Share Auctions with Linear Demands

by

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ABSTRACT

Buyers have private information on auctioning divisible goods. Linearity could be a useful property on measuring their marginal utility on those goods or on their bidding strategies under such a share auction environment. This paper establishes an auction model with independent private-values paradigm (IPVP) where bidders have linear demand. A mechanism design approach is applied to explore the optimal share auction in this model. I discuss the most popular auction formats in practice, including Vickrey auction (VA), uniform-price auction (UPA) and discriminatory price auction (DPA). The ex-post equilibriums on explicit solutions are achieved. I found VA does not generally constitute an optimal mechanism as expected even in a symmetric scenario. Furthermore, I rank the different auction formats in terms of revenue and social efficiency. The more private information bidders keep, the lower revenue VA generates to seller, and it could be even inferior to UPA or DPA. My study aggregates dispersed private information with linearity and is robust to distributional assumption.

DEDICATION

To Shugang, Yimei and Daisy

In memory of Dr. Lin Chen (1981-2009)

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CHAPTER 1

INTRODUCTION

Auctions of divisible goods are common in many markets, especially in markets for financial securities, which include the auctions of treasury bills and bonds, defaulted bonds and loans in the settlement of credit default swaps, energy products, and environmental emission permits, say Carbon Dioxide Pollution Permit CDPP. In those auctions, the bids specify quantities of the divisible goods: the share of stock, megawatt-hours of electricity, or the tons of emissions, and also, the unit or total price to pay for those shares. We called such selling mechanisms share auctions.

Rather than selling the whole unit to one individual buyer (called the unit auction or indivisible good auction), share auction is designed to enable smaller firms and more risk-averse firms to participate in the auctions of highly risky leases by allowing them to bid for fractional working interest shares, thereby reducing their capital requirements for payment of the sale price, and also reducing their exposure to risk. Analyzing the bidding behavior in these auctions helps us better understand information aggregation, allocative efficiency, and market design.

In a typical share auction, bidders submit demand schedules. The auctioneer computes the aggregate demand and the market clearing (or stop-out) price by equalizing demand and available supply of shares in aggregation. Then the target is divided to those bidders by specific rules. Three sealed-bid auction formats for the sale of divisible goods are of particular interest. The first two are important on practical grounds, they are widely used in real world, and the last, although not widely used, is of special interest for theoretical reasons. All of them are based on standard auction pricing rules where the highest bids win, but different on payment rules.

• **Uniform Price Auction UPA**: each bidder pays the stop-out price for the total share he wins.

• **Discriminatory Price DPA**: each bidder pays what he bids, up to the share he wins. The rule is comparable to the first price auction for indivisible good.

• **Vickrey Auction VA**: each bidder pays the highest losing bids, up to the share he wins. It is comparable to the second price auction for indivisible good. Ausubel (2004) proposes a dynamic ascending-bid auction for homogeneous goods. The auctioneer calls a price, bidders respond with quantities and the process iterates with increasing price until demand is no greater than supply. Items are awarded at the current price whenever they are clinched. It shows that with private values, this auction yield exactly the same outcome as sealed-bid Vickrey auction, but has advantage through its operability, simplicity and privacy preservation.

There has been a longstanding debate between the two mechanisms most commonly used to sell divisible goods: the discriminatory-price ("pay-as-bid" or "multiple-price") auction and the uniform-price ("single price") auction. Rostek, Weretka and Pycia (2010) cite an interesting cross-country study on Treasury practices. Out of 48 countries it surveyed, 24 use a DPA to finance public debt, 9 use a UPA, 9 employ both auction formats, depending on the type of security being issued, and the remaining 6 countries use a different mechanism. Apart from financial securities, the two formats have also become standard designs when selling divisible goods in other markets. For energy products, U.K. electricity generators adopted UPA in 1990 and switch to DPA in 2000.

The national exchange for sulfur-dioxide emission permits uses a discriminatory price format. The European Union implemented the Emission Trading Scheme ETS for trading on CDPP and other climate policy relevant products. Their auctions are conducted as a sealed bid uniform price auction.

Two general questions will naturally be inquired by both practitioners and economics theorists. First, among UPA, DPA, VA and any potential others, which one is the best rule or mechanism to sell a divisible good at the seller's interests? Second, what is the best rule at society's interest, is a specific auction Pareto-optimal or efficient? Of course the answers depend on the bidding environment including bidder's information structure, risk preference and the bidding rules allowed where the seller cares about the combination of efficiency (allocating the multi units divisible goods to the bidders who value them the most) and the revenue maximization.

Due to the complexity of modeling a divisible goods market that covering every detail of trading process, quite few conclusions are known about the optimality of a share auction with strategic bidders, even about the superiority of either UPA or DPA. Our study provides those results and examines the optimal design of divisible good markets in a simple setting, without losing the key features under preliminary environment. The core assumptions of our model are illustrated in the following.

Linearity is the first and the most critical assumption we make on our model. Holmstrom and Milgrom (1987) argue the linear contract could be the optimal compensation scheme even if the agent has a rich set of actions to choose from, including highly nonlinear scheme. Linear equilibria are tractable, particularly in the presence of independent private information, have desirable properties like simplicity, and have proved to be very useful as a basis for empirical analysis. Hortacsu (2002) uses detailed bidder-level data from Turkish treasury auction market to show that bidders' demand functions can be represented quite closely by a simple linear interpolation of the bids. A straight line interpolation through the multiple price-quantity pairs can explain 92% of the variation in the multiple price and quantity pairs submitted by the bidders. Hence, a divisible-good auction model that would generate linear demand functions for each bidder as equilibrium bidding strategies would provide a good description of the data. Kastl (2012) shows that on the share auction where bidders are required to submit multiple price-quantity pairs as their bidding strategies, they would submit few step bid functions than the maximal number are allowed to use. It is another evidence to support the bidding equilibriums on divisible goods could be as simple as a linear combination.

In our model, a given quantity of a perfectly divisible good is sold to strategic buyers who have (weakly) decreasing linear marginal utility (e.g., mean-variance preferences). So it is a simple and smart way to account for bidder's risk preference. Maskin and Riley (2000) assumed bidders have a general decreasing demand on divisible goods and they tried to characterize the optimal auction in their model under independent private-values paradigm IPVP. But no explicit solution can be derived due to the generality of demand function. Tenorio (1999) in a private value but indivisible multiple unit auction gave a geometric form to bidders' utility function so analytical solution could be derived easier. Similar to that idea, we take the setting as Rostek, Weretka and Pycia (2010), bidders have a quadratic utility function on the share they get. Such a function is concave so bidders are risk averse, which is equivalent to the setting that each bidder has a constant marginal value for the good, up to a fixed capacity. Similar setup does also appear in Vives (2010, 2011), Du and Zhu (2013) and Ollikka (2011).

We examine the comparative design of three auction formats for divisible goods: the commonly used uniform and discriminatory price auctions, and as a theoretical benchmark, Vickrey auction. By analyzing linear Nash equilibria, we are able to characterize bidder equilibrium price impact and deliver sharp comparisons.

Another important perspective that highly discussed among different auction models is the information structure of bidders. The common value environment is proper to model the auctioning object with the *ex-post* fixed valuation, say the lease for gas or oil exploration, and most of financial securities. However, some empirical studies suggested that in practice, players in those common value divisible goods may well have private values. Hortacsu and Kastl (2008) exploited data from Canadian Treasury bill auctions. They developed a test applying structural and nonparametric estimation to see if bidder's values are private. They cannot reject the null hypothesis of private values in 3-months treasury bills, but reject private values for 12-months treasury bills. Intuitively, in the long run, the security price goes stable and its valuation seems to be same for everyone, but in short run, price fluctuates a lot so bidders have to value its risk and their personal preference on investment. In this scenario, a private value model may work better. What is more, there is some share auction examples fit into the IPVP perfectly. Consider a CDPP auction mentioned above. Factories sell their redundant CDPP quota to different buyers, including industry competitors, NGO or even individual environmentalists. Buyers are not profit-incentive at this auction because there is no second-hand market exists. They value the object independently for different purposes, but not the common price it worth on market, if there exists so.

In our model, bidders start with private valuations of the divisible goods and have diminishing marginal values for owning it. Due to the independence, a bidder's one-dimensional demand schedule handles the $(n-1)$ dimensional uncertainty regarding all other bidders' valuations.

Working with linearity and independent private value paradigm allows us to overcome the complexity of optimal auction design. But given the lack of dominance of equilibrium sets for the different auction formats, it is hard to compare the revenue or other criteria without an explicit equilibrium selection. For instance, in the uniform price design, the equilibrium selection that has become the workhorse model in the financial microstructure and industrial organization literature is the linear Bayesian Nash equilibrium as Kyle (1985, 1989). This paper implements *ex post* equilibrium as the solution concept. The bidder would not deviate from his strategy even if he would observe the private information or signals of others. It is robust and implies no regret. Beyond this, *ex post* equilibrium is also unshakeable with different distributional assumptions and increasing transformation of bidders' utility functions, see Du and Zhu (2013), in which *ex post* equilibrium is characterized in uniform price double auctions of divisible assets. Other representing papers study equilibria that are *ex post* optimal with respect to supply shocks when bidders have symmetric information regarding the asset value. Related papers include Klemperer and Meyer (1989), Ausubel, Cramton, Pycia, Rostek, and Weretka (2011) and Rostek and Weretka (2012), among others.

We now proceed to the main predictions of this article. When bidders have linear marginal utility under the IPVP as the information structure, we achieve the explicit optimal share auction. Most notably, the optimal solution performs as the competitive market equilibrium. The seller aggregates bidders' demand with virtual valuations as the estimation of the private signals. Market equilibrium price, or we call it the shadow price in this optimization process, will be decided by equalizing exogenous total supply and endogenous aggregate demand. To compensate information advantage that bidders hold, seller could not fully extract all of surplus but leave positive information rent as the consumer surplus through the payment rules. Since the asymmetric and independent private value bidders have, one can expect the solution we have is neither universal nor anonymous that each bidder's type distribution counts for the optimal result. It is also not efficient because the virtual valuation could not fully reveal the true information of bidders, where the similar arguments applied in the canonical optimal unit auction design by Myerson (1981). The unexpected result is that even in a symmetric environment with much stronger assumption, Vickrey auction could not attain to the optimal solution, which not analogy to the second price auction in indivisible goods. Some previous studies shows the VA or VA with reserve price could constitute an optimal auction. Those include independent private value models like Harris and Raviv (1981-1, 1981-2), Segal (2004); interdependent model like Ausubel and Cramton (2004). We show VA would approximate the solution if the inverse hazard rate is low in which bidders lose their information advantage.

Following the symmetric environment, we look at the comparative performance of various practical designs of divisible good markets, including UPA, DPA and VA. With the strong linear *ex-post* equilibrium concept applied, we could do the revenue ranking for those different auction formats. Some studies on common value divisible goods auctions predict the ranking is ambiguous due to the specific properties of distributions, *e.g.* Back and Zender (1993), Wang and Zender (2002) or Ausubel, Cramton, Pycia, Rostek and Weretka (2011). We conclude under IPVP, the DPA dominates the UPA for all distributions considered, but how the VA is ranked depends on the variance of bidders' private signals. This finding coincides with the optimal mechanism we derived. When the variance of private values is small which means bidders keep relatively lower information advantages, then the revenue ranking $VA \succ DPA \succ UPA$ holds. In the limit if variance is pretty closed to zero, the VA is approaching the optimal solution that revenue dominance of DPA will diminish. We would have the revenue equivalence of $VA \succ DPA{\sim}UPA$. On the opposite way, if the variance is relatively high enough, we could have $DPA \succ UPA \succ VA$. To understand the expected-revenue dominance of the DPA over the UPA in non-competitive auctions, it is useful to pin down the effects from which the revenue differences derive in the two formats. On the robust *ex-post* equilibriums we derived, the UPA design indeed fosters more aggressive bidding than DPA, measured by lower demand reduction.

Further, our analysis draws attentions to the social welfare. Unlike most of literature discussing common value divisible goods market with symmetric bidders, that the equilibrium allocations are trivially efficient, we can compare the efficiency of auction formats under IPVP. Since truth-telling constitute the dominant strategy, the VA obviously generate the potential highest total welfare for both seller and bidders. We also predict the UPA is more efficient than DPA by maintaining higher *ex-post* aggregate social welfare.

Our study is related to share auction design under IPVP. So we would make couple of policy suggestions to the relevant markets which best fit into our model, especially those where assets liquidity is relevantly low, like emission allowance markets, landing slots or import quota licenses. On the best interests of seller, the auction designer should evaluate how much they know about the potential auction participants before he/she choose the relevant formats. If the auctioneer get enough information concern bidders' reservation price in advance, the sealed-bid VA or on equivalence the ascending bid clinging auction proposed by Ausubel (2004) which is more straight forward in practice, sounds more dominant in expected revenue. Otherwise, without any information advantage, the seller should pursue the DPA rather than VA or UPA. But on the best interests of the whole society, Ausubel's auction is better than any other share auction format.

There is lack of general theoretical guidelines for the optimal design of divisible good markets. The reasons behind include the multi-dimensional uncertainty in mechanism design for multi-unit auctions, the complexity of bidders' demand and so on. Harris and Raviv (1981-1, 1981-2) look at the optimal mechanism when bidders' type space is discrete. Under the IPVP environment, the optimal auction approaches the VA or some modified VA (Harris and Raviv 1981-2) if the number of private types going to infinity. Segal (2004) allows the continuous type space but assume bidders have unit demand.[1] It shows that when bidders' valuations are independently draw from the same

[1] The demand of each bidder is either one or zero.

known distribution (symmetric IPVP), the optimal mechanism is equivalent to the Vickrey-Groves-Clarke mechanism in which each buyer pays the externality he imposes on others. So efficiency can be achieved there. Maskin, Riley and Hahn (1989) build up their model with symmetric IPVP but allow generally downward-sloping demand of bidders. They start with unit demand and conclude the standard selling procedure that winners are the highest bidders equal to supply and they pay their bids constitute an optimal solution. But when it extends to multiunit demands, the story changes that those standard formats are not optimal anymore. The new optimal selling procedure involves a nonlinear pricing scheme which implicitly predetermined, but could not explicitly be solved in the model.

There is relatively more literature focus on comparison of different share auction formats on ground, with for instance seller's revenue ranking. In the canonical and one of the earliest analyses of price-quantity choice issue in divisible good, Wilson (1979) shows by examples that compared to unit auction, the seller might experience a remarkable reduction in revenue. It is a common value environment with constant marginal utility that each bidder submits a decreasing bidding strategy $q(p)$ as demand schedule. The inferiority of share auction exists in all formats, UPA, DPA and VA. The multiplicity of equilibrium strategies enables the bidders to choose one that could be severely disadvantageous to the seller. Following Wilson's setup, Back and Zender (1993) conclude UPA might be worse than DPA on seller's expected revenue. But there may be other equilibira where the ranking of the auctions is reversed. Same revenue ranking is also characterized in Wang and Zender (2002). They prove when bidders are risk averse, there may exist equilibira of UPA that provide higher expected revenue than DPA.

Rostek, Weretka and Pycia (2010) further extend this common value model with bidders have decreasing marginal utility and linear Bayesian Nash equilibrium. Their model accommodates small and large markets, as well as different risk preferences of the buyers and the sellers. The revenue ranking that $DPA \succ VA \succ UPA$ with strategic risk-averse bidders is concluded. Ollikka (2011), which has the similar model setup, verified the same revenue ranking by various numerical examples. Kremer and Nyborg (2004-1, 2004-2) illustrate the underpricing exist in UPA and provide a simple allocation rule (*pro rata*) that specify the way divided in cases of excess demand to eliminate this effect.

Some other papers have the models of private value. Kastl (2012) restricts bidders' strategy into finite number of bids with price-quantity combinations. The equilibrium existence is discussed in both UPA and DPA. Tenorio (1999) looks at the case that seller auction-off multi identical units with bidders have different risk attitude (risk averse, risk neutral or risk prefer). He concludes both reservation price and reservation quantity would increase the expected revenue. Kyle (1989) introduced private information into a double auction for a risky asset of unknown liquidation value and derived a unique symmetric linear Bayesian equilibrium in demand schedules when traders have constant absolute risk aversion, there is noise trading, and uncertainty follows a Gaussian distribution.

The balance of paper is organized as follows. Section 2 presents the general model. Section 3 defines and characterizes the optimal share auction under our framework. The linear *ex-post* equilibria of different auction formats and the revenue ranking and efficiency are discussed in Section 4. Section 5 concludes. All proofs are included in Appendices.

CHAPTER 2

MODEL

There is totally I unit divisible good to be auctioned off from a risk-neutral seller (she) and n potential risk-averse bidders (he) are bidding for it. Seller is seeking to maximize her expected profit from the payments. Bidder i's utility is characterized by the amount of divisible good received q_i, where $\sum_{i=1}^{n} q_i \leq I, q_i \geq 0\ \forall i$ as the feasibility conditions and the money he pays. His utility u_i is given by a quasi-linear consumer revenue function $v_i(q_i) + money_i$. We assume bidder's willingness to pay or the marginal value of q_i is linear,

$$\frac{\partial v_i(q_i)}{\partial q_i} = \alpha_i - \beta_i q_i$$

where the stochastic intercept $\alpha_i > 0$ is the private information of him, and the parameter $\beta_i > 0$ that measures the convexity of his utility function is public information. Vives (2010) interpret β_i as an adjustment for transaction cost or opportunity cost. Rostek, Weretka and Pycia (2010) further argue that when the auctioned good is a risky asset with a normally distributed payoff, and the bidders have CARA utility functions, then β_i measures risk aversion. Put this framework under the competitive market environment, it is exactly the same to say each bidder face a linear demand $p = \alpha_i - \beta_i q_i$ that $p \geq 0$ is the market price on the target goods. Then we have a linear-quadratic consumer revenue function.

$$v_i(q_i) = \int_0^{q_i} (\alpha_i - \beta_i q_i)\, dq_i = \alpha_i q_i - \tfrac{1}{2}\beta_i q_i^2$$

Figure 1 shows this relationship clearly.

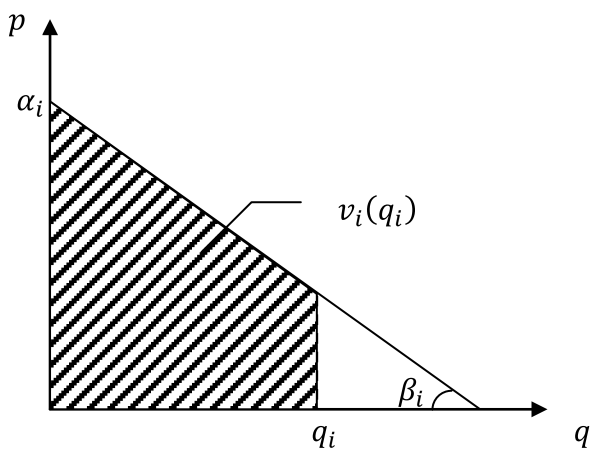


Figure 1: The Linear Demand and Consumer Revenue

The linear-quadratic utility function is widely utilized in Vives (2011), Rostek and Weretka (2012), Rostek, Weretka and Pycia (2010), Du and Zhu (2013) and Ollikka (2011), but the models have different information structures.

The information structure we adopt here is called independent private values paradigm IPVP. Bidder knows his type α_i, but β_i is common knowledge. They are asymmetric. Their private information is independently distributed on the interval $\chi_i = [\underline{w}_i, \overline{w}_i]$ according to the density function f_i. Let players' type space be $\chi = \prod_{i=1}^{n}[\underline{w}_i, \overline{w}_i]$, realized types $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_{-i} = (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n)$ and $\chi_{-i} = \prod_{j \neq i}[\underline{w}_j, \overline{w}_j]$, for notational ease. Define $f(\alpha)$ to be the joint density of α, and $f_{-i}(\alpha_{-i})$ to be the joint density of α_{-i}.

In a share auction, each bidder is required to submit a bidding strategy $b_i(\alpha_i)$ simultaneously. The seller should decide allocation rules $q_i(b_1, b_2, \dots, b_n)$, where $\sum_{i=1}^{n} q_i \leq I, q_i \geq 0\ \forall i$, and payment rules $m_i(b_1, b_2, \dots, b_n)$ for each bidder. For instance, the most popular divisible good auction in practice is called uniform-price auction UPA. It requires bidder to submit his demand schedule $b_i(p;\alpha_i)$ which is unobservable to others except the seller. The demand schedule specifies that he wishes to buy a quantity $b_i(p;\alpha_i)$ of the target good at price p. Then the seller determines the selling price p^* from market clearing condition

$$\sum_{i=1}^{n} b_i(p^*;\alpha_i) = I$$

After this, bidder i is awarded the quantity $q_i = b_i(p^*;\alpha_i)$ and he pays $m_i = b_i(p^*;\alpha_i)p^*$ as the payment. Other auction formats people interested in are discriminatory auction DA and Vickrey Auction, for instance. We will discuss them all in details later on.

CHAPTER 3

OPTIMAL AUCTIONS DESIGN

Among different kind of auction rules, let's look at what is the optimal one for maximizing seller's revenue.

3.1 Setup

First of all, in the language of mechanism design, we are interested in a direct auction scheme that all bidders will submit their types rather than demand schedules.

Definition 1. From seller's perspective, a **direct mechanism** (q, m) is a pair of functions $q: \chi \to \Delta; m: \chi \to \mathbb{R}^N$, where $\Delta = \{(q_1, q_2, \dots, q_n): \sum_{i=1}^{n} q_i \le I, q_i \ge 0\ \forall i\}$, $q_i(\alpha)$ is the share that i will get and $m_i(\alpha)$ is the relative payment.

By Revelation Principle, we can just restrict our attention to direct mechanisms to find out an optimal one.

Each bidder's utility is his consumer revenue v_i and money transfer m_i. Given a direct mechanism (q, m), if i reports z_i and other bidders truthfully reveal their types, his *ex-post*[2] utility would be the consumer revenue add money transfer

$$u_i(z_i, \alpha_{-i}) = v_i(q_i) + money_i = \alpha_i q_i(z_i, \alpha_{-i}) - \tfrac{1}{2}\beta_i q_i^2(z_i, \alpha_{-i}) - m_i(z_i, \alpha_{-i}) \qquad (1)$$

Then the *ex-mid* utility of him when reporting z_i can be represented as $\alpha_i Q_i(z_i) - \tfrac{1}{2}\beta_i R_i(z_i) - M_i(z_i)$, where

$$Q_i(z_i) = \int_{\chi_{-i}} q_i(z_i, \alpha_{-i})\, f_{-i}(\alpha_{-i}) d\alpha_{-i}$$

[2] We adopt the phrases *ex-post, ex-mid, ex-ante* to express the different stages of information revelation. *Ex-post* means when all bidders private information are revealed; *ex-mid* means bidder is unknown about others' types except his own; *ex-ante* is the stage that all kinds of private information are unknown for seller.

is the expected share he would get when he reports z_i and all others tell the truth. Similarly, define

$$M_i(z_i) = \int_{\chi_{-i}} m_i(z_i, \alpha_{-i})\, f_{-i}(\alpha_{-i}) d\alpha_{-i}$$

to be the expected payment of i when he reports z_i and all other bidders tell the truth. Last, let

$$R_i(z_i) = \int_{\chi_{-i}} q_i^2(z_i, \alpha_{-i})\, f_{-i}(\alpha_{-i}) d\alpha_{-i}$$

just for ease of notation purpose.

By Revelation Principle, given equilibrium for any auction format, there always exists a direct mechanism that the truth telling is equilibrium and the outcomes are the same as the original auction format. So in the process of finding an optimal auction, we could only restrict our attention on those direct mechanisms where bidders have no incentive to hide their types.

Definition 2. The direct mechanism (q, m) is said to be **incentive compatible (IC)** if and only if for all buyer i, all his type α_i and other potential types z_i,

$$U_i(\alpha_i) \equiv \alpha_i Q_i(\alpha_i) - \tfrac{1}{2}\beta_i R_i(\alpha_i) - M_i(\alpha_i) \geq \alpha_i Q_i(z_i) - \tfrac{1}{2}\beta_i R_i(z_i) - M_i(z_i)$$

Now we are interested in what kind of direct auction formats are satisfied IC condition. The following proposition will tell you that.

Proposition 1. (Incentive Compatible) *The direct mechanism* (q, m) *is incentive compatible* (IC) *if and only if the following two conditions are satisfied:*

(1) Allocation rule: $Q_i(\alpha_i)$ *is increasing on* α_i

(2) Payment rule: $M_i(\alpha_i) = \alpha_i Q_i(\alpha_i) - \frac{1}{2}\beta_i R_i(\alpha_i) - \int_{\underline{w_i}}^{\alpha_i} Q_i(t_i) dt_i - U_i(\underline{w_i})$

This conclusion is very intuitive. Basically, it says a direct auction scheme would implement truth telling if it awards more shares of goods, on expectation, to bidder who has higher reported marginal value. Also, please notice that through the definition of IC condition, the payment rule is equivalent to

$$U_i(\alpha_i) = U_i(\underline{w_i}) + \int_{\underline{w_i}}^{\alpha_i} Q_i(t_i) dt_i \tag{2}$$

It tells us the expected utility to bidders depends only on the allocation rule q, with an additive constant comes from $U_i(\underline{w_i})$. Due to this specific property of an incentive compatible direct auction scheme, we need a boundary condition to figure out the payment rule. Following the general literature of mechanism design, we assume the bidder will not participate in the auction if he could not expect to get a positive payoff.

Definition 3. The direct mechanism (q, m) is said to be **individual rational** (**IR)** if for all buyer i, all his type α_i, the expected payoff $U_i(\alpha_i) \geq 0$ on the equilibrium.

By equation (2), it is straight forward to conclude that for any direct mechanism satisfied IC, the individual rationality IR is equivalent to $U_i(\underline{w_i}) \geq 0, \forall i$.

3.2 Solution

Now let us view the seller as the designer of the auction and looking for the one with maximal expected revenue among all direct mechanisms that are incentive compatible and individually rational. Under our information structure, we should forget that seller does not know any potential bidder's type before the auction. So she is designing the optimal share auction (q, m) which would generate the highest *ex-ante* expected revenue or payment through the aggregation of all bidders.

Proposition 2.(Designer's Problem) *Suppose* $q = (q_1(\alpha), q_2(\alpha), \dots, q_n(\alpha))$ *maximize*

$$\sum_{i=1}^{n} \{\varphi_i(\alpha_i) q_i(\alpha) - \tfrac{1}{2}\beta_i q_i^2(\alpha)\},$$ [3]

subject to the feasibility conditions $\sum_{i=1}^{n} q_i(\alpha) \leq I, q_i(\alpha) \geq 0\ \forall i, \forall \alpha$ *and* $q_i(\alpha)$ *is increasing on* α_i *for any* α_{-i}. *Suppose also that for* $\forall i, \forall \alpha$

$$m_i(\alpha) = \alpha_i q_i(\alpha) - \tfrac{1}{2}\beta_i q_i^2(\alpha) - \int_{\underline{w}_i}^{\alpha_i} q_i(t_i, \alpha_{-i}) dt_i. \tag{3}$$

then (q, m) *is an optimal share auction.*

To better interpret the process of optimal auction design, here are some preliminary works including assumption and technical lemma. For ease of notation reason, we utilize φ_i to stand for $\varphi_i(\alpha_i)$ and q_i for $q_i(\alpha)$ in the left of this section and relative proof.

Assumption 1. (Regularity) The virtual valuation φ_i is increasing on α_i for any bidder i. Such an auction design problem is called **regular**.

[3] $\varphi_i(\alpha_i) \equiv \alpha_i - \frac{1-F_i(\alpha_i)}{f_i(\alpha_i)}$ is called virtual valuation of a bidder with his type as α_i.

The regularity assumption on φ_i is generally taken for most of classic mechanism design approach. It include most of popular continuous random distribution F_i on type space $\chi_i = [\underline{w}_i, \overline{w}_i]$. For instance, say F_i is uniform distribution. We have $f_i(\alpha_i) = \frac{1}{\overline{w}_i - \underline{w}_i}$ and $F_i(\alpha_i) = \frac{\alpha_i - \underline{w}_i}{\overline{w}_i - \underline{w}_i}$. So $\varphi_i(\alpha_i) = 2\alpha_i - \overline{w}_i$ that is increasing on α_i. Please notice that $\gamma_i(\alpha_i) \equiv \frac{1-F_i(\alpha_i)}{f_i(\alpha_i)}$ is actually the reciprocal function of hazard rate of F_i, so we call it **inverse hazard rate**. Bulow and Roberts (1989) gives very intuitive explanation of what does the virtual valuation φ_i mean in traditional economics that we will discuss it later on.

Without loss of generality, we rank all virtual valuation as

$$\varphi_1 \geq \varphi_2 \geq \cdots \geq \varphi_M > 0 \geq \varphi_{M+1} \geq \cdots \geq \varphi_n$$

where $M \in \{1,2,\ldots,n\}$. Please be aware that it does not necessarily imply the ranking of bidder's type α_i even though we assume φ_i is increasing. Think of the asymmetric information environment that different bidder have different virtual valuation. Then the following important technical lemma is characterized for the purpose of optimality planning.

Lemma 1. *Let* $\sigma_K \equiv \sum_{i=1}^{K} \frac{\varphi_i - \varphi_K}{\beta_i} - I$, *where* $K \in \{1,2,\ldots,M\}$. *Then there is a unique* $L \in \{1,2,\ldots,M\}$ *such that* $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_L < 0 \leq \sigma_{L+1} \leq \cdots \leq \sigma_M$, *or* $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_M < 0$ such that $\mathrm{L} = \mathrm{M}$.

We called this specific bidder with Lth highest virtual valuation φ_L the **pivotal bidder**. It is an important intermediate result for solving the optimality problem. We will discuss more intuition later on that why we need such a definition.

Thus, we obtain the main result of this section as following

Proposition 3.(Optimal Share Auction) *Suppose the auction design problem is regular, then the following direct auction scheme* (q, m) *constitute an optimal share auction:*

(1) Allocation rules:

If $\sum_{i=1}^{M} \frac{\varphi_i}{\beta_i} \leq I$, $q_i = \frac{\varphi_i}{\beta_i}$ *for* $\forall i \leq M$ *and* $q_i = 0$ *for* $\forall i > M$*;*

If $\sum_{i=1}^{M} \frac{\varphi_i}{\beta_i} > I$, $q_i = \frac{\varphi_i - \lambda}{\beta_i}$ *for* $\forall i \leq L$ *where* $\lambda = \frac{\sum_{i=1}^{L} \frac{\varphi_i}{\beta_i} - I}{\sum_{i=1}^{L} \frac{1}{\beta_i}}$ *and* $q_i = 0$ *for* $\forall i > L$*;*

(2) Payment rules:

$$m_i(\alpha) = \alpha_i q_i(\alpha) - \frac{1}{2}\beta_i q_i^2(\alpha) - \int_{\underline{w}_i}^{\alpha_i} q_i(t_i, \alpha_{-i}) dt_i.$$

To better understand the optimal share auction we obtain, let's look at what is the virtual valuation $\varphi_i(\alpha_i)$. It is well known that in optimal indivisible good design under the independent private values paradigm IPVP, the target will be awarded to the bidder with highest $\varphi_i(\alpha_i)$ if it is positive. Bulow and Roberts (1989) argue $\varphi_i(\alpha_i)$ is actually the marginal revenue of seller when she charges the bidder i a take-it-or-leave-it offer at price α_i for this indivisible good. After though, the seller choose the winner with the highest marginal revenue.

In our model, the I units divisible good is not necessary to be awarded to one individual buyer. The seller is trying to assign her good to bidders who have relatively stronger incentive to buy, that is to say, to bidders who has higher demand $p = \alpha_i - \beta_i q_i$. Under the asymmetric information environment, since α_i is unknown before trading, seller will use $\varphi_i(\alpha_i)$ instead. The virtual valuation here could be interpreted as the marginal revenue of seller when she charges α_i at the last traded small amount from $q_i = 0$.

On the optimality, the seller will choose bidder with positive $\varphi_i(\alpha_i)$ as candidates and their demands or marginal revenue as $p = \varphi_i(\alpha_i) - \beta_i q_i$. In the scenario of indivisible good, the winner must be with $\varphi_i(\alpha_i) > 0$ since the seller would lose money when facing negative marginal revenue. Similar to that, in our optimal solution of divisible good, we only assign the share to bidder who has positive virtual valuation, or positive demand. Then to be more specifically on the allocation rule, if the market cannot be cleared, which is $\sum_{i=1}^{M} \frac{\varphi_i}{\beta_i} \leq I$, then each candidate will be awarded the maximal potential demand as $q_i = \frac{\varphi_i}{\beta_i}$. But if $\sum_{i=1}^{M} \frac{\varphi_i}{\beta_i} > I$, the market will be cleared by aggregate demand equaling total supply. Then the market price

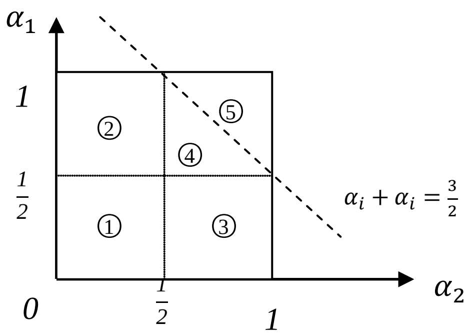
$$p = \frac{\sum_{i=1}^{L} \frac{\varphi_i}{\beta_i} - I}{\sum_{i=1}^{L} \frac{1}{\beta_i}}$$

where $L \leq M$ is the number of winning bidders. At mean while $p = \lambda$, the market price is actually the Shadow Price from the optimality problem. So we are not surprised the shares are allocated by demand at the price $q_i = \frac{\varphi_i - \lambda}{\beta_i} \, for \; \forall i \leq L$.

Here is an example with a couple of simple numbers to show how this auction works optimally on allocation, from seller's perspective.

Example 1. (Uniform Distribution) Suppose the auction is under a symmetric environment that every bidder has the marginal value of q_i as $\alpha_i - q_i$ where his type α_i is dropped onto a uniform distribution in $[0,1]$. The market supply $I = 1$. Now the virtual valuation $\varphi_i = 2\alpha_i - 1$.

First of all, suppose there are two bidders competing in this auction. **Figure 2** illustrates the optimal allocation rule for different type spaces.



① $q_1 = q_2 = 0$
② $q_1 = 2\alpha_1 - 1; q_2 = 0$
③ $q_1 = 0; q_2 = 2\alpha_2 - 1$
④ $q_1 = 2\alpha_1 - 1; q_2 = 2\alpha_2 - 1$
⑤ $q_1 = \alpha_1 - \alpha_2 + \frac{1}{2}; q_2 = \alpha_2 - \alpha_1 + \frac{1}{2}$

Figure 2: Optimal Allocation Rule with Two Bidders

Basically, seller will not trade any in ①; sell to only one bidder on his maximal demand in ② and ③; sell to both bidders on their maximal demands in ④ and sell to both bidders by market clear condition in ⑤.

Secondly, one more bidder join the auction. We have types $\alpha_1 = 1, \alpha_2 = 0.75, \alpha_3 = 0.6$, then their virtual valuation φ_i are 1, 0.5 and 0.2 respectively. All of them have a positive virtual valuation, but it does not necessarily guarantee their winning. In this case, $\sigma_2 = -0.5 < 0$, $\sigma_3 = 0.1 > 0$, the pivotal bidder is the one with 2nd highest φ_i, so the 3rd player is ruled out of the auction. The clearance price $\lambda = 0.25$. Then $q_1 = 0.75$ and $q_2 = 0.25$. ▲

3.3 Discussion

Before we proceed to the next section, there are still couples of interested questions including model assumption, solution and extension to be discussed in more details.

First, how to interpret the optimal payment rules? By **Proposition** 3 (Optimal Share Auction), we get:

$$\int_{\underline{w}_i}^{\alpha_i} q_i(t_i, \alpha_{-i}) dt_i = \alpha_i q_i(\alpha) - \tfrac{1}{2}\beta_i q_i^2(\alpha) - m_i(\alpha) = v_i\big(q_i(\alpha)\big) - m_i(\alpha) \geq 0$$

So the winning bidders will walk away with positive consumer surplus. The surplus $\int_{\underline{w}_i}^{\alpha_i} q_i(t_i, \alpha_{-i}) dt_i$ can be referred as the informational rent for the compensation of bidders holding their private information in the auction. Take all others' types α_{-i} as given, the higher of bidder i's type α_i, the more rent he benefited from getting more share. The seller could not fully extract all the bidders' surplus, in order to implement an incentive compatible scheme.

Second, we should be careful on the feasibility conditions. The feasible constraints $\sum_{i=1}^{n} q_i(\alpha) \leq I, q_i(\alpha) \geq 0 \ \forall i, \forall \alpha$ is reasonable in the manner of we are selling something concrete and physical. For instance, a large Pepperoni pizza, Outer Continental Shelf OCS lease or the right for oil exploration. But in a more general auction format where both buying and selling are involved from bidders (to be more specific, the auction participants), $q_i(\alpha) < 0$ is possible. Sometimes it is referred as a Double Auction. Most of these scenarios happen in capital market, like trading of securities or other derivatives, both longing and shorting are generally allowed in advance market. In such a setting, we do not need this feasibility condition anymore, for instance, Du & Zhu (2013). Our result can easily be extended over there.

Third, the linear demands $p = \alpha_i - \beta_i q_i$ of bidders implicitly imply $q_i \leq \frac{\alpha_i}{\beta_i}$, the maximal quantity demand at $p = 0$. If $q_i > \frac{\alpha_i}{\beta_i}$, bidder could not get any additional payoff through more share of goods. So the *ex-post* consumer revenue

$$v_i(q_i) = \alpha_i q_i - \tfrac{1}{2}\beta_i q_i^2 = \frac{\alpha_i^2}{2\beta_i}$$

will be the same for any $q_i > \frac{\alpha_i}{\beta_i}$. The specific case is not included in the previous analysis, especially for IC conditions. Next, we show the optimal solution derived before will not be contradict with this. By **Proposition** 3 (Optimal Share Auction), since $\varphi_i(\alpha_i) = \alpha_i - \frac{1-F_i(\alpha_i)}{f_i(\alpha_i)} < \alpha_i$, the *ex-post* allocation share $q_i \leq \frac{\alpha_i}{\beta_i}$ in any case. Then let's look at the payment rule m_i, do the derivative on α_i, we have:

$$\frac{\partial m_i}{\partial \alpha_i} = q_i + \alpha_i \frac{\partial q_i}{\partial \alpha_i} - \beta_i q_i \frac{\partial q_i}{\partial \alpha_i} - q_i(\alpha_i, \alpha_{-i}) = (\alpha_i - \beta_i q_i)\frac{\partial q_i}{\partial \alpha_i} \geq 0$$

with the increasing property of q_i on α_i. Now consider we are on the truth-telling equilibrium of the optimal direct mechanism we have derived. Suppose bidder i reports a type z_i' rather than α_i to achieve a larger share $q_i(z_i', \alpha_{-i})$ such that $q_i(z_i', \alpha_{-i}) > \frac{\alpha_i}{\beta_i}$. With $\frac{\partial q_i}{\partial \alpha_i} \geq 0$ and $\frac{\partial m_i}{\partial \alpha_i} \geq 0$, there always exists $z_i'' < z_i'$, such that $q_i(z_i', \alpha_{-i}) > q_i(z_i'', \alpha_{-i}) \geq \frac{\alpha_i}{\beta_i}$ and $m_i(z_i', \alpha_{-i}) > m_i(z_i'', \alpha_{-i})$. So switching from z_i' to z_i'' can strictly reduce the payment but keep the bidder the revenue as the same, which is equivalent to say bidder i has no incentive to hidden his true type, in order to get share higher than $\frac{\alpha_i}{\beta_i}$.

Fourth, the optimal mechanism is not an efficient one on maximizing the aggregate social welfare include seller and bidders. It is simply because that in some circumstance, the divisible good is not traded even though some bidders have positive value while it is zero value for seller.

Last, in the canonical model of optimal indivisible good auction design with IPVP as the information structure, it is well known that a second-price auction a reserve price is an optimal auction if the problem is regular and symmetric. Similarly, in the framework of share auction, the Vickrey auction which could also implement the truth-telling strategies is the parallel second-price auction for divisible goods. People might be interested in whether Vickrey auction is an optimal one based on our model? We will study it further more in later sections.

CHAPTER 4

EQUILIBRIUMS AND EFFICIENCY

A common aspect of auction is that they extract information solely from bids, which is simple to be executed. The scheme rules should capture potential buyers regarding their willingness to pay, but no matter whom they are and what they are bidding for. Auctions are **universal** in the sense that they might be utilized to sell any good. Also, they are called **anonymous** if the identities of the buyers play no role in determining who wins the object and who pays how much. But the optimal share auction we derived last section is neither universal (the allocation rules depend on the value distribution to the item for sale) nor anonymous (bidders' type distribution play an important role). From the practical standpoint, beyond a direct mechanism, we are going to look at some specific indirect auction mechanism where bidders submit their bidding schemes instead of revealing their own types.

We study the most popular auction formats in markets for divisible goods, the commonly used uniform price auction UPA and discriminatory price auction DPA, as well as the theoretical benchmark of the Vickrey auction VA. Among those auction formats, bidders submit their downward-sloping bid schedules, or their demand to specify the quantity q_i for each price p. Then the seller will find the market price or the **stop-out price** p^* to equalize aggregate demand and total units of supply. To be more careful, we figure out if there is no such price or multiple stop-out prices exist, the seller will keep the good herself. But we could show later on that there is a unique market price in our model. In all three auctions, they implement the same allocation rules that shares bidders get are decided by $q_i(p^*)$. The difference are coming from the payment rules:

• **UPA**: each bidder pays the stop-out price for the total share he wins.

• **DPA**: each bidder pays the valuation he revealed through the submitted bidding schedule, up to the share he wins.

• **VA**: each bidder pays the valuation below the residual supply he faced, up to the share he wins.

Throughout the left of section, we are interested in three different perspectives: first, what are the equilibriums for those auction formats under our model setting? Second, which one will generate relative higher revenue for the seller? Last, which one is more efficient through maximizing the total social welfare?

4.1 Linear ex-post Equilibirums

The equilibrium concept we implement here is *ex-post* equilibrium. First of all, we will proceed to define the notion of the equilibrium, and then characterize it in all auction formats.

Definition 4. An ***ex-post*** **equilibrium** in our model of share auctions is a profile of bidding schedule $(q_1(p), q_2(p), \dots, q_n(p))$ such that for any profile of bidders' types α, bidder i has no incentive to deviate from $q_i(p)$. That is for any alternative strategy $\widetilde{q_i}(p)$ of bidder i

$$u_i(q_i(p), p^*) \geq u_i(\widetilde{q_i}(p), \tilde{p}^*)$$

where u_i is the ***ex-post*** utility, p^* is the stop-out price with $q_i(p)$ and $\tilde{p}^*$ is the new stop-out price given $\widetilde{q_i}(p)$.

This definition further emphasizes the robustness of *ex-post* equilibrium. It does not require bidders to have common knowledge about other's types or even distribution.

Another advantage of this concept is that the equilibrium is less sensitive to bidder's preference. If a profile of bidding schedule $(q_1(p), q_2(p), \dots, q_n(p))$ build up an *ex-post* equilibrium with the linear-quadratic utility function u_i, then it is still the equilibrium for preference utility function of the form $g(u_i)$ where g is a strictly increasing function.

We follow the same model framework of last section, except some of variations or additional technical conditions for analytical purpose. All of those assumptions will better serve our characterization of uniqueness equilibriums for different auction formats.

Assumption 2. (Linearity) Among any possible *ex-post* equilibrium with downward-sloping bid schedule $q_i(p)$, we are looking for such equilibriums where the strategy is linear on price p, such that $q_i(p) = a_i - b_i p,\ a_i > 0,\ b_i > 0$.

Assumption 3. (Symmetry) Bidders have the same convexity of their preference, which is $\beta_i = \beta_j = \beta, \forall i \neq j$. They also have the same types $\alpha_i = \alpha_j = \alpha, \forall i \neq j$. on the same interval $\bar{\chi} = [\underline{w}, \overline{w}]$ following F as the distribution.

All of the auction formats, VA, UPA and DPA, are implementing the same allocation rules. Bidders are asked to submit demand schedule as $q_i(p) = a_i - b_i p$, then the stop-out price is generated through market clearing condition:

$$p^* = \frac{\sum_{i=1}^{n} a_i - I}{\sum_{i=1}^{n} b_i} \tag{4}$$

and the share bidder i obtains is

$$q_i^* = \frac{a_i \sum_{i=1}^{n} b_i - b_i \sum_{i=1}^{n} a_i + b_i I}{\sum_{i=1}^{n} b_i} \tag{5}$$

With that information, we have a couple of more assumptions regarding the trading feasibility of share auctions.

Assumption 4. (Market Clearance) There exists positive market clearing price $p^* > 0$ on equilibriums of any auction format. In another word, the market supply I is not as high as the maximal aggregate demand $I \leq \sum_{i=1}^{n} a_i^*$, where a_i^* is on equilibriums $q_i^*(p) = a_i^* - b_i^* p$.

Assumption 5. (Participation) The auctions end up with all bidders participate into the allocation with $q_i^* > 0$ or $I > \sum_{i=1}^{n} a_i^* - \frac{a_i^*}{b_i^*} \sum_{i=1}^{n} b_i^*$ at equilibriums.

The purpose of participation assumption is to set up the environment to be analytically convenient. It is also equivalent to assume a double auction background where $q_i^* > 0$ for any i need not to be satisfied.

With the allocation rules above, the payment rules of different auction formats are defined respectively.

4.1.1 Vickrey Auction

First of all, in Vickrey auction, bidder i is facing the residual supply as [4]

$$\widehat{S}_i(p) = I - \Sigma_{-i} q_i(p) = I - \Sigma_{-i} a_i + (\Sigma_{-i} b_i) p$$

The bidder is required to pay the highest losing bids *ex-post.* So the money transfer would be the area below inverse residual supply function up to the winning share q_i^*:

[4] We have the notations $\Sigma_i \equiv \Sigma_{i=1}^{n}$, and $\Sigma_{-i} \equiv \Sigma_{j=1, j \neq i}^{n}$ in what following.

$$m_i^{VA} = \int_0^{q_i^*} \widehat{S_\iota}^{-1}(p)dq$$

$$= \int_0^{q_i^*} \frac{(q - I + \sum_{-i} a_i)}{\sum_{-i} b_i} dq$$

$$= \frac{1}{\sum_{-i} b_i}(-\left(I - \sum_{-i} a_i\right)q_i^* + \frac{1}{2}{q_i^*}^2)$$

Now we can start to determine the linear bidding strategies as *ex-post* equilibriums.

Proposition 4.(Equilibrium in VA) *In a unique linear ex-post equilibrium, the strategy of each bidder with type* α_i *is equal to*

$$q_i^*(p) = \frac{1}{\beta}(\alpha_i - p)$$

It is not surprised that the familiar truth-telling strategies are obtained in VA. We are more interested in whether it is an optimal share auction under this symmetric environment, that parallel to the second-price auction with reserve price constituting an optimal one in indivisible goods. Since $a_i = \frac{\alpha_i}{\beta}$, $b_i = \frac{1}{\beta}$, the share bidder i wins on equilibrium is

$$q_i^{VA} = \frac{n\alpha_i - \sum_i \alpha_i + \beta I}{\beta n}$$

Comparing with optimal solution we derive in last section by **Proposition** 3 (Optimal Share Auction), plus the symmetric, market clearance and participation assumptions above, the respective optimal allocated share is

$$q_i^{OPT} = \frac{\varphi_i - \lambda}{\beta} = \frac{n\varphi_i - \sum_i \varphi_i + \beta I}{\beta n}$$

So we conclude, VA is not an optimal share auction in general. But in extreme case where the information asymmetric are fully eliminated such that $\varphi_i = \alpha_i$ or the inverse hazard rate $\gamma(\alpha_i) = \frac{1-F(\alpha_i)}{f(\alpha_i)} = 0$, for $\forall \alpha_i \in [\underline{w}, \overline{w}]$, $\forall i$, then $q_i^{VA} = q_i^{OPT}$ and we can easily verify that $m_i^{VA} = m_i^{OPT}$ in this symmetric scenario.

We know the inverse hazard rate should not always be zero since the optimal share auction would not hold anymore under complete information. The relationship between the optimal one and the VA could be addressed more formal as following:

$$\gamma(\alpha_i) \longrightarrow 0, \Longrightarrow q_i^{VA} \longrightarrow q_i^{OPT}, m_i^{VA} \longrightarrow m_i^{OPT}$$

As long as the inverse hazard rate function of bidders' private information closed to zero, the VA will be infinitely approximate to an optimal share auction. More discussion of this finding will be emphasized in later sections.

4.1.2 Uniform-Price Auction

Secondly, the payment in UPA is as simply as the multiplication of stop-out price and the share bidders received.

$$m_i^{UPA} = p^* q_i^*$$

So the linear bidding strategies as *ex-post* equilibriums are derived as following.

Proposition 5.(Equilibrium in UPA) *In a unique linear ex-post equilibrium, the strategy of each bidder with type α_i is equal to*

$$q_i^*(p) = \frac{1}{\beta}\left(\frac{n-2}{n-1}\right)(\alpha_i - p)$$

Equilibrium exists if and only if n>2.

There is the demand reduction for equilibrium in UPA where the bidding demand they submit is

$$p = \alpha_i - \beta \left(\frac{n-1}{n-2}\right) q_i$$

It is from the fact that, with positive probability, the bid for the share $q_i > 0$ will determine the price paid on all shares he wins. So any overbid may result in potential loss *ex-post*.

We observe several important properties of this equilibrium. First of all, there is no demand reduction present at $q_i = 0$. This is coinciding with the famous result in indivisible multiple units auction, e.g. Krishna (2002). Since price impact externality is linear and increasing in q_i, the demand deduction is increasing as so. To be more precisely, we could measure the demand reduction by

$$\Delta D_i = \alpha_i - \beta q_i - \left(\alpha_i - \beta \left(\frac{n-1}{n-2}\right) q_i\right) = \left(\frac{\beta}{n-2}\right) q_i$$

So the second property is that the magnitude of the reduction is decreasing as more bidders joining the auction. When the number of bidders gets arbitrarily large, the equilibrium is converging to truth-telling strategy profiles that no demand reduction exists. Because the more players compete for the target good, the less effect will be taken on the stop-out price from each individual bidding schedule. Bidders will gradually incline to truly reveals their real demand rather than manipulate it through demand reduction. Last, the equilibrium bidding strategies in our model does not depend on the total supply I. We know the number of units available matters in indivisible multi-units auction, e.g. Krishna (2002). Bidders will decide how defensive (the demand reduction) they bid on each different units while knowing the total number of target goods. But in the divisible property and linearity bidding strategies make this process smoothly. The participation assumption further dilutes the effect of exogenous supply on equilibriums. The linear marginal utility assumption might be also relevant but not critical. In the similar research like Ollikka (2011) and Du, Zhu (2013) with linear quadratic utility on share auction, both of their solutions on UPA are related on market supply.

4.1.3 Discriminatory Price Auction

Next, we will look at discriminatory price auction. Again, in the indivisible case, the equilibrium bidding involve “flat demand”, that bidders have incentive to reduce their winning bids on the last units since it will not influence their winning shares but will do decrease their payments.

While for Discriminatory Price Auction, winners pay what they bid, so

$$m_i^{DPA} = \int_0^{q_i^*} q_i{}^{-1}(p)dq = \int_0^{q_i^*} \frac{(\,a_i - q)}{b_i} dq = \frac{a_i}{b_i} q_i^* - \frac{1}{2b_i} q_i^{*2}$$

The equilibrium solution is more complex than VA and UPA.

Proposition 6.(Equilibrium in DPA) *There is no linear ex-post equilibrium in the general format of* $q_i(p) = a_i - b_i p$*, but in a special case where bidding strategy is proportional to* $\alpha_i - p$*, then the following demand schedule constitutes an ex-post equilibrium profile:*

$$q_i^*(p) = \frac{1}{2\beta}\left(\frac{n-2}{n-1}\right)(\alpha_i - p)$$

Equilibrium exists if and only if n>2.

As we expect, due to the pay-as-you-bid rule, bidding for one realization of the market clearance does affect payments for other realization, so it is very challenging to characterize equilibriums, especially on *ex-post*. Some related literature, say Rostek, Weretka and Pycia (2010), Ollikka (2011), deduce the explicit solution on *ex-ante* with solution concept as symmetric linear Bayesian equilibrium, which implement different kinds of technical assumption on information structure. The IPVP framework could reduce these interactive effects on different states but not eliminate them at all. The specific solutions we provide here avoid the incentive of “flat demand”. Since the linearity of bidding profiles restrict their strategies on winning shares. Whenever they reduce their bids, they should weigh the tradeoff between decreasing of payment as well as losing shares. It is also comparable with other *ex-post* equilibriums in VA and UPA.

The demand reduction of DPA is further enforced as twice large as UPA

$$\Delta D_i = \left(\frac{2\beta}{n-2}\right) q_i$$

It is intuitive because the bidder will pay all consumer surpluses as he bids rather than the universal stop-out price. **Figure 3** summarizes the ex-post equilibriums we derive for different auction formats.

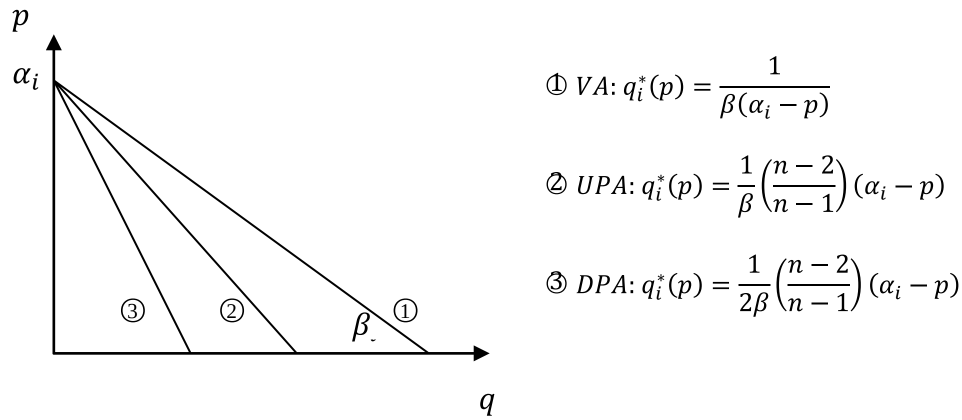


Figure 3: The Demand Reduction of Different Auction Formats

Table 1 further summarize the comparison of VA, UPA and DPA with respective to *ex-post* equilibriums, the market clearing or stop-out prices, which are derived by market clearing conditions as equation (4), and magnitude of demand reduction.

Table 1: Comparisons of Different Auction Formats

	VA	**UPA**	**DPA**
Equilibriums ($\boldsymbol{q_i^*}$)	$\frac{1}{\beta}(\alpha_i - p^*)$	$\frac{1}{\beta}\left(\frac{n-2}{n-1}\right)(\alpha_i - p^*)$	$\frac{1}{2\beta}\left(\frac{n-2}{n-1}\right)(\alpha_i - p^*)$
Stop-out Prices ($\boldsymbol{p^*}$)	$\frac{\sum_i \alpha_i - \beta I}{n}$	$\frac{\sum_i \alpha_i - \left(\frac{n-1}{n-2}\right)\beta I}{n}$	$\frac{\sum_i \alpha_i - 2\left(\frac{n-1}{n-2}\right)\beta I}{n}$
Demand Reduction ($\Delta\boldsymbol{D_i}$)	0	$\left(\frac{\beta}{n-2}\right)q_i$	$\left(\frac{2\beta}{n-2}\right)q_i$

4.2 Revenue Ranking

In this section, we present the revenue comparison of VA, UPA and DPA. From the seller's perspective, we are interested in which auction format will bring higher *ex-ante* expected revenue, based on fixed total divisible goods supply I and the same number of attendance n for different auctions.

Proposition 7.(Revenue Ranking) *In the respective linear ex-post equilibriums of different auction formats, the ex-ante expected total revenues for seller are ranked as following:*

(1) $E[R^{DPA}] \geq E[R^{UPA}]$

(2) $E[R^{VA}] \geq E[R^{UPA}]$ *if and only if* $Var[\alpha] \leq \frac{\beta^2 I^2}{n(n-1)(n-2)}$

(3) $E[R^{VA}] \geq E[R^{DPA}]$ *if and only if* $Var[\alpha] \leq \frac{2\beta^2 I^2}{n^2(n-1)(n-2)}$

In our model, the DPA is dominant of UPA on expected payment regardless of uncertainty on bidders' type space. This conclusion could be understood from two points. First, the DPA is trying to extract the full surplus from bidders while UPA only charges at the minimal winning price. Second, the larger demand reduction in DPA will lead to less equilibrium price impact.

For VA, the ranking of the expected payment depend on number of bidders, total exogenous supply, the magnitude of bidder's risk aversion and the volatility of other competitor's types. The higher degree of bidder's risk-aversion, the larger of divisible shares and the less players participate in the auction, the VA is more preferred. On an extreme case where no asymmetric information exists as $Var[\alpha] = 0$, then VA is absolute dominant all other two auctions. Moreover, as we discuss in last section, VA is closer to the optimal auction format as long as more bidders' private information is revealed publicly. Intuitively, since VA is the mechanism that will implement truthful revealing of bidder's private information, the revenue seller transfers to bidder in this auction is the information rent she pays. So we could capture this rent by measuring the uncertainty of bidders' type by $Var[\alpha]$. So the revenue ranking reinforces our finding about the comparison of VA and the optimal auction. When $Var[\alpha]$ is as small as zero, the inverse hazard rate $\gamma(\alpha_i)$ for any bidder i is approximate to zero as well, then VA constitute the optimal share auction on the limit.

When we look at the difference between UPA and DPA:

$$E[R^{DPA}] - E[R^{UPA}] = \frac{(n-2)}{4\beta} Var[\alpha]$$

So the revenue equivalence of UPA and DPA would hold on limit if the uncertainty of bidders' private information is approaching zero.

4.3 Efficiency

Due to the varied purpose of holding an auction, it might also be interesting to look at the efficiency, measuring by total social welfare. For instance, in selling of the oil exploration right, the seller would like to maximize her expected revenue. But for some environmental related goods, such as Carbon Dioxide Pollution Permit CDPP, the auctioneer might consider a lot of factors to balance the interests of different groups, rather than pursuing a maximal profit.

We measure the social welfare by adding the total surplus all bidders maintain and the aggregate payments seller earned at the end of auction where all private information is revealed.

Definition 5. An **efficient** share auction is the auction with the allocation rules q that

$$\max_{q_1, q_2, \ldots, q_n} W = \sum_{i=1}^{n} v_i(q_i) = \sum_{i=1}^{n} (\alpha_i q_i - \tfrac{1}{2}\beta_i q_i^2)$$

subject to the feasibility conditions $\sum_{i=1}^{n} q_i \leq I, q_i \geq 0\ \forall i$ on *ex-post*.

The optimality process of total social welfare defined above is exactly the same as a market clearing process. The stop-out price is derived by equalizing market supply I to total market demands where every bidder just submit their true marginal utility as $p = \alpha_i - \beta_i q_i$. It is also equivalent to a VA since truth-telling strategies are implemented there. So the next proposition will illustrate the efficiency comparison of different auctions.

Proposition 8.(Efficiency Ranking) *In the symmetric environment, VA is an efficient share auction on equilibriums, the ex-post total social welfare of different auction formats on equilibriums are ranked as following:*

$$W^{VA} \geq W^{UPA} \geq W^{DPA}$$

This result can be understood with the demand reduction of three different auction formats, like what illustrated in **Figure 3**. The higher degree of defensive bidding, the lower equilibrium price would be expected to beyond the stop-out price deduced from true demands. Then we would expect more efficiency loss on the equilibriums.

CHAPTER 5

CONCLUSIONS

This paper explore the optimal share auction with bidders have linear marginal utility or demands. The information structure is IPVP. On optimality, the seller designs the allocation rules through a competitive equilibrium process. The stop-out price or shadow price is deduced by equalizing market supply and aggregate market demands which are represented as individual's marginal revenue, or the so called virtual valuation. We show the mechanism that implements truth-telling, like Vickrey auction, is not generally an optimal one in this class of share auctions. But the difference between the VA and optimal solution will converge to zero when bidders keep almost null private information.

We also compare the most popular auction formats for divisible goods, including VA, UPA and DPA. The linear *ex-post* equilibriums are characterized. We found the ranking of magnitude of demand reduction would be DPA>UPA>VA, which would affect their performance on revenue generating and social efficiency achieving. With further ranking formats in term of revenue, the result show DPA would dominant UPA regardless of information distribution. VA could be ranked higher or lower than the other two, depends on the volatility of private information. Last, we compare the efficiency of different auctions. The VA turns out to be an efficient share auction on ex-post while DPA is the format which loss the most of social welfare through the largest demand reduction on bidding, then the largest twist on stop-out price.

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APPENDIX A

PROOFS

Proof. Proposition 1 (Incentive Compatible) The technique applied here is similar to the canonical analysis of optimal auction design in indivisible good with information structure as IPVP, say Myerson (1981).

The definition of incentive compatibility condition is equivalent to that

$$U_i(\alpha_i) = \max_{z_i \in \chi_i} \{\alpha_i Q_i(z_i) - \tfrac{1}{2}\beta_i R_i(z_i) - M_i(z_i)\}$$

where $\alpha_i Q_i(z_i) - \frac{1}{2}\beta_i R_i(z_i) - M_i(z_i)$ is an affine function of α_i. Therefore U_i is a convex function. Also, by Envelope Theory, $U_i{}'(\alpha_i) = Q_i(\alpha_i)$. So we have the first condition that $Q_i(\alpha_i)$ is increasing on α_i and

$$U_i(\alpha_i) = U_i(\underline{w}_i) + \int_{\underline{w}_i}^{\alpha_i} Q_i(t_i) dt_i$$

Then by the definition of $U_i(\alpha_i)$, it is equivalent to the second condition.

For the converse way, the following inequality hold for any α_i, z_i,

$$\int_{\alpha_i}^{z_i} Q_i(t_i) dt_i \geq Q_i(\alpha_i)(z_i - \alpha_i)$$

if $Q_i(\alpha_i)$ is increasing. With the second condition applied to both α_i and z_i, we have

$$U_i(z_i) \geq U_i(\alpha_i) + Q_i(\alpha_i)(z_i - \alpha_i)$$

Since it is hold for any α_i, z_i, we also have

$$U_i(\alpha_i) \geq U_i(z_i) + Q_i(z_i)(\alpha_i - z_i) = \alpha_i Q_i(z_i) - \tfrac{1}{2}\beta_i R_i(z_i) - M_i(z_i)$$

which build up the IC condition. ■

Proof. Proposition 2 (Designer's Problem) The seller is seeking maximizing her *ex-ante* expected revenue

$$E[R] = \sum_{i=1}^{n} E[M_i(\alpha_i)]$$

For each individual bidder under payment rule of IC

$$E[M_i(\alpha_i)] = \int_{\underline{w}_i}^{\overline{w}_i} M_i(\alpha_i) f_i(\alpha_i) d\alpha_i$$

$$= -U_i(\underline{w}_i) + \int_{\underline{w}_i}^{\overline{w}_i} \left(\alpha_i Q_i(\alpha_i) - \tfrac{1}{2}\beta_i R_i(\alpha_i)\right) f_i(\alpha_i) d\alpha_i$$

$$- \int_{\underline{w}_i}^{\overline{w}_i} \int_{\underline{w}_i}^{\alpha_i} Q_i(t_i) f_i(\alpha_i) dt_i \, d\alpha_i$$

By interchanging the order of integration in the last term:

$$\int_{\underline{w}_i}^{\overline{w}_i} \int_{\underline{w}_i}^{\alpha_i} Q_i(t_i) f_i(\alpha_i) dt_i \, d\alpha_i = \int_{\underline{w}_i}^{\overline{w}_i} \int_{t_i}^{\overline{w}_i} Q_i(t_i) f_i(\alpha_i) d\alpha_i dt_i$$

$$= \int_{\underline{w}_i}^{\overline{w}_i} (1 - F_i(t_i)) Q_i(t_i) dt_i$$

Then we write the *ex-ante* expected payment from bidder i as

$$E[M_i(\alpha_i)] = -U_i(\underline{w}_i)$$

$$+ \int_{\underline{w}_i}^{\overline{w}_i} \left(\alpha_i Q_i(\alpha_i) - \tfrac{1}{2}\beta_i R_i(\alpha_i) - \frac{1 - F_i(\alpha_i)}{f_i(\alpha_i)} Q_i(\alpha_i)\right) f_i(\alpha_i) d\alpha_i$$

$$= -U_i(\underline{w}_i) + \int_{\chi} \left(\alpha_i q_i(\alpha) - \tfrac{1}{2}\beta_i q_i^2(\alpha) - \frac{1 - F_i(\alpha_i)}{f_i(\alpha_i)} q_i(\alpha)\right) f(\alpha) d\alpha$$

Define the virtual valuation of bidder i with type α_i as $\varphi_i(\alpha_i) \equiv \alpha_i - \frac{1-F_i(\alpha_i)}{f_i(\alpha_i)}$, so now the auction design problem is turning to be

$$\max_{(q,m)} \{ \int_{\chi} \sum_{i=1}^{n} \left(\varphi_i(\alpha_i) q_i(\alpha) - \tfrac{1}{2}\beta_i q_i^2(\alpha)\right) f(\alpha) d\alpha - \sum_{i=1}^{n} U_i(\underline{w}_i) \}$$

subject to

(IC-Allocation Rule): $Q_i(\alpha_i)$ *is increasing on* α_i

(IC-Payment Rule): $M_i(\alpha_i) = \alpha_i Q_i(\alpha_i) - \frac{1}{2}\beta_i R_i(\alpha_i) - \int_{\underline{w_i}}^{\alpha_i} Q_i(t_i)dt_i - U_i(\underline{w_i})$

(IR): $U_i(\underline{w_i}) \geq 0, \forall i$

(Feasibility): $\sum_{i=1}^{n} q_i(\alpha) \leq I, q_i(\alpha) \geq 0\ \forall i, \forall \alpha$

The idea to solve this optimality problem is like this. If we can find the direct mechanism (q, m) that maximizing the revenue *ex-post*, then it will be also the solution *ex-ante*.

First of all, we looks for increasing $q_i(\alpha)$ on α_i that will maximize $\sum_{i=1}^{n}\left(\varphi_i(\alpha_i)q_i(\alpha) - \frac{1}{2}\beta_i q_i^2(\alpha)\right)$ under feasibility conditions. The monotone property of $q_i(\alpha)$ *will imply* the allocation rule of IC. Second, we set $U_i(\underline{w_i}) = 0, \forall i$, then IR is satisfied while maximizing $-\sum_{i=1}^{n} U_i(\underline{w_i})$. Third, the *ex-post* payment rule is derived from allocation rule with $U_i(\underline{w_i}) = 0$ as

$$m_i(\alpha) = \alpha_i q_i(\alpha) - \frac{1}{2}\beta_i q_i^2(\alpha) - \int_{\underline{w_i}}^{\alpha_i} q_i(t_i, \alpha_{-i})dt_i$$

That will give you the *ex-mid* expected payment $M_i(\alpha_i)$ of bidder i as condition IC-Payment Rule characterized. ∎

Proof. Lemma 1 First of all, σ_K is increasing on K since

$$\sigma_K - \sigma_{K-1} = \left(\sum_{i=1}^{K-1} \frac{1}{\beta_i}\right)(\varphi_{K-1} - \varphi_K) \geq 0$$

There is a unique L such that $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_L < 0 \leq \sigma_{L+1} \leq \cdots \leq \sigma_M$.

Secondly, $\sigma_1 = -I < 0$, so it is possible that $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_M < 0$. ∎

Proof. Proposition 3 (Optimal Share Auction) By **Proposition 2** (Designer's Problem), the payment rules is characterized as

$$m_i(\alpha) = \alpha_i q_i(\alpha) - \tfrac{1}{2}\beta_i q_i^2(\alpha) - \int_{\underline{w}_i}^{\alpha_i} q_i(t_i, \alpha_{-i}) dt_i.$$

We only need to solve the allocation rules q from the following problem:

$$\max_{q_1, q_2, \dots, q_n} \sum_{i=1}^{n} (\varphi_i q_i - \tfrac{1}{2}\beta_i q_i^2)$$

subject to the the feasibility conditions $\sum_{i=1}^{n} q_i \leq I, q_i \geq 0\ \forall i, \forall \alpha$ and q_i is increasing on α_i for any α_{-i}. The solution is separated into several steps:

Step.1: $q_i = 0\ for\ \forall i > M$.

Think about M is coming from

$$\varphi_1 \geq \varphi_2 \geq \cdots \geq \varphi_M > 0 \geq \varphi_{M+1} \geq \cdots \geq \varphi_n$$

by ranking of virtual valuation. For $\forall i > M$, $\varphi_i q_i - \tfrac{1}{2}\beta_i q_i^2 \leq 0$. Then we further reduce the optimal problem into:

$$\max_{q_1, q_2, \dots, q_M} \sum_{i=1}^{M} (\varphi_i q_i - \tfrac{1}{2}\beta_i q_i^2)$$

subject to $\sum_{i=1}^{M} q_i \leq I\ and\ q_i \geq 0\ \forall i$

Step.2: *The first order conditions FOCs are both necessary and sufficient to derive the solutions.*

The objective function is concave on the arguments $(q_1, q_2, \dots, q_M)$. What is more, it is separable additive concave since for each $i \in \{1,2, \dots, M\}$, $\varphi_i q_i - \tfrac{1}{2}\beta_i q_i^2$ is concave on q_i.

Step.3: *FOCs*

(q_i): $\varphi_i - \beta_i q_i = \lambda - \mu_i$, for $\forall i \in \{1,2, \dots, M\}$;

(λ): $\lambda \geq 0$, $\sum_{i=1}^{M} q_i - I \leq 0$ and $\lambda(\sum_{i=1}^{M} q_i - I) = 0$;

(μ_i): $\mu_i \geq 0$, $q_i \geq 0$, $\mu_i q_i = 0$, for $\forall i \in \{1,2, \ldots, M\}$,

where λ and μ_i are Lagrange Multiplier to respective inequality constraints. Specifically, λ is the shadow price of optimization problem.

Step.4*: Solutions*

(1) If $\lambda = 0$, $\varphi_i - \beta_i q_i = -\mu_i \leq 0$, then $q_i \geq \frac{\varphi_i}{\beta_i} \forall i$.

If $\mu_i > 0 \Rightarrow q_i = 0$, contradict with $q_i \geq \frac{\varphi_i}{\beta_i} \forall i$;

If $\mu_i = 0 \Rightarrow q_i = \frac{\varphi_i}{\beta_i}$, that we need $\sum_{i=1}^{M} \frac{\varphi_i}{\beta_i} - I \leq 0$ to be satisfied.

(2) If $\lambda > 0$, then the feasibility constraint is binding as $\sum_{i=1}^{M} q_i = I$.

If $q_i > 0 \Rightarrow \mu_i = 0 \Rightarrow q_i = \frac{\varphi_i - \lambda}{\beta_i} > 0$, where $\lambda < \varphi_i$ need to be checked;

If $q_i = 0 \Rightarrow \varphi_i = \lambda - \mu_i$, then $\mu_i = \lambda - \varphi_i \geq 0 \Rightarrow \lambda \geq \varphi_i > 0$.

Scenario (1) indicates the allocation rule when feasibility constraint is not binding, so if $\sum_{i=1}^{M} \frac{\varphi_i}{\beta_i} \leq I$, $q_i = \frac{\varphi_i}{\beta_i}$ for $\forall i \leq M$ and $q_i = 0$ for $\forall i > M$.

Scenario (2) tells us whether bidder i could get a positive share or not depends on his virtual valuation. The bidder get $q_i > 0$ if and only if $\varphi_i > \lambda$. Suppose there are totally K winners as the result of auction. So

$$\varphi_1 \geq \varphi_2 \geq \cdots \geq \varphi_K > \lambda \geq \varphi_{K+1} \geq \cdots \geq \varphi_M$$

Hence by the binding budget constraint, $\sum_{i=1}^{K} q_i = \sum_{i=1}^{K} \frac{\varphi_i - \lambda}{\beta_i} = I$, we have the shadow price:

$$\lambda = \frac{\sum_{i=1}^{K} \frac{\varphi_i}{\beta_i} - I}{\sum_{i=1}^{K} \frac{1}{\beta_i}}$$

First of all, $\lambda > 0$ since $\lambda \geq \varphi_M > 0$. In the extreme case where $K = M$, then $\sum_{i=1}^{M} \frac{\varphi_i}{\beta_i} - I > 0 \Rightarrow \lambda > 0$. Otherwise, if $\sum_{i=1}^{M} \frac{\varphi_i}{\beta_i} - I \leq 0$, we go back to Scenario (1).

Secondly, $\lambda < \varphi_K$ is equivalent to $\sigma_K = \sum_{i=1}^{K} \frac{\varphi_i - \varphi_K}{\beta_i} - I < 0$. Hence by **Lemma 1**, the unique number of winning bidders is set by $K = L$.

So in this scenario, we have the solution of q_i as below:

$$If \sum_{i=1}^{M} \frac{\varphi_i}{\beta_i} > I,\ q_i = \frac{\varphi_i - \lambda}{\beta_i}\ for\ \forall i \leq L\ where\ \lambda = \frac{\sum_{i=1}^{L} \frac{\varphi_i}{\beta_i} - I}{\sum_{i=1}^{L} \frac{1}{\beta_i}}\ and\ q_i = 0\ for\ \forall i > L;$$

Combining Scenario (1) and (2), we have fully characterized the optimal allocation rules of share auction. What left is to check if the direct mechanism (q, m) we derived above constitutes an incentive compatible auction scheme.

***Step.5**: q_i is increasing on α_i, for $\forall i, \forall \alpha_{-i}$.*

With regularity assumption and the allocation rule q get above, what we need to prove is that q_i increasing on φ_i, for $\forall i, \forall \varphi_{-i}$.

Given a realized *ex-post* φ_{-i}, let us re-rank their virtual valuations among $n - 1$ bidders.

$$\varphi_1 \geq \varphi_2 \geq \cdots \geq \varphi_M > 0 \geq \varphi_{M+1} \geq \cdots \geq \varphi_{n-1}$$

We can also find the pivotal bidder as the Lth highest virtual valuation. To avoid any unnecessary confusion, we adopt $(\breve{L}, \breve{M})$, instead of (L, M), to index the group φ_{-i}. Then we discuss different scenarios same as above through whether the total-supply constraint is binding or not.

(1) If $\sum_{j=1}^{\breve{M}} \frac{\varphi_j}{\beta_j} > I$

By **Lemma 1**, there exists unique $\varphi_{\check{L}} > \varphi_i^* \geq \varphi_{\check{L}+1}$, such that equation (*) holds

$$(*) \quad \sum_{j=1}^{\check{L}} \frac{\varphi_j}{\beta_j} + \frac{\varphi_i^*}{\beta_i} - I - \left(\sum_{j=1}^{\check{L}} \frac{1}{\beta_j} + \frac{1}{\beta_i} \right) \varphi_i^* = 0$$

We define inequalities $(\check{L})$ and $(\check{L}+1)$ for further reference

$$(\check{L}) \quad \sum_{j=1}^{\check{L}} \frac{\varphi_j}{\beta_j} - I - \left(\sum_{j=1}^{\check{L}} \frac{1}{\beta_j} \right) \varphi_{\check{L}} < 0$$

$$(\check{L}+1) \quad \sum_{j=1}^{\check{L}} \frac{\varphi_j}{\beta_j} + \frac{\varphi_{\check{L}+1}}{\beta_{\check{L}+1}} - I - \left(\sum_{j=1}^{\check{L}} \frac{1}{\beta_j} + \frac{1}{\beta_{\check{L}+1}} \right) \varphi_{\check{L}+1} \geq 0$$

(1.1) If $\varphi_i \leq \varphi_i^*$, then $q_i = 0$

(1.2) If $\varphi_i^* < \varphi_i \leq \varphi_{\check{L}}$, then $q_i = \frac{\varphi_i - \lambda}{\beta_i}$, where $q_1, q_2, \ldots, q_{\check{L}}, q_i > 0$

$$(*) \Rightarrow \sum_{j=1}^{\check{L}} \frac{\varphi_j}{\beta_j} + \frac{\varphi_i}{\beta_i} - I - \left(\sum_{j=1}^{\check{L}} \frac{1}{\beta_j} + \frac{1}{\beta_i} \right) \varphi_i < 0 \;\Rightarrow\; q_i > 0$$

$$(\check{L}+1) \Rightarrow \sum_{j=1}^{\check{L}} \frac{\varphi_j}{\beta_j} + \frac{\varphi_i}{\beta_i} + \frac{\varphi_{\check{L}+1}}{\beta_{\check{L}+1}} - I - \left(\sum_{j=1}^{\check{L}} \frac{1}{\beta_j} + \frac{1}{\beta_i} + \frac{1}{\beta_{\check{L}+1}} \right) \varphi_{\check{L}+1} \geq 0 \;\Rightarrow\; q_{\check{L}+1} = 0$$

Then

$$\frac{\partial q_i}{\partial \varphi_i} = \frac{1}{\beta_i} \left(1 - \frac{\frac{1}{\beta_i}}{\sum_{j=1}^{\check{L}} \frac{1}{\beta_j} + \frac{1}{\beta_i}} \right) > 0$$

(1.3) If $\varphi_{\check{L}} < \varphi_i \leq \varphi_{\check{L}-1}$, then $q_i = \frac{\varphi_i - \lambda}{\beta_i}$, where $q_1, q_2, \ldots, q_{\check{L}-1}, q_i > 0$, with the possibility that $q_{\check{L}} = 0$.

$$(\check{L}) \Rightarrow \sum_{j=1}^{\check{L}-1} \frac{\varphi_j}{\beta_j} + \frac{\varphi_i}{\beta_i} - I - \left(\sum_{j=1}^{\check{L}-1} \frac{1}{\beta_j} + \frac{1}{\beta_i} \right) \varphi_i < 0 \;\Rightarrow\; q_i > 0$$

But

$$\sum_{j=1}^{\breve{L}} \frac{\varphi_j}{\beta_j} + \frac{\varphi_i}{\beta_i} - I - \left(\sum_{j=1}^{\breve{L}} \frac{1}{\beta_j} + \frac{1}{\beta_i}\right)\varphi_{\breve{L}} = \sum_{j=1}^{\breve{L}} \frac{\varphi_j}{\beta_j} - I - \left(\sum_{j=1}^{\breve{L}} \frac{1}{\beta_j}\right)\varphi_{\breve{L}} + \frac{1}{\beta_i}(\varphi_i - \varphi_{\breve{L}})$$

could be either positive or negative. If it is negative, $q_{\breve{L}} > 0$, we have the same winning group with q_1,q_2,…, $q_{\breve{L}}, q_i > 0$; if it is positive, $q_{\breve{L}} = 0$

$$\frac{\partial q_i}{\partial \varphi_i} = \frac{1}{\beta_i}\left(1 - \frac{\frac{1}{\beta_i}}{\sum_{j=1}^{\breve{L}-1} \frac{1}{\beta_j} + \frac{1}{\beta_i}}\right)$$

is still positive but less than the previous case.

(1.4) Keep increasing φ_i and repeat the analysis above, there might be less and less winning bidders left. But first of all, q_i will be positive always. Secondly, q_i will be increasing also, even though $\frac{\partial q_i}{\partial \varphi_i}$ get smaller when more bidders are excluded. **Figure 4** shows the relationship between q_i and φ_i in this scenario.

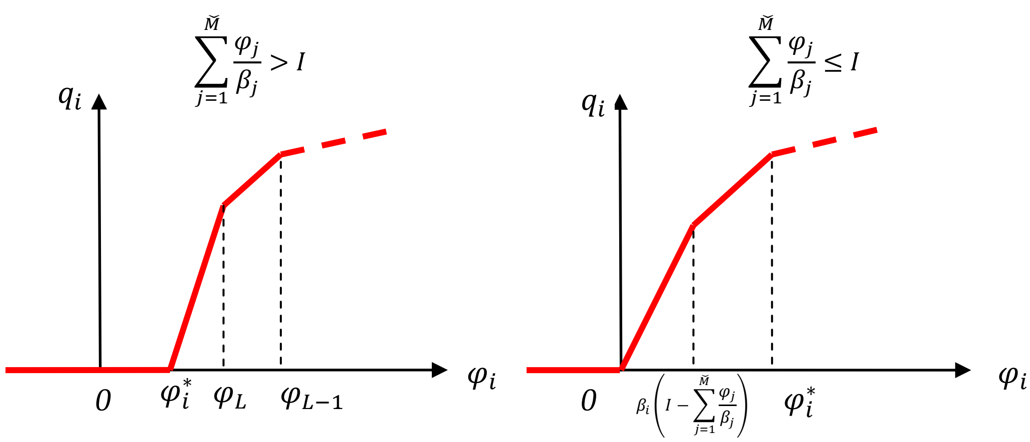

Figure 4: The Positive Relationship between q_i and φ_i

(2) If $\sum_{j=1}^{\breve{M}} \frac{\varphi_j}{\beta_j} \leq I$

(2.1) If $\varphi_i \leq 0$, then $q_i = 0$

(2.2) If $0 < \varphi_i \leq \beta_i(I - \sum_{j=1}^{\breve{M}} \frac{\varphi_j}{\beta_j})$, then $q_i \geq \frac{\varphi_i}{\beta_i}$. $\frac{\partial q_i}{\partial \varphi_i} = \frac{1}{\beta_i} > 0$

(2.3) If $\varphi_i > \beta_i(I - \sum_{j=1}^{\breve{M}} \frac{\varphi_j}{\beta_j})$, then we go back to the same situation as scenario (2), $q_i = \frac{\varphi_i - \lambda}{\beta_i}$. When keep increasing on φ_i, more and more bidders are exclude. We always have an increasing q_i but smaller slope. **Figure 4** shows how q_i will be increasing on φ_i in this scenario.

Combining Scenario (1) and (2), we show why the allocation rule q_i we characterize is increasing on α_i, for $\forall i, \forall \alpha_{-i}$.

Through step.1 to step.5, we conclude this proposition. ■

Proof. Proposition 4 (Equilibrium in VA) First of all, we derive have some preliminary comparative statics, which could be applied not only in VA, but all other auction formats. By equation (4) of market clearing conditions,

$$\frac{\partial p^*}{\partial a_i} = \frac{1}{\sum_i b_i} > 0$$

$$\frac{\partial q_i^*}{\partial a_i} = 1 - b\frac{\partial p^*}{\partial a_i} = \frac{\sum_{-i} b_i}{\sum_i b_i} > 0$$

$$\frac{\partial p^*}{\partial b_i} = \frac{I - \sum_i a_i}{(\sum_i b_i)^2} = \frac{\partial p^*}{\partial a_i}(-p^*) < 0$$

$$\frac{\partial q_i^*}{\partial b_i} = -p^* - b_i\frac{\partial p^*}{\partial b_i} = \frac{(I - \sum_i a_i)\sum_{-i} b_i}{(\sum_i b_i)^2} = \frac{\partial q_i^*}{\partial a_i}(-p^*) < 0$$

Secondly, we look at the *ex-post* utility of bidder i under VA,

$$u_i^{VA}(\alpha) = v_i(q_i) - m_i^{VA} = \left(\alpha_i + \frac{I - \sum_{-i} a_i}{\sum_{-i} b_i}\right)q_i^* - \frac{1}{2\left(\beta + \frac{1}{\sum_{-i} b_i}\right)q_i^{*2}}$$

The first order conditions FOCs of a_i and b_i are respectively as :

$$(a_i)\text{:}\quad \left(\alpha_i + \frac{I - \sum_{-i} a_i}{\sum_{-i} b_i} - \left(\beta + \frac{1}{\sum_{-i} b_i}\right) q_i^*\right) \frac{\partial q_i^*}{\partial a_i} = 0$$

$$(b_i)\text{:}\quad \left(\alpha_i + \frac{I - \sum_{-i} a_i}{\sum_{-i} b_i} - \left(\beta + \frac{1}{\sum_{-i} b_i}\right) q_i^*\right) \frac{\partial q_i^*}{\partial b_i} = 0$$

By $-(a_i)p^* - (b_i) = 0$, they are equivalent to

$$\alpha_i + \frac{I - \sum_{-i} a_i}{\sum_{-i} b_i} - \left(\beta + \frac{1}{\sum_{-i} b_i}\right) q_i^* = 0$$

with the expression of market clearing share q_i^*, it is also equivalent to

$$\left((\alpha_i - \beta I)b_i - a_i + I\right) + (\beta b_i - 1)\sum\nolimits_{-i} a_i - (\beta a_i - \alpha_i)\sum\nolimits_{-i} b_i = 0$$

This should be hold for any profile of bidders' types $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, then for any competitors' bidding strategies $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ and $b_{-i} = (b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)$, following the definition of *ex-post* equilibrium. So there is

$$(\alpha_i - \beta I)b_i - a_i + I = \beta b_i - 1 = \beta a_i - \alpha_i = 0$$

where we could derive $a_i = \frac{\alpha_i}{\beta}$; $b_i = \frac{1}{\beta}$, that conclude the truth-telling as the unique *ex-post* linear equilibrium in VA. ■

Proof. Proposition 5 (Equilibrium in UPA) The *ex-post* utility of bidder i under UPA is:

$$u_i^{UPA}(\alpha) = v_i(q_i) - m_i^{UPA} = (\alpha_i - p^*)q_i^* - \frac{1}{2}\beta {q_i^*}^2$$

The first order conditions FOCs of a_i and b_i are respectively as:

$$(a_i)\text{:}\quad -q_i^* \frac{\partial p^*}{\partial a_i} + (\alpha_i - p^* - \beta q_i^*) \frac{\partial q_i^*}{\partial a_i} = 0$$

$$(b_i): \quad -q_i^* \frac{\partial p^*}{\partial b_i} + (\alpha_i - p^* - \beta q_i^*) \frac{\partial q_i^*}{\partial b_i} = 0$$

By equation (4), $-(a_i)p^* - (b_i) = 0$, and $q_i^* = a_i - b_i p^*$, they are equivalent to

$$\left(-a_i - \beta a_i \sum_{-i} b_i + \alpha_i \sum_{-i} b_i\right) + \left(b_i + \beta b_i \sum_{-i} b_i - \sum_{-i} b_i\right) p^* = 0 \tag{6}$$

Similar as the proof of equilibriums in VA, following the definition of *ex-post* equilibrium, this must be hold for any profile of bidders' types $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, then for any competitors' bidding strategies $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ and $b_{-i} = (b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)$. So for any bidder i

$$\begin{aligned} b_i + \beta b_i \sum_{-i} b_i - \sum_{-i} b_i &= 0 \\ -a_i - \beta a_i \sum_{-i} b_i + \alpha_i \sum_{-i} b_i &= 0 \end{aligned} \tag{7}$$

It is generally a non-linear system that we could not solve a_i and b_i explicitly, but some quick observations can make the solution much simpler as we image.

First of all, it is easier to derive a symmetric solution where $b_i = b_j \ \forall i \neq j$, then $b_i = \frac{1}{\beta}(\frac{n-2}{n-1})$ with $n > 2$. When $n = 2$, the FOC of (6) can not be hold that there is no *ex-post* solution exists in this scenario. From equation (7), $a_i = \alpha_i b_i$, then we prove the liner bidding strategy on equilibrium would be

$$q_i^*(p) = \frac{1}{\beta}\left(\frac{n-2}{n-1}\right)(\alpha_i - p)$$

Secondly, we show the symmetric solution is the unique one under this setup. If there are $b_i \neq b_j$, say $b_i > b_j$, then FOC of (6) is equivalent to $\frac{1}{b_i} = \beta + \frac{1}{\sum_{-i} b_i}$. Thus

$$\frac{1}{b_i}-\frac{1}{b_j}=\frac{b_j-b_i}{b_ib_j}=\frac{b_i-b_j}{(\sum_{-i}b_i)(\sum_{-j}b_j)}\Longrightarrow -\frac{1}{b_ib_j}=\frac{1}{(\sum_{-i}b_i)(\sum_{-j}b_j)}$$

That is the contradiction with $b_i > 0, \forall i$. ■

Proof. Proposition 6 (Equilibrium in DPA) The *ex-post* utility of bidder i under DPA is:

$$u_i^{DPA}(\alpha)=v_i(q_i)-m_i^{DPA}=(\alpha_i-\frac{a_i}{b_i})q_i^*-\frac{1}{2}(\beta-\frac{1}{b_i})q_i^{*2}$$

The first order conditions FOCs of a_i and b_i are respectively as :

$$(a_i):\quad -\frac{1}{b_i}q_i^*+(\alpha_i-\frac{a_i}{b_i}-\beta q_i^*+\frac{q_i^*}{b_i})\frac{\partial q_i^*}{\partial a_i}=0$$

$$(b_i):\quad \frac{a_i}{b_i^2}q_i^*-\frac{1}{2b_i^2}q_i^{*2}+(\alpha_i-\frac{a_i}{b_i}-\beta q_i^*+\frac{q_i^*}{b_i})\frac{\partial q_i^*}{\partial b_i}=0$$

By equation (4), $-(a_i)p^*-(b_i)=-\frac{1}{2b_i^2}q_i^{*2}<0$, which is contradict with $(a_i)=(b_i)=0$. So there is no linear *ex-post* equilibrium exist in the general format $q_i(p)=a_i-b_ip$. But comparing with the solution of last two auction formats VA and UPA, if we look at a special case where the submitted demand schedule is proportional to α_i-p or we have $a_i=\alpha_ib_i$, then the equilibrium is achievable. Now we are looking for the optimal bidding strategy $q_i(p)=b_i(\alpha_i-p)$ to maximize the *ex-post* utility of bidder i under DPA

$$u_i^{DPA}(\alpha)=-\frac{1}{2}(\beta-\frac{1}{b_i})q_i^{*2}$$

The first order conditions FOC of b_i can be represented as:

$$(\frac{1}{b_i}-\beta)q_i^*\frac{\partial q_i^*}{\partial b_i}-\frac{1}{2b_i^2}q_i^{*2}=0$$

Where $q_i^* = b_i(\alpha_i - p^*)$ and the new stop-out price $p^* = \frac{\sum_i b_i \alpha_i - I}{\sum_i b_i}$. So we can further reduce it into

$$(-\frac{1}{2}b_i - \beta b_i \sum_{-i} b_i + \frac{1}{2}\sum_{-i} b_i)(\alpha_i - p^*)^2 = 0 \tag{8}$$

Then for any bidder i, $-\frac{1}{2}b_i - \beta b_i \sum_{-i} b_i + \frac{1}{2}\sum_{-i} b_i = 0$ must be hold.

It is generally a non-linear system that we could not solve b_i explicitly, but mimic as the technique applied in the solution of UPA, we could derive a unique symmetric solution where $b_i = b_j \; \forall i \neq j$, then $b_i = \frac{1}{2\beta}(\frac{n-2}{n-1})$ with $n > 2$. Since if $n = 2$, the FOC of (8) can not be hold that there is no *ex-post* solution exists in this scenario. We prove the linear bidding strategy on equilibrium would be

$$q_i^*(p) = \frac{1}{2\beta}\left(\frac{n-2}{n-1}\right)(\alpha_i - p)$$

If there are $b_i \neq b_j$, say $b_i > b_j$, then FOC of (8) is equivalent to $\frac{1}{b_i} = 2\beta + \frac{1}{\sum_{-i} b_i}$. Thus

$$\frac{1}{b_i} - \frac{1}{b_j} = \frac{b_j - b_i}{b_i b_j} = \frac{b_i - b_j}{(\sum_{-i} b_i)(\sum_{-j} b_j)} \Longrightarrow -\frac{1}{b_i b_j} = \frac{1}{(\sum_{-i} b_i)(\sum_{-j} b_j)}$$

That is the contradiction with $b_i > 0, \forall i$.

Last, we show FOC of b_i is not necessary but also sufficient condition to maximize the *ex-post* utility $u_i^{DPA}(\alpha)$. Look at the FOC of (8), since $\frac{1}{2} + \beta \sum_{-i} b_i > 0$, if $b_i < \frac{1}{2\beta}(\frac{n-2}{n-1})$, then FOC is greater than zero while $b_i > \frac{1}{2\beta}(\frac{n-2}{n-1})$ it is less than zero. So we conclude $b_i = \frac{1}{2\beta}(\frac{n-2}{n-1})$ is the global maximization solution of $u_i^{DPA}(\alpha)$. ■

Proof. Proposition 7 (Revenue Ranking) In order to compare the *ex-ante* expected revenue generated by different auctions, we will check their individual *ex-post* payments or money transfer first. Then we put them on the aggregate level and move the time window to *ex-ante.*

***Step.1**: ex-post payments*

We have achieved the *ex-post* equilibriums of VA, UPA and DPA respectively. Refer to the summery of **Table 1,** their *ex-post* payments on equilibriums are following:

$$m_i^{VA} = \left(\frac{\sum_{-i}\alpha_i}{n-1} - \frac{\beta I}{n-1}\right) q_i^* + \frac{\beta}{2(n-1)} {q_i^*}^2$$

$$m_i^{UPA} = p^* q_i^*$$

$$m_i^{DPA} = \alpha_i q_i^* - \beta\left(\frac{n-1}{n-2}\right) {q_i^*}^2$$

***Step.2**: preliminary algebra on aggregation*

Before we advance to the expected aggregation payments of all competitors, here are some important arithmetic identities equations for further reference in reduction.

$$\sum_i \Big(\sum_j (\alpha_i - \alpha_j)\Big)^2 = \sum_i \Big(n\alpha_i - \sum_i \alpha_i\Big)^2 = n^2 \sum_i \alpha_i^2 - n\Big(\sum_i \alpha_i\Big)^2$$

$$\sum_i \sum_j (\alpha_i - \alpha_j) = \sum_i \Big(n\alpha_i - \sum_i \alpha_i\Big) = n\sum_i \alpha_i - n\sum_i \alpha_i = 0$$

$$\sum_i \Big(\sum_{-i} \alpha_i \cdot \sum_j (\alpha_i - \alpha_j)\Big) = \sum_i \Big(\sum_{-i} \alpha_i \cdot \Big(n\alpha_i - \sum_i \alpha_i\Big)\Big)$$

$$= n\sum_{i \neq j} \alpha_i \alpha_j - (n-1)\Big(\sum_i \alpha_i\Big)^2$$

$$\sum_i \sum_{-i} \alpha_i = (n-1)\sum_i \alpha_i$$

$$\sum_i \Big(\sum_i \alpha_i \cdot \sum_j (\alpha_i - \alpha_j)\Big) = \sum_i \alpha_i \cdot \sum_i \sum_j (\alpha_i - \alpha_j) = 0$$

$$\sum_i \Big(\alpha_i \cdot \sum_j (\alpha_i - \alpha_j)\Big) = \sum_i \Big(\alpha_i \cdot \big(n\alpha_i - \sum_i \alpha_i\big)\Big) = n\sum_i \alpha_i^2 - \Big(\sum_i \alpha_i\Big)^2$$

$$\Big(\sum_i \alpha_i\Big)^2 = \sum_i \alpha_i^2 + \sum_{i \neq j} \alpha_i \alpha_j$$

$$(n-1)\sum_i \alpha_i^2 - \sum_{i \neq j} \alpha_i \alpha_j = \sum_{i<j} (\alpha_i - \alpha_j)^2$$

***Step.3**: aggregate ex-post payment*

With the equilibrium solutions, stop-out prices from **Table 1,** plus the relevant algebra derived from ***Step.1*** and ***Step.2,*** we get the aggregate *ex-post* payment of different auction formats.

$$\sum_i m_i^{VA} = \sum_i \left[\frac{1}{(n-1)}\left(\frac{\sum_{-i}\alpha_i}{\beta} - I\right)\left(\frac{\sum_j(\alpha_i - \alpha_j) + \beta I}{n}\right) + \frac{1}{2\beta(n-1)}\left(\frac{\sum_j(\alpha_i - \alpha_j) + \beta I}{n}\right)^2\right]$$

$$= \frac{1}{2\beta(n-1)}\sum_i \alpha_i^2 + \frac{1}{\beta(n-1)}\sum_{i\neq j}\alpha_i\alpha_j - \frac{2n-1}{2\beta n(n-1)}\left(\sum_i \alpha_i\right)^2 + \frac{I}{n}\sum_i \alpha_i - \frac{(2n-1)}{2n^2(n-1)}\beta I^2$$

$$= -\frac{1}{2\beta n(n-1)}\sum_{i<j}(\alpha_i - \alpha_j)^2 + \frac{I}{n}\sum_i \alpha_i - \frac{(2n-1)}{2n^2(n-1)}\beta I^2$$

$$\sum_i m_i^{UPA} = \sum_i \left[\frac{1}{\beta}\left(\frac{n-2}{n-1}\right)\left(\frac{\sum_i \alpha_i - \beta\left(\frac{n-1}{n-2}\right)I}{n}\right)\left(\frac{\sum_j(\alpha_i - \alpha_j) + \beta\left(\frac{n-1}{n-2}\right)I}{n}\right)\right]$$

$$= \frac{I}{n}\sum_i \alpha_i - \frac{(n-1)}{n^2(n-2)}\beta I^2$$

$$\sum_i m_i^{DPA} = \sum_i \left[\frac{\alpha_i}{2\beta}\left(\frac{n-2}{n-1}\right)\left(\frac{\sum_j(\alpha_i - \alpha_j) + 2\beta\left(\frac{n-1}{n-2}\right)I}{n}\right) \right.$$

$$\left. - \frac{1}{4\beta}\left(\frac{n-2}{n-1}\right)\left(\frac{\sum_j(\alpha_i - \alpha_j) + 2\beta\left(\frac{n-1}{n-2}\right)I}{n}\right)^2 \right]$$

$$= \frac{1}{4\beta}\left(\frac{n-2}{n-1}\right)\sum_i \alpha_i^2 - \frac{1}{4\beta n}\left(\frac{n-2}{n-1}\right)\left(\sum_i \alpha_i\right)^2 + \frac{I}{n}\sum_i \alpha_i - \frac{(n-1)}{n^2(n-2)}\beta I^2$$

$$= \frac{1}{4\beta n}\left(\frac{n-2}{n-1}\right)\sum_{i<j}(\alpha_i - \alpha_j)^2 + \frac{I}{n}\sum_i \alpha_i - \frac{(n-1)}{n^2(n-2)}\beta I^2$$

Step.4*: UPA versus DPA*

The *ex-post* aggregate payment of DPA is higher than UPA by what we derive

$$\sum_i m_i^{DPA} - \sum_i m_i^{UPA} = \frac{1}{4\beta n}\left(\frac{n-2}{n-1}\right)\sum_{i<j}(\alpha_i - \alpha_j)^2 > 0$$

So we conclude DPA, which is preferred by seller, will generate higher *ex-ante* expected revenue as $E[R^{DPA}] \geq E[R^{UPA}]$.

Step.5*: VA versus Others*

Last, let us look at the aggregate payment of VA versus other auction formats. With the *ex-post* aggregate payments we have respectively, we can compare them on expectation to the uncertainty of bidder's type distribution.

$$E[R^{VA}] - E[R^{UPA}] = E\left[\sum_i m_i^{VA} - \sum_i m_i^{UPA}\right]$$

$$= \frac{\beta I^2}{2n(n-1)(n-2)} - \frac{1}{2\beta n(n-1)} E\left[\sum_{i<j} (\alpha_i - \alpha_j)^2\right]$$

$$E[R^{VA}] - E[R^{DPA}] = E\left[\sum_i m_i^{VA} - \sum_i m_i^{DPA}\right]$$

$$= \frac{\beta I^2}{2n(n-1)(n-2)} - \frac{1}{4\beta(n-1)} E\left[\sum_{i<j} (\alpha_i - \alpha_j)^2\right]$$

So in order to rank the revenue of different auctions, it is critical to interpret $E\left[\sum_{i<j}(\alpha_i - \alpha_j)^2\right]$. We know α_i, α_j are all independent random draw from $\bar{\chi} = [\underline{w}, \overline{w}]$ following F as the same distribution.

$$E\left[\sum_{i<j} (\alpha_i - \alpha_j)^2\right] = \frac{n(n-1)}{2} E\left[(\alpha_i - \alpha_j)^2\right]$$

$$= \frac{n(n-1)}{2} \cdot 2(E[\alpha^2] - E^2[\alpha])$$

$$= n(n-1) Var[\alpha]$$

By now, we know whether VA is a preferred auction for seller could be highly dependent on the volatility of bidder's private information. To be more specific,

$$E[R^{VA}] \geq E[R^{UPA}] \text{ if and only if } Var[\alpha] \leq \frac{\beta^2 I^2}{n(n-1)(n-2)}$$

$$E[R^{VA}] \geq E[R^{DPA}] \text{ if and only if } Var[\alpha] \leq \frac{2\beta^2 I^2}{n^2(n-1)(n-2)}$$

which conclude the proposition. ■

Proof. Proposition 8 (Efficiency Ranking) Through the definition of efficient share auction in our model, we have already known that VA could achieve the full efficiency.

The only question left is which one can avoid lower welfare loss on either UPA or DPA. The total social welfare it generates on *ex-post* would be $W = \sum_i (\alpha_i q_i - \frac{1}{2}\beta_i q_i^2)$. Refer to equilibriums and relative stop-out prices listed in **Table 1,**

$$W^{UPA} = \frac{1}{\beta}\left(\frac{n-2}{n-1}\right)\sum_i \alpha_i(\alpha_i - p^*) - \frac{1}{2\beta}\left(\frac{n-2}{n-1}\right)^2 \sum_i (\alpha_i - p^*)^2$$

$$= \frac{1}{\beta}\left(\frac{n-2}{n-1}\right)\sum_i \alpha_i \left(\frac{\sum_j(\alpha_i - \alpha_j) + \left(\frac{n-1}{n-2}\right)\beta I}{n}\right)$$

$$-\frac{1}{2\beta}\left(\frac{n-2}{n-1}\right)^2 \sum_i \left(\frac{\sum_j(\alpha_i - \alpha_j) + \left(\frac{n-1}{n-2}\right)\beta I}{n}\right)^2$$

$$W^{DPA} = \frac{1}{2\beta}\left(\frac{n-2}{n-1}\right)\sum_i \alpha_i(\alpha_i - p^*) - \frac{1}{8\beta}\left(\frac{n-2}{n-1}\right)^2 \sum_i (\alpha_i - p^*)^2$$

$$= \frac{1}{\beta}\left(\frac{n-2}{n-1}\right)\sum_i \alpha_i \left(\frac{\frac{1}{2}\sum_j(\alpha_i - \alpha_j) + \left(\frac{n-1}{n-2}\right)\beta I}{n}\right)$$

$$-\frac{1}{2\beta}\left(\frac{n-2}{n-1}\right)^2 \sum_i \left(\frac{\frac{1}{2}\sum_j(\alpha_i - \alpha_j) + \left(\frac{n-1}{n-2}\right)\beta I}{n}\right)^2$$

Furthermore, refer the algebra on aggregation we derive in the proof of **Proposition 7**, we can compare the total welfare of UPA and DPA as:

$$W^{UPA} - W^{DPA} = \frac{1}{\beta}\left(\frac{n-2}{n-1}\right)\sum_i \left[\frac{\alpha_i}{2n}\sum_j(\alpha_i - \alpha_j)\right]$$

$$-\frac{1}{2\beta}\left(\frac{n-2}{n-1}\right)^2 \sum_i \left(\frac{3}{4n^2}\left(\sum_j(\alpha_i - \alpha_j)\right)^2 + \left(\frac{n-1}{n-2}\right)\frac{\beta I}{n}\sum_j(\alpha_i - \alpha_j)\right)$$

$$= \frac{1}{\beta}\left(\frac{n-2}{n-1}\right)\frac{1}{2n}\left(n\sum_i \alpha_i^2 - \left(\sum_i \alpha_i\right)^2\right)$$

$$-\frac{1}{2\beta}\left(\frac{n-2}{n-1}\right)^2 \frac{3}{4n^2}\left(n^2\sum_i \alpha_i^2 - n\left(\sum_i \alpha_i\right)^2\right)$$

$$= \frac{1}{2n\beta}\left(\frac{n-2}{n-1}\right)\left(1-\frac{3}{4}\left(\frac{n-2}{n-1}\right)\right)\left(n\sum_i \alpha_i^2 - \left(\sum_i \alpha_i\right)^2\right)$$

Since $n > 2$, then $1-\frac{3}{4}\left(\frac{n-2}{n-1}\right) > 0$, plus $n\sum_i \alpha_i^2 - (\sum_i \alpha_i)^2 = \sum_{i<j}(\alpha_i - \alpha_j)^2 > 0$, we conclude $W^{VA} > W^{UPA} > W^{DPA}$. ■