v.$$

By (5.8),

$$[F(y)]_v \geq (1-\eta)r[y]_v.$$

Since $F(y) = F^{M+1}(x)$, $\|F(y)\| \leq \delta$. So we have $F^2(y) = F(F(y)) \geq (1-\eta)B(F(y))$. We repeat the previous argument and obtain

$$[F^2(y)]_v \geq r^2(1-\eta)^2[y]_v.$$

By induction, we have $[F^n(y)]_v \geq r^n(1-\eta)^n[y]_v$ for all $n \geq M$.

Since $y \in X_+$ and $r(1-\eta) > 1$, we have $[y]_v > 0$ and $[F^n(y)]_v \to \infty$ as $n \to \infty$. But $([F^n(x)]_v)$ is bounded because $(F^n(x))$ is bounded and $[\cdot]_v$ is continuous and homogeneous. □

5.3.2 Proof of Main Result

Let $X = C(\bar{\Omega})$ be the Banach space of continuous functions on $\bar{\Omega}$ with supremum norm and $X_+ = C_+(\bar{\Omega})$ be the cone of nonnegative functions. X_+ has nonempty interior; in fact, every strictly positive continuous functions on $\bar{\Omega}$ is an interior point of X_+. Further X_+ is normal; actually, the supremum norm is monotone itself.

Part of the proof of the main result will be that the homogeneous map B is an approximation of the next-year-offspring map F at the origin in the following sense: For any $\eta \in (0,1)$, there exists some $\delta > 0$ such that

$$(1-\eta)B(f) \leq F(f) \leq (1+\eta)B(f), \qquad f \in X_+, \ \|f\| \leq \delta.$$

This will make the application of the results in the previous section possible.

Proof of Theorem 5.4. Recall that $B(f)(x) = \phi(x, \tilde{u}_1(1,x), \tilde{u}_2(1,x))$ where $\tilde{u}_1$ and $\tilde{u}_2$ are the solutions of

$$\begin{aligned} (\partial_t - d_i \triangle)\tilde{u}_i &= -\mu_i(t,x,0)\tilde{u}_i, \qquad 0 \leq t \leq 1, x \in \Omega, \\ \tilde{u} &= (\tilde{u}_1, \tilde{u}_2) \end{aligned} \tag{5.10}$$

with Neumann boundary conditions $\partial_\nu \tilde{u}_i(t,x) = 0, \quad x \in \partial\Omega$ and initial conditions $\tilde{u}_i(0,x) = p_i(x)f(x)$

It follows from the smoothing properties of parabolic differential equations that the map $f(\tilde{u}_1(1,\cdot), \tilde{u}_2(1,\cdot))$ is compact from X to X^2. Since the solution of (5.18) depend continuously on their initial data, this map is also continuous. It is linear because the equations are linear. By a standard comparison principle (see (Smith and Thieme, 2011, Theorem.A.23), e.g.), this map is also order-preserving. It follows from the properties of the mating and reproduction function ϕ that B is compact, homogeneous, continuous, and order-preserving on X_+.

We first prove the second result: We assume $\mu_i(t,x,0) \le \mu_i(t,x,u)$. By a standard comparison principle (see (Smith and Thieme, 2011, Theorem.A.23), e.g.), we have $u_i \le \tilde{u}_i$ and so $F(f) \le B(f)$ for all $f \in C_+(\bar{\Omega})$. The second result now follows from Theorem 5.6.

Next we prove the third result. Let $\mathbf{r}_+(B) > 1$, apply Theorem 5.7. We check the first condition. If $f \in C_+(\bar{\Omega})$ and $f \not\equiv 0$, then $u_i(t,x) > 0$ for all $t \in (0,1]$, $x \in \bar{\Omega}$ for the solutions of (5.2),(5.3) and (5.4). It follows from the properties of the mating and reproduction function that $\phi(x,u_1,u_2) > 0$ for all $x \in \bar{\Omega}$, $u_1 > 0$, $u_2 > 0$. So $F(f)$ is strictly positive on $\bar{\Omega}$ and a point in $\bar{X}_+$ for all $f \in X_+$, $f \neq 0$.

To verify the third condition of Theorem 5.7, we use Theorem 2.15. Since B is compact, there exists some $v \in X_+$, $v \neq 0$, such that $Bv = rv$, $r = \mathbf{r}_+(B) > 1$. The map B has similar positivity properties as F; so v is an interior point of X_+.

Finally, we check the second condition of Theorem 5.4: For any $\eta \in (0,1)$ there exists some $\delta > 0$ such that $F(x) \ge (1-\eta)B(x)$ for all $x \in X_+$ with $\|x\| \le \delta$.

We know that $F(f)(x) = \phi(x, u_1(1,x), u_2(1,x))$ where u_1 and u_2 are the solutions of

$$\begin{aligned} (\partial_t - d_i \triangle)u_i &= -\mu_i(t,x,u)u_i, \qquad 0 \le t \le 1, x \in \Omega, \\ u &= (u_1, u_2) \end{aligned} \tag{5.11}$$

with Neumann boundary conditions $\partial_\nu u_i(t,x) = 0, \quad x \in \partial\Omega$ and initial conditions $u_i(0,x) = p_i(x)f(x)$, while the operator B is associated with the solution of (5.18)

We also consider solutions of

$$\begin{aligned}(\partial_t - d_i\Delta)\bar{u}_i =& 0, \qquad 0 \le t \le 1, x \in \Omega \\ \partial_\nu \bar{u}_i(t,x) =& 0, \qquad x \in \partial\Omega, \\ \bar{u}_i(0,x) =& p_i(x)f(x).\end{aligned} \tag{5.12}$$

Since $\mu_i \ge 0$, it follows from a comparison principle (Smith and Thieme, 2011, Theorem.A.23) that

$$\tilde{u}_i \le \bar{u}_i, \qquad u_i \le \bar{u}_i \tag{5.13}$$

and, for any $\delta > 0$ and newborn density $f \in C_+(\bar{\Omega})$,

$$f \le \delta \implies \bar{u}_i \le \delta. \tag{5.14}$$

Let $\eta \in (0,1)$ and $\epsilon > 0$ such that $e^{-\epsilon} = 1 - \eta$. Since μ_i is locally Lipschitz in u, by choosing $\delta \in (0,1)$ small enough,

$$\mu_i(t,x,u) \le \mu_i(t,x,0) + \epsilon, \qquad t \ge 0, x \in \Omega, u = (u_1,u_2), 0 \le u_i \le \delta.$$

Set $u_\epsilon(t,x) = e^{\epsilon t}u(t,x)$. Since $u_i \le \bar{u}_i \le \delta$,

$$(\partial_t - d_i\Delta)u_\epsilon(t,x) = -(\mu_i(t,x,u) - \epsilon)u_\epsilon(t,x) \ge -\mu_i(t,x,0)u_\epsilon.$$

We recall Mallet-Paret and Nussbaum (2010), use a comparison principle again (Smith and Thieme, 2011, Theorem.A.23) and obtain,

$$u_\epsilon(t,x) \ge \tilde{u}(t,x)$$

which implies that

$$u(t,x) \ge e^{-\epsilon t}\tilde{u}(t,x) = (1-\eta)\tilde{u}(t,x), \qquad 0 \le t \le 1, x \in \Omega.$$

Since $\phi(x,\cdot)$ is order preserving and homogeneous,

$$F(f)(x) = \phi(x,u(1,x)) \ge \phi(x,(1-\eta)\tilde{u}(1,x)) = (1-\eta)\phi(x,\tilde{u}(1,x)) = (1-\eta)B(f)(x).$$

We have verified all condition of Theorem 5.7 which implies the persistence result of Theorem 5.4.

Similarly, by considering $e^{-\epsilon t}u(t,x)$ with $e^{\epsilon} = 1 + \eta$, we find that $u(t,x) \leq (1+\eta)\tilde{u}(t,x)$ and so $F(f) \leq (1+\eta)B(f)$. The first result now follows from Theorem 5.5. □

5.4 The Model with Dirichlet Boundary Condition

5.4.1 General Results

Guided by Lewis and Li (2012), now we consider the model within one year is represented by a system of partial differential equations

$$\begin{aligned} (\partial_t - d_i\Delta)u_i &= -\mu_i(t,x,u)u_i, \qquad 0 < t \leq 1, x \in \Omega, i = 1,2 \\ u &= (u_1, u_2) \end{aligned} \tag{5.15}$$

with Dirichlet boundary condition

$$u_i(t,x) = 0, \qquad x \in \partial\Omega, i = 1,2 \tag{5.16}$$

and initial conditions

$$u_i(0,x) = p_i(x)f(x), \qquad i = 1,2. \tag{5.17}$$

The linear operator B which is the approximation of F at 0. For map B, partial differential equations become to

$$\begin{aligned} (\partial_t - d_i\triangle)\tilde{u}_i &= -\mu_i(t,x,0)\tilde{u}_i, \qquad 0 \leq t \leq 1, x \in \Omega, i = 1,2 \\ \tilde{u} &= (\tilde{u}_1, \tilde{u}_2) \end{aligned} \tag{5.18}$$

with Dirichlet boundary conditions $\tilde{u}_i(t,x) = 0, x \in \partial\Omega$ and initial conditions $\tilde{u}_i(0,x) = p_i(x)f(x)$, $i = 1,2$.

By Section 5.3.2, we still have the conclusion: For any $\eta \in (0,1)$, there exists some $\delta > 0$ such that

$$(1+\eta)B(f) \geq F(f) \geq (1-\eta)B(f), f \in X_+, \|f\| \leq \delta.$$

All results of previous model discussion hold except for Theorem 5.7. For model with Neumann boundary condition, we can assume that the cone X_+ is solid, but this assumption does not hold any more in Dirichlet boundary condition. Therefore we have another result:

Theorem 5.8. *Assume the norm $\|\cdot\|$ is monotone. Let $F : X_+ \to X_+$ and $B : X_+ \to X_+$, and let B be homogeneous, order preserving and continuous, $r = \mathbf{r}_+(B) > 1$. Assume*

(a) There is some $r > 1$ and some $v \in X_+ \setminus \{0\}$, such that $Bv \geq rv$

(b) Let $n \in \mathbb{N}$, there exists some $N > n$ and c which relates to N such that $cv < F^N(x)$ if $x \in X_+$, $x \neq 0$.

(c) For any $\eta \in (0,1)$ there exists some $\delta > 0$ such that $F(x) \geq (1-\eta)B(x)$ for all $x \in X_+$ with $\|x\| \leq \delta$.

Then the semiflow induced by F is uniformly weakly norm-persistent: There exists some $\epsilon > 0$ such that $\limsup_{n\to\infty} \|F^n(x)\| \geq \epsilon$ for any $x \in X_+$ with $x \neq 0$.

Proof by contradiction. According to assumption (a), we choose $r > 0$ and $v \in X_+$ such that $Bv \geq rv$.

Choose $\eta \in (0,1)$ such that $r(1-\eta) > 1$. According to assumption (c), choose some $\delta > 0$ such that $F(x) \geq (1-\eta)B(x)$ for all $x \in X_+$ with $\|x\| \leq \delta$.

Suppose the semiflow is not uniformly weakly norm-persistent. Then, for $\epsilon \in (0,\delta)$, there exists some $x \in X_+$ with $x \neq 0$ and some $m \in \mathbb{N}$ such that $\|F^n(x)\| \leq \epsilon$ for all $n \geq m$.

By assumption (b), we can find some $M > m$ such that $F^M(x) \in X_+$. Let $y = F^M(x)$. Since $\|y\| \le \epsilon < \delta$, $F(y) \ge (1-\eta)B(y)$.

Since $y \in X_+$, we have $c_y v \le y$. Therefore

$$B(y) \ge B(c_y v) = c_y B(v) \ge c_y r v.$$

We combine the inequalities,

$$F(y) \ge c_y(1-\eta) r v.$$

Since $F(y) = F^{M+1}(x)$, $\|F(y)\| \le \delta$. So we have $F^2(y) = F(F(y)) \ge (1-\eta)B(F(y))$. We repeat the previous argument and obtain

$$F^2(y) \ge c_y(1-\eta)^2 r^2 v.$$

By induction, we have $F^n(y) \ge c_y(1-\eta)^n r^n v$ for all $n \ge M$.

Since $y \in X_+$ and $(1-\eta)r > 1$, we have $\|F^n(y)\| \ge c_y(1-\eta)^n r^n \|v\| \to \infty$ as $n \to \infty$ which caused contradiction. $\square$

5.4.2 Simplified Model

Now we simplify our model by considering $\mu_i(t,x,0)$ only relates to time t, thus $\mu_i(t,x,0)$ becomes $\mu_i(t,0)$.

$$\begin{aligned}(\partial_t - d_i\triangle)\tilde{u}_i &= -\mu_i(t,0)\tilde{u}_i, \quad 0 \le t \le 1, x \in \Omega,\\ \tilde{u} &= (\tilde{u}_1, \tilde{u}_2)\end{aligned} \tag{5.19}$$

with Dirichlet boundary conditions

$$\tilde{u}_i(t,x) = 0, \qquad x \in \partial\Omega \tag{5.20}$$

and initial conditions

$$\tilde{u}_i(0,x) = p_i v(x). \tag{5.21}$$

To solve the partial differential equation (5.19)(5.20)(5.21). We consider the solutions satisfy the following condition:

$$\begin{cases} \tilde{u}_i(t,x) &= g_i(t)v(x), \qquad g_i(0) = p_i \\ \\ \triangle v &= -\lambda v \end{cases} \tag{5.22}$$

$\lambda > 0$, v is positive on Ω and $v = 0$ on $\partial\Omega$.

Plug (5.22) into (5.19), we have

$$(\partial_t - d_i\triangle)(g_i(t)v(x)) = -\mu_i(t,0)(g_i(t)v(x)),$$

$$g_i'(t)v(x) - g_i(t)d_i\triangle v(x) = -\mu_i(t,0)g_i(t)v(x),$$

$$g_i'(t)v(x) + \lambda d_i g_i(t)v(x) = -\mu_i(t,0)g_i(t)v(x).$$

For $v(x) \neq 0$, we have

$$g_i'(t) + \lambda d_i g_i(t) = -\mu_i(t,0)g_i(t).$$

Solving this differential equation, we have

$$\frac{d(g_i(t))}{g_i(t)} = [-\lambda d_i - \mu_i(t,0)]dt.$$

Take integral for both sides, we have

$$\ln(g_i(t)) = -\lambda d_i t - \int_0^t \mu_i(s,0)ds.$$

Then

$$g_i(t) = p_i e^{-\lambda d_i t - \int_0^t \mu_i(s,0)ds}.$$

Therefore

$$\tilde{u}_i(t,x) = p_i e^{-\lambda d_i t - \int_0^t \mu_i(s,0)ds} v(x).$$

For $t = 1$, set $\tilde{k}_1 = g_1(1)$, $\tilde{k}_2 = g_2(1)$, then $\tilde{u}_1(1,x) = \tilde{k}_1 v(x)$, $\tilde{u}_2(1,x) = \tilde{k}_2 v(x)$.

Now we consider the mating function does not depend on x. Apply the mating function ϕ on $\tilde{u}_1(1,x)$, $\tilde{u}_2(1,x)$, we have $\phi(\tilde{u}_1(1,x),\tilde{u}_2(1,x)) = \phi(\tilde{k}_1 v(x), \tilde{k}_2 v(x)) = v(x)\phi(\tilde{k}_1,\tilde{k}_2)$.

Therefore $B(v) = v(x)\phi(\tilde{k}_1,\tilde{k}_2)$. Set $r_0 = \phi(\tilde{k}_1,\tilde{k}_2)$, then $B(v) = r_0 v$.

Suppose $r = \mathbf{r}_+(B) \geq 1$. Since B is compact under reasonable assumptions, there exists some $v \in X_+$ with $B(v) = rv$. Then v has the properties above, and it follows that $r_0 \geq 1$. By contraposition, if $r_0 < 1$, then $\mathbf{r}_+(B) < 1$.

Now we prove that if $r_0 > 1$, then the conditions of Theorem 5.8 are satisfied. Assumption (a) is proven by previous discussion. To show assumption (b), by Thieme (1988), let w be the solution of

$$-\triangle_x w(x) = 1, x \in \Omega,$$
$$w(x) = 0, x \in \partial\Omega.$$

According to (Thieme, 1988, Lemma 6.6), we find $\delta_0, c_0 > 0$ such that

$$\delta_0 u_i(1,x) \leq w(x)\int_\Omega u_i(0,y)w(y)dy \leq c_0 u_i(1,x), \qquad i = 1,2.$$

where $u = (u_1,u_2)$ is the solution of $(\partial_t - d_i\triangle_x)u_i = -\mu_i(t,x,u)u_i$.

Therefore there exists some $m_i > 0$ such that $u_i(1,x) \geq m_i w(x)$,$i = 1,2$. To show that $w(x) \geq n_i\tilde{u}_i(1,t)$ for some $n_i > 0$, $i = 1,2$, we set $\tilde{v}(x) = \frac{v}{\lambda\|v\|}$. ($\|\cdot\|$ is supreme norm). Then $\|\tilde{v}\| \leq \frac{1}{\lambda}$. Plug $\tilde{v}(x)$ into equation systems:

$$-\triangle_x v(x) = \lambda v, x \in \Omega,$$
$$v(x) = 0, x \in \partial\Omega.$$

We have

$$-\triangle_x \tilde{v}(x) = \lambda\tilde{v}, x \in \Omega,$$
$$\tilde{v}(x) = 0, x \in \partial\Omega.$$

Since $\tilde{v}(x) \leq \|\tilde{v}\| \leq \frac{1}{\lambda}$, $-\triangle_x \tilde{v}(x) = \lambda \tilde{v} \leq 1$. Let $z = \tilde{v} - w$. Then

$$-\triangle_x z(x) \leq 0, x \in \Omega,$$

$$z(x) = 0, x \in \partial\Omega.$$

Apply maximum principle, we have

$$\max_{\bar{\Omega}} z = \max_{\partial\Omega} z = 0.$$

Therefore $z(x) = \tilde{v}(x) - w(x) \leq 0$ for all $x \in \bar{\Omega}$, we have $\tilde{v} \leq w$. Then $\frac{v}{\lambda\|v\|} \leq w$. Since ϕ is order preserving, we have:

$$\begin{aligned} F(f)(x) &= \phi(u_1(1,x), u_2(1,x)) \\ &\geq \phi(m_1 w(x), m_2 w(x)) \\ &= w(x)\phi(m_1, m_2) \\ &\geq \tilde{v}\phi(m_1, m_2) \\ &= \frac{v}{\lambda\|v\|}\phi(m_1, m_2) \\ &= cv(x) \end{aligned}$$

where $c = \dfrac{\phi(m_1, m_2)}{\lambda\|v\|}$.

Hence assumption (b) is satisfied. Using the same strategy in the Proof of Theorem 5.4, we have assumption (c).

Since all assumptions of Theorem 5.8 are satisfied, we have the conclusion that if $r = \mathbf{r}_+(B) > 1$, the semiflow induced by F is uniformly weakly norm-persistent.

If $r = \mathbf{r}_+(B) < 1$, we can apply Theorem 5.5, Theorem 5.6 to get corresponding extinction results.

Summary: For Dirichlet boundary condition problem, we study the homogeneous map B which is the approximation of F at 0 again. The main result Theorem 5.4 still works for this case. By getting rid of space term x, we can find a way to calculate $\mathbf{r}(B)$ which actually depends on the size of the domain.

Chapter 6

APPLICATION TO A RANK-STRUCTURED POPULATION MODEL WITH MATING

6.1 Model

Let $X \subseteq \mathbb{R}^{\mathbb{N}}$ be an ordered normed vector space with cone $X_+ = X \cap \mathbb{R}^{\mathbb{N}}_+$. Assume that the norm has the property that $x_j \le \|x\|$ for all $x = (x_j) \in X_+$ and all $j \in \mathbb{N}$. This implies that $X \subset l^\infty$ and $\|x\|_\infty \le \|x\|$ for all $x \in X$.

Define a map $F : X_+ \to \mathbb{R}^{\mathbb{N}}_+$, $F(x) = (F_j(x))$, by

$$\left.\begin{aligned} F_1(x) = &\quad q_1(x)x_1 + \sum_{j,k=1}^{\infty} \beta_{jk}(x)\min\{x_j, x_k\} \\ F_j(x) = &\quad \max\left\{p_{j-1}(x)x_{j-1},\ q_j(x)x_j\right\}, \qquad j \ge 2 \end{aligned}\right\} x = (x_j) \in X_+. \tag{6.1}$$

Here $p_j, q_j : X_+ \to \mathbb{R}_+$, $\beta_{j,k} : X_+ \to \mathbb{R}$ are continuous functions of $x \in X_+$ for all $j, k \in \mathbb{N}$.

The dynamical system $(F^n)_{n\in\mathbb{N}}$ can be interpreted as the dynamics of a rank-structured population and, in a way, is a discrete version (in a double sense) of the two-sex models with continuous age-structure in Hadeler (1989); Iannelli *et al.* (2005). The number x_j is the amount of individuals at rank $j \in \mathbb{N}$ and the sequence $x = (x_j)$ is the rank distribution of the population. At rank distribution x, $F_1(x)$ is the number of newborn individuals who all have the lowest rank 1. Procreation is assumed to require some mating. Mating is assumed to be rank-selective and is described by taking the minimum of individuals in two ranks. The functions $\beta_{jk}(x)$ represent the fertility of a pair when the female has rank j and the male has rank k where a 1:1 sex ratio is assumed at each rank. The maps $F_j, j \ge 2$, describe how individuals survive and move upwards in the ranks from year to year where one cannot mover by more than one rank within a year. If the population size is modeled in number of individuals one would assume that $p_j(x), q_j(x) \leq 1$, but if it is modeled in biomass, then such an assumption would not be made. However, an individuals at rank j is at least $j-1$ years old, and so mortality eventually wins the upper hand such that the assumption

$$p_j(x) \to 0 \text{ and } q_j(x) \to 0, \qquad j \to \infty, \text{ uniformly for } x \in X_+, \tag{6.2}$$

is natural. From a population dynamics point of view, $X = l^1$ is the most natural space because the norm of $x \in X_+$ gives the total population size. So we choose $X = l^1$. We also assume that there is a sequence $(\tilde{\beta}_k) \in l^1_+$ such that

$$\beta_{jk}(x) + \beta_{kj}(x) \leq \tilde{\beta}_k, \qquad j, k \in \mathbb{N}, x \in X_+. \tag{6.3}$$

6.2 Main Results

6.2.1 Homogeneous Function B

Now we define the map $B : X_+ \to X_+$ by the same formula, but with $\beta_{jk}(0)$ replacing $\beta_{jk}(x)$, $p_j(0)$ replacing $p_j(x)$ and $q_j(0)$ replacing $q_j(x)$. So $B(x)$ is the approximation of $F(x)$ at $x = 0$. $B(x) = (B_j(x))$ has the form:

$$\left.\begin{aligned} B_1(x) = &\quad q_1(0)x_1 + \sum_{j,k=1}^{\infty} \beta_{jk}(0) \min\{x_j, x_k\} \\ B_j(x) = &\quad \max\left\{p_{j-1}(0)x_{j-1},\ q_j(0)x_j\right\}, \qquad j \geq 2 \end{aligned}\right\} x = (x_j) \in X_+. \tag{6.4}$$

We assume that $\frac{\beta_{jk}(x)}{\beta_{jk}(0)} \to 1$ as $x \to 0$ uniformly for those $j, k \in \mathbb{N}$ for which $\beta_{jk}(0) > 0$.

Also we assume that $\frac{p_j(x)}{p_j(0)} \to 1$ as $x \to 0$ uniformly for those $j \in \mathbb{N}$ for which $p_j(0) > 0$.

Further we assume that $\frac{q_j(x)}{q_j(0)} \to 1$ as $x \to 0$ uniformly for those $j \in \mathbb{N}$ for which $q_j(0) > 0$.

This implies that for each $\epsilon \in (0,1)$ there exists some $\delta > 0$ such that $(1-\epsilon)B(x) \leq F(x) \leq (1+\epsilon)B(x)$ for all $x \in X_+$ with $\|x\| \leq \delta$.

6.2.2 Point-Dissipative and Compact Attractor of Bounded Sets

We assume that $p_{j-1}(x) + q_j(x) \to 0$ uniformly for $j \in \mathbb{N}$ as $\|x\| \to \infty$, further $\sum_{j,k=1}^{\infty} \beta_{jk}(x) \to 0$ as $\|x\| \to \infty$. Let $\epsilon > 0$ still be chosen and choose $c > 0$ such that

$$\left.\begin{aligned} p_{j-1}(x) + q_j(x) \leq \epsilon \\ \sum_{j,k=1}^{\infty} \beta_{jk}(x) \leq \epsilon \end{aligned}\right\} \|x\| \geq c. \tag{6.5}$$

Choose $A : X \to X$, $A(x) = A_j(x)$,

$$\left.\begin{aligned} A_1(x) = {} & \epsilon x_1 + \sum_{j,k=1}^{\infty} \tilde{\beta}_{jk}(x_j) \\ A_j(x) = {} & \epsilon(x_{j-1} + x_j), \qquad j \geq 2 \end{aligned}\right\} x = (x_j) \in X_+. \tag{6.6}$$

Here $\tilde{\beta}_{jk} = \sup_{\|x\| \geq c} \beta_{jk}(x)$. Then $B(x) \leq A(x)$ whenever $\|x\| \geq c$. Choose $\epsilon < 1/3$ if $X = l^1$. Then $\|A\| < 1$ and F is point-dissipative by Proposition 3.3 and Corollary 3.4.

To make F compact in $X = l^1$, we assume that for any $c > 0$ there exists $u \in l^1$ such that $\sup_{k \geq 1} \sum_{j=1}^{\infty} \beta_{jk}(x) \leq u_1$ and $p_{j-1}(x) + q_j(x) \leq u_j$ for $j \geq 2$ and all $x \in X_+$ with $\|x\| \leq c$. Under these assumptions, the discrete semiflow (F^n) has a compact attractor of bounded sets.

6.2.3 *Spectral Radius of Map B*

Suppose there exists some $v \in \dot{X}_+$ and $r \geq 1$ with $B(v) = rv$. We drop the zero from $\beta_{jk}(0)$, similarly for p_j and q_j. We assume that $0 < p_j \leq 1$ and $0 \leq q_j < 1$ for all $j \in \mathbb{N}$. We obtain the equations,

$$\begin{aligned} rv_1 =& q_1 v_1 + \sum_{j,k=1}^{\infty} \beta_{jk} \min\{v_j, v_k\} \\ rv_j =& \max\left\{p_{j-1} v_{j-1},\ q_j v_j\right\}, \qquad j \geq 2. \end{aligned} \tag{6.7}$$

Suppose that $v_{j-1} > 0$ and $p_{j-1} v_{j-1} < q_j v_j$. Then $rv_j = q_j v_j$. Since $r \geq 1$ and $q_j < 1$, we have $v_j = 0$, a contradiction.

Suppose that $p_{j-1} v_{j-1} \geq q_j v_j$. Then

$$v_j = \frac{p_{j-1}}{r} v_{j-1}. \tag{6.8}$$

By iteration,

$$v_j = r^{1-j} P_j v_1, \qquad P_j = \prod_{i=1}^{j-1} p_i, \quad j \geq 2, \quad P_1 = 1. \tag{6.9}$$

We notice that $v_1 = 0$ implies $v = 0$; so we can assume $v_1 > 0$. We substitute this formula into the one for v_1 and divide by rv_1,

$$1 = r^{-1} q_1 + \sum_{j,k=1}^{\infty} \beta_{jk} \min\{r^{-j} P_j, r^{-k} P_k\}. \tag{6.10}$$

Assume that $(P_j) \in X$ and set

$$\mathcal{R}_0 := q_1 + \sum_{j,k=1}^{\infty} \beta_{jk} \min\{P_j, P_k\}. \tag{6.11}$$

It follows that $\mathcal{R}_0 \geq 1$. Since (P_j) is a decreasing sequence,

$$\mathcal{R}_0 = q_1 + \sum_{j=1}^{\infty} \beta_{jj} P_j + \sum_{j=2}^{\infty} \sum_{k=1}^{j-1} \beta_{jk} P_j + \sum_{j=1}^{\infty} \sum_{k=j+1}^{\infty} \beta_{jk} P_k. \tag{6.12}$$

We change the order or summation in the last series and switch j and k,

$$\mathcal{R}_0 = q_1 + \sum_{j=1}^{\infty} \beta_{jj} P_j + \sum_{j=2}^{\infty} P_j \sum_{k=1}^{j-1} (\beta_{jk} + \beta_{kj}). \tag{6.13}$$

We now assume that $\mathcal{R}_0 > 1$. Since our state space is $X = \ell^1$, we assume that $(P_j) \in \ell_1$. Under these assumptions and by (6.3), there exists some $m \in \mathbb{N}$ and $r > 1$ such that

$$q_1 r^{-1} + \sum_{j=1}^{m} \beta_{jj} r^{-j} P_j + \sum_{j=2}^{m} r^{-j} P_j \sum_{k=1}^{j-1} (\beta_{jk} + \beta_{kj}) > 1. \tag{6.14}$$

We define

$$w_j = r^{1-j} P_j, \qquad j = 1, \dots, m, \qquad w_j = 0, \quad j \geq m+1. \tag{6.15}$$

Then, for $j \geq 2, \dots, m$,

$$B_j(w) = r \max\{p_{j-1} r^{1-j} P_{j-1}, q_j r^{-j} P_j\} = r \max\{r^{1-j} P_j, q_j r^{-j} P_j\} = r w_j. \tag{6.16}$$

For $j \geq m+1$,

$$B_j(w) \geq 0 = r w_j. \tag{6.17}$$

Finally

$$B_1(w) = q_1 + \sum_{j,m=1}^{\infty} \beta_{jk} \min\{w_j, w_k\}. \tag{6.18}$$

Since (w_j) is decreasing, as before,

$$B_1(w) = q_1 + \sum_{j=1}^{\infty} \beta_{jj} w_j + \sum_{j=2}^{\infty} w_j \sum_{k=1}^{j-1} (\beta_{jk} + \beta_{kj}). \tag{6.19}$$

We substitute the definition of w_j,

$$B_1(w) = q_1 + \sum_{j=1}^{m} \beta_{jj} r^{1-j} P_j + \sum_{j=2}^{m} r^{1-j} P_j \sum_{k=1}^{j-1} (\beta_{jk} + \beta_{kj}). \tag{6.20}$$

By (6.14), $B_1(w) \geq r = r w_1$. We combine these results and obtain $Bw \geq rw$.

Suppose $r = \mathbf{r}_+(B) \geq 1$. Since B is compact under reasonable assumptions (see Section 6.2.2), there exists some $v \in \dot{X}_+$ with $B(v) = rv$. Then v has the properties above, and it follows that $\mathcal{R}_0 \geq 1$. By contraposition, if $\mathcal{R}_0 < 1$, then $\mathbf{r}_+(B) < 1$.

6.2.4 Persistence Results

Assume that the semiflow induced by F has a compact attractor of points, e.g. under the assumption of Section 6.2.2. Choose the persistence function $\rho : X_+ \to \mathbb{R}_+$ as

$$\rho(x) = x_1, \qquad x = (x_j) \in X_+. \tag{6.21}$$

Theorem 6.1. *If $r = \mathbf{r}_+(B) > 1$, then the semiflow induced by F is uniformly weakly ρ-persistent where ρ is defined above.*

Proof. Let

$$X_0 = \{x \in X_+; \forall n \in \mathbb{Z}_+ : \rho(F^n(x)) = 0\}. \tag{6.22}$$

Consider the ω-limit set ω of an orbit in X_0. Let $x \in \omega$. Since ω is invariant, there exist a total trajectory $(x^k)_{k\in\mathbb{Z}}$ in $\omega \subseteq X_0$ with $x^0 = x$. Then $x_1^k = 0$ for all $k \in \mathbb{Z}$. This implies that $x_2^0 = q_2^k x_2^{-k}$ for all $k \in \mathbb{N}$. Since (x^k) is bounded and $q_2 < 1$, we obtain $x_2^0 = 0$. So $x_2 = 0$ for all $x \in \omega$. A similar consider provides that $x_3 = 0$ for all $x \in \omega$ and finally $x_n = 0$ for all $x \in \omega$.

By (Smith and Thieme, 2011, Theorem.8.20), to prove uniform ρ-persistence, it is sufficient to show that the zero sequence is uniformly weakly ρ-repelling.

Assume that there is some $j \in \mathbb{N}$ such that $\beta_{jj}(x) > 0$ for all $x \in X_+$. Let $x \in X_+$ and $x_1 > 0$. Then $(F^n(x))_{n+1} > 0$ for all $n \in \mathbb{N}$ and $(F^j(x))_1 > 0$. This way we see that, if $x_1 > 0$, $\rho(F^n(x)) > 0$ for infinitely many $n \in \mathbb{N}$.

Suppose that the zero sequence is not weakly uniformly ρ-repelling. Choose $\epsilon > 0$ such that $r(1-\epsilon) > 1$. Choose $\delta > 0$ such that $F(x) \geq (1-\epsilon)B(x)$ for all $x \in X_+$ with $\|x\| \leq \delta$.

After a shift in time, we have some $x \in X_+$ with $x_1 > 0$ and $\|F^n(x)\| \leq \delta$ for all $\delta > 0$. We assume that also $\beta_{j+1,j+1}(0) > 0$. Then for some sufficiently large

n we have that $(F^n(x))_1 > 0$ and $(F^n(x))_2 > 0$. For some even bigger $n \in \mathbb{N}$, we have that $(F^n(x))_k > 0$ for $k = 1, 2, 3$. Finally, for some sufficiently big $n \in \mathbb{N}$, we have $(F^n(x))_k > 0$ for $k = 1, \ldots j$. For this n we then also have the $F^{(n+i)}(x)_k > 0$ for $k = 1, \ldots, j+i$, $i \in \mathbb{N}$. Eventually, we find some $i \in \mathbb{N}$ such that $F^{n+i}(x)$ is w-positive. So, after a shift in time, we can assume that $x \in X_+$ is w-positive and $\|F^n(x)\| \leq \delta$ for all $n \in Z_+$. Then we obtain that $[F^n(x)]_w \geq (r(1-\epsilon))^{n-1}[x]_w > 0$ and so $[F^n(x)]_w \to \infty$, a contradiction. Hence the semiflow induced by F is uniformly weakly ρ-persistence.

□

Assume that there exists $m \in \mathbb{N}$ such that $\beta_{m,m}(x) > 0$, $\beta_{m+1,m+1}(x) > 0$ for all $x \in X_+$. Also assume that $p_j(x) > 0$ for $j = 1, \ldots, m$. Define the persistence function $\tilde{\rho} : X_+ \to \mathbb{R}_+$ as

$$\tilde{\rho}(x) = \sum_{j=1}^{m} x_j, \qquad x = (x_j) \in X_+. \tag{6.23}$$

Then we have following results:

Theorem 6.2. *If $r = \mathbf{r}_+(B) > 1$, then the semiflow induced by F is uniformly $\tilde{\rho}$-persistent where $\tilde{\rho}$ is defined above. If the semiflow F has a compact attractor of points, then the semiflow induced by F is uniformly $\tilde{\rho}$-persistent.*

Proof. Since $\tilde{\rho}(x) \geq \rho(x)$ in (6.21) and the semiflow induced by F is uniformly weakly ρ-persistent in Theorem 6.1, there exists some $\epsilon > 0$ such that

$$\lim_{n\to\infty} \sup \tilde{\rho}(F^n(x)) \geq \lim_{n\to\infty} \sup \rho(F^n(x)) \geq \epsilon.$$

Therefore the semiflow induced by F is uniformly weakly $\tilde{\rho}$-persistent.

Also if $\tilde{\rho}(x) > 0$, then $\rho(F^n(x)) > 0$ for some $n \in \mathbb{N}$. Since $\tilde{\rho}(x) > 0$ implies at least one of $x_i > 0$ where $i \in 1, \ldots, m$, by assumption $p_j(x) > 0$ for $j = 1, \ldots, m$, there exists some $t \in \mathbb{N}$ such that $F^t(x)_m > 0$. By assumption $\beta_{m,m}(x) > 0$, we have $F^{t+1}(x)_1 > 0$. Thus $\rho(F^{t+1}(x)) > 0$.

To prove uniform $\tilde{\rho}$-persistence, we show that if $\tilde{\rho}(x) > 0$, then $\tilde{\rho}(F(x)) > 0$.

Let x be an element in total trajectory $(x^t)_{t\in\mathbb{Z}}$ such that $\tilde{\rho}(x) > 0$, then at least one of $x_i > 0$ where $i \in 1, \ldots, m$. If $i < m$, then by assumption $p_j(x) > 0$ for $j = 1, \ldots, m$, we have $F(x)_{i+1} > 0$ where $i + 1 \in 2, \ldots, m$. Hence

$$\tilde{\rho}(F(x)) > F(x)_{i+1} > 0.$$

If $i = m$, then by assumption $\beta_{m,m}(x) > 0$, we have $F(x)_1 > 0$. Hence

$$\tilde{\rho}(F(x)) > F(x)_1 > 0.$$

Therefore we have if $\tilde{\rho}(x) > 0$, then $\tilde{\rho}(F(x)) > 0$. If there exists some $s \in \mathbb{N}$ such that $\tilde{\rho}(x^{-s}) > 0$, then

$$\tilde{\rho}(F^n(x^{-s})) = \tilde{\rho}(x^{n-s}) > 0$$

for all $n \in \mathbb{N}$. This contradict with $\tilde{\rho}(x^0) = 0$ since we can choose $n = s$. So (H1) in (Smith and Thieme, 2011, Sec.5.1) has been checked.

Now we can apply (Smith and Thieme, 2011, Theorem 4.5) to get the conclusion that the semiflow induced by F is uniformly $\tilde{\rho}$-persistent. $\square$

Theorem 6.3. *Let the assumptions of Theorem 6.1 be satisfied, further assume that the semiflow induced by F has a compact attractor of points, then the semiflow induced by F is uniformly ρ-persistent.*

Proof. By Theorem 6.1 and Theorem 6.2, the semiflow induced by F is uniformly $\tilde{\rho}$-persistent. To apply (Smith and Thieme, 2011, Corollary 4.22), we need to check assumption (i), the semiflow induced by F is point-dissipative and asymptotically smooth, this is given by Section 6.2.2.

Then we need to check assumption (ii), if we have a total trajectory $(x^t)_{t\in\mathbb{Z}}$ with pre-compact range and

$$\inf_{t\in\mathbb{Z}} \tilde{\rho}(x^t) > 0,$$

then $\rho(x^0) > 0$. Since $\tilde{\rho}(x^t) > 0$ for all $t \in \mathbb{Z}$, at least one of $x_i^t > 0$ where $i \in 1, \ldots, m$. By assumptions $p_j(x) > 0$ for $j = 1, \ldots, m$ and $\beta_{m,m}(x) > 0$, after a shift in time, we have $x_1^0 > 0$. Therefore $\rho(x^0) > 0$.

Both assumptions are checked, we have the semiflow induced by F is uniformly ρ-persistent by (Smith and Thieme, 2011, Corollary 4.22). □

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