ALEA, Lat. Am. J. Probab. Math. Stat. 9 (2), 383–402 (2012)

The critical value of the Deffuant model equals one half

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Abstract. Similarly to the popular voter model, the Deffuant model describes opinion dynamics taking place in spatially structured environments represented by a connected graph. Pairs of adjacent vertices interact at a constant rate. If the opinion distance between the interacting vertices is larger than some confidence threshold $\epsilon > 0$, then nothing happens, otherwise, the vertices' opinions get closer to each other. It has been conjectured based on numerical simulations that this process exhibits a phase transition at the critical value $\epsilon_c = 1/2$. For confidence thresholds larger than one half, the process converges to a global consensus, whereas coexistence occurs for confidence thresholds smaller than one half. In this article, we develop new geometrical techniques to prove this conjecture.

1. Introduction

The past decade has experienced a rapidly growing interest in spatially explicit models of social dynamics, where space takes the form of local interactions through the edges of a graph representing either an actual physical space or a social network where agents are located. See for instance the article of Castellano et al. (2009) which gives a thorough review of over five hundreds, mostly very recent, research papers published in this field. While there is a common effort from sociologists, economists, psychologists, and statistical physicists to understand models of either opinion, cultural or language dynamics, the field has so far been essentially ignored by mathematicians, with the notable exception of the voter model introduced independently in Clifford and Sudbury (1973); Holley and Liggett (1975). In other words, there is already a copious amount of numerical results in this important topic but almost no analytical results to confirm or disprove these conjectures. At the same time, stochastic spatial simulations are known to be difficult to interpret, which suggests the need of rigorous mathematical analyses. The effort to obtain

Received by the editors June 13, 2011; accepted June 7, 2012.
2000 *Mathematics Subject Classification.* 60K35.
Key words and phrases. Interacting particle system, random walks, social dynamics.

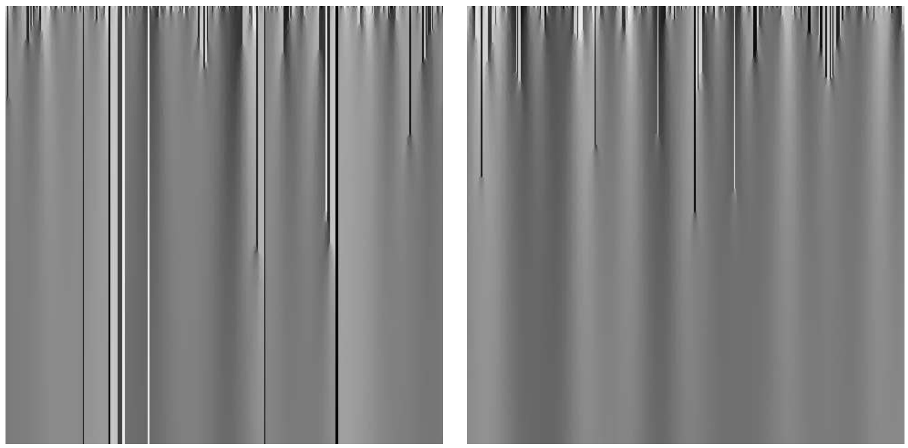

Figure 1.1. Realizations of the Deffuant model on the one-dimensional torus with 400 vertices. Time goes down from time 0 to time 400, and the opinions are function of the brightness with 1 = white and 0 = black. In both realizations, the convergence parameter is $\mu = 0.25$ while the threshold is $\epsilon = 0.48$ and 0.52, respectively.

analytical results in the field of opinion and cultural dynamics has been initiated by the author in Lanchier (2010, 2012) and pursued with co-workers in Lanchier and Scarlatos (2012); Lanchier and Schweinsberg (2012) for the popular Axelrod model for the dissemination of culture and in Lanchier and Neufer (2012) for a spatial version of the majority rule model proposed by socio-physicist Galam to describe public debates. This article goes in this direction by proving the main conjecture about the model introduced by sociologist Deffuant and co-workers in Deffuant et al. (2001). We refer to Section III.F in Castellano et al. (2009) for a survey of this stochastic process.

In the one-dimensional Deffuant model, each vertex $x \in \mathbb{Z}$ is permanently occupied by an agent characterized by an opinion, with the set of all possible opinions ranging from zero to one. As in the voter model, pairs of adjacent vertices interact at a constant rate, say one to fix the time scale, but unlike in the voter model, the interaction is symmetric and results in an update of the process only if a certain compatibility condition is satisfied. More precisely, if the opinion distance between both vertices is larger than some confidence threshold $\epsilon > 0$ at the time of the interaction, then nothing happens. Otherwise, the vertices follow a compromise strategy: their opinions get closer to each other by the relative amount $\mu \in (0, 1/2]$. Figure 1.1 gives simulation pictures of the model on the one-dimensional torus starting from the configuration in which opinions are independent and uniformly distributed over the interval $[0, 1]$. This defines a continuous-time Markov chain whose state space consists of all functions that map $\mathbb{Z}$ into the interval $[0, 1]$ and whose dynamics are described by the Markov generator

$$Lf(\eta) \ = \ \sum_{x \in \mathbb{Z}} \mathbf{1}\{|\eta(x) - \eta(x-1)| < \epsilon\} \ [f(\sigma_x \, \eta) - f(\eta)]$$

where configuration $\sigma_x \eta$ is defined by

$$(\sigma_x \eta)(z_1) = \eta(z_1) + \mu(\eta(z_2) - \eta(z_1)) \quad \text{whenever} \quad \{z_1, z_2\} = \{x, x-1\}$$

while the opinion at all other vertices is unchanged. The main conjecture about the Deffuant model is that, regardless of the value of the convergence parameter μ, the process approaches a global consensus when the confidence threshold is larger than one half whereas coexistence occurs when the threshold is smaller than one half. In other words, there is a dichotomy in the long-term behavior of the process with a phase transition at $\epsilon = 1/2$. This paper gives a rigorous proof of this conjecture that mainly relies on new geometrical techniques, but we also refer to the recent paper of Häggström (2012) who, shortly after the present article was written, proposed an interesting alternative proof based on more traditional techniques.

To state rigorously our main result, we first point out that the meaning of consensus in the case of the Deffuant model is different from the traditional notion of consensus for the voter model since, at all times, the probability that two vertices share the same opinion when starting from initial opinions which are independent continuous random variables is equal to zero. Also, we say that the process converges to a consensus if all pairs of adjacent vertices are ultimately compatible:

$$\lim_{t \to \infty} P(-\epsilon < \eta_t(x) - \eta_t(x+1) < \epsilon) = 1 \quad \text{for all } x \in \mathbb{Z}. \tag{1.1}$$

Our proof is organized around three propositions. The first two state that, under certain assumptions on the initial distribution, the process converges to a consensus when the confidence threshold ϵ is strictly larger than one half but not when it is strictly smaller than one half. More precisely, letting

$$\begin{aligned} \Omega_j = \Big\{ & x \in \mathbb{Z} : \text{for all integers } n > 0 \text{ we have} \\ & \frac{1}{2} - \theta < \frac{1}{n} \sum_{y=0}^{n-1} \eta_0(x+j+y),\ \frac{1}{n} \sum_{y=1}^{n} \eta_0(x-y) < \frac{1}{2} + \theta \Big\} \end{aligned} \tag{1.2}$$

for all integers $j \geq 0$ and all $\theta > 0$, we have the following proposition.

Proposition 1.1. *Assume that for all $\theta > 0$,*

$$P(\operatorname{card}(\Omega_0 \cap \mathbb{Z}_+) = \infty) = P(\operatorname{card}(\Omega_0 \cap \mathbb{Z}_-) = \infty) = 1. \tag{1.3}$$

Then, for all $\mu > 0$ and all $\epsilon > 1/2$, the process converges to a consensus.

In contrast, letting $\mathcal{F}_1 = \{x \in \Omega_1 : |2\eta_0(x) - 1| > 1 - \theta\}$, we have

Proposition 1.2. *Assume that for all $\theta > 0$,*

$$P(\operatorname{card}(\mathcal{F}_1 \cap \mathbb{Z}_+) = \infty) = P(\operatorname{card}(\mathcal{F}_1 \cap \mathbb{Z}_-) = \infty) = 1. \tag{1.4}$$

Then, for all $\mu > 0$ and all $\epsilon < 1/2$, coexistence occurs:

$$P(\operatorname{card}\{x \in \mathbb{Z} : |\eta_t(x) - \eta_t(x+1)| > \epsilon \text{ for all } t \geq 0\} = \infty) = 1. \tag{1.5}$$

Note that the conclusion (1.5) of Proposition 1.2 is somewhat stronger than the lack of convergence to a consensus, since it also indicates that the number of opinion clusters is infinite, and that each cluster is almost surely finite. In particular, defining the critical value of the system as

$$\epsilon_c = \inf\{\epsilon \in [0,1] : \text{consensus (1.1) holds}\},$$

the combination of both propositions implies that, for all initial distributions that simultaneously satisfy condition (1.3) and condition (1.4), we have $\epsilon_c = 1/2$. Also, to deduce the conjecture, it suffices to prove that the configuration in which opinions are independent and uniformly distributed indeed satisfies the two conditions above, which is given by our third proposition.

Proposition 1.3. *Assume that the initial opinions are independent and uniformly distributed over the interval* $[0, 1]$. *Then,* (1.3) *and* (1.4) *hold.*

From all three propositions, we conclude that

Theorem 1.4. *If the initial opinions are independent and uniformly distributed then, regardless of the value of* $\mu > 0$*, the critical threshold* $\epsilon_c = 1/2$.

Since our proof relies on new geometrical techniques which are not standard in the field of interacting particle systems, we start by giving a brief overview of our approach in the next section. The details of the proof are then provided in the subsequent sections, and organized in the same order as Propositions 1.1–1.3.

2. The broken line representation

The traditional technique used to study various aspects, such as the stationary distributions, cluster size, and occupation times, of the voter model is duality. The basic idea of duality is to deduce the individuals' opinion at the current time from the initial configuration of the system by keeping track of the opinion's genealogy going backwards in time, i.e., the spatial locations the opinion originates from. In the case of the voter model, this genealogy, which is encoded in the so-called dual process, simply consists of a system of coalescing random walks. In the case of the Deffuant model, however, this technique fails due to the inclusion of a confidence threshold: while keeping track of an opinion going backwards in time, upon interaction with a neighbor, whether the genealogy branches or not to include the neighbor's opinion depends on whether or not the interacting pair is within the confidence threshold which, in turn, depends on the initial configuration. This key aspect of the model prevents us from defining a dual process. In particular, our proof relies on new, mainly geometrical, tailor-made techniques that we have developed especially for the Deffuant model. The first step is a step of visualization that allows to identify the state of the system at any given time with a doubly-infinite broken line: for every space-time point $(x, t) \in \mathbb{Z} \times \mathbb{R}_+$ we define

$$\xi_t(x) \ = \sum_{0 \leq y \leq x-1} (2\eta_t(y) - 1) \ \mathbf{1}\{x > 0\} \ - \sum_{x \leq y \leq -1} (2\eta_t(y) - 1) \ \mathbf{1}\{x < 0\},$$

and introduce the following process that we shall call profile

$$\zeta_t : \mathbb{Z} \ \longrightarrow \ \mathbb{Z} \times \mathbb{R} \ \text{ defined by } \ \zeta_t(x) = (x, \xi_t(x)) \ \text{ for all } x \in \mathbb{Z}.$$

The profile ζ_t can be seen as a doubly-infinite sequence of points in the Euclidean plane, and it is convenient, in order to get a visual representation, to connect consecutive points of this sequence by a line segment in order to obtain a doubly-infinite broken line. We point out that this broken line always goes through the origin, while the slope of the line segment connecting the points with first coordinates x and $x + 1$ is equal to $2\eta_t(x) - 1$. This indicates that the opinion at vertex x is related to the slope of the corresponding line segment: centrist implies that the slope equals zero whereas extremist implies that the slope equals ± 1. The main reason for introducing this representation is that somewhat complicated properties related to the opinion model can be translated into much simpler geometrical properties related to its broken line representation. To briefly explain the strategy of our proof, we first let, for all $\theta > 0$ and $A = (a_1, a_2) \in \mathbb{Z} \times \mathbb{R}$,

$$\begin{aligned} D(A, +\theta) &= \{(z_1, z_2) \in \mathbb{R} \times \mathbb{R} : z_2 - a_2 = \theta\,(z_1 - a_1)\} \\ D(A, -\theta) &= \{(z_1, z_2) \in \mathbb{R} \times \mathbb{R} : z_2 - a_2 = \theta\,(a_1 - z_1)\} \end{aligned}$$

be the straight lines going through A with slope θ and $-\theta$, respectively. These two straight lines divide the Euclidean plane into four infinite triangles:

$$\begin{array}{rclrcl} \text{West}\,(A, \theta) &=& \text{left triangle} & \text{East}\,(A, \theta) &=& \text{right triangle} \\ \text{North}\,(A, \theta) &=& \text{upper triangle} & \text{South}\,(A, \theta) &=& \text{lower triangle.} \end{array}$$

From a topological point of view, we assume that West and East are closed sets whereas North and South are open sets. The main objective of Section 3 is to study certain dynamical properties of the profile. First, we will prove that, if the left part of the profile starting at vertex x is initially contained in the left triangle with corner $\zeta_0(x)$, then the inclusion remains true at all times provided the profile at vertex x is static. More precisely, conditioned on the events that

$$\zeta_0((-\infty, x]) \subset \text{West}\,(\zeta_0(x), \theta) \quad \text{and} \quad \zeta_t(x) = \zeta_0(x) \ \text{ for all } t > 0 \tag{2.1}$$

the inclusion remains true at all times:

$$\zeta_t((-\infty, x]) \subset \text{West}\,(\zeta_0(x), \theta) \ \text{ for all } \ t > 0. \tag{2.2}$$

Note that this last inclusion implies that the slope of the line segment connecting vertices x and $x + 1$ is bounded in absolute value by θ, hence the opinion at x is within distance θ of the centrist opinion at all times. This property will be used in Section 5 together with the fact that (1.2) can be simply written as

$$\Omega_j \ = \ \{x \in \mathbb{Z} : \zeta_0(\mathbb{Z}) \subset \text{West}\,(\zeta_0(x), \theta) \cup \text{East}\,(\zeta_0(x + j), \theta)\}$$

and obvious symmetry properties to prove that, for all $\epsilon < 1/2$, there exists θ small such that vertices that belong to the set $\mathcal{F}_1$ are never compatible with their right neighbors. This is the key to proving the lack of convergence to a consensus and thus Proposition 1.2. The consensus part given by Proposition 1.1 is more challenging. Since in this case all vertices eventually interact with their neighbors, the main difficulty is to prove a property of the type (2.1) implies (2.2) in the more general case when the opinion at each vertex is not static. This is the second objective of Section 3 where we prove that, whenever

$$\zeta_0(\mathbb{Z}) \subset \text{West}\,(\zeta_0(x), \theta) \cup \text{East}\,(\zeta_0(x), \theta), \tag{2.3}$$

we have, uniformly over all realizations of the Deffuant dynamics,

$$\zeta_t(\mathbb{Z}) \subset \text{West}\,(\zeta_t(x), 3\theta) \cup \text{East}\,(\zeta_t(x), 3\theta) \ \text{ for all } t > 0. \tag{2.4}$$

This will be used in Section 4 to prove that, for all $\epsilon > 1/2$, there exists θ small such that vertices in the set Ω_0 keep interacting with their neighbors to spread an almost centrist opinion, which is the key to proving convergence of the system to a consensus and Proposition 1.1. More precisely, the fact that Ω_0 is almost surely infinite induces a partition of $\mathbb{Z}$ into finite intervals whose boundaries are arbitrarily close to the centrist opinion at all times, and thus compatible with all other vertices when $\epsilon > 1/2$. These boundaries act as sources from which the centrist opinion spreads, which implies that all the opinions in any given finite region reach eventually a consensus.

To deduce that $\epsilon_c = 1/2$ when starting with opinions which are independent and uniformly distributed, the last step is to establish Proposition 1.3, which is carried out in Section 6 where we study certain asymptotic properties of the initial profile using random walk estimates to get conditions like (2.1) and (2.3). The connection with random walks is obtained by thinking of the spatial structure $\mathbb{Z}$ as a doubly-infinite temporal structure, in which case the initial profile can be seen as the concatenation of the realizations of two symmetric random walks, both starting at zero at time 0 and both having increments which are uniformly distributed over the interval $(-1, 1)$, but with one random walk evolving forward in time and the other one evolving backwards in time. Using this approach and dissecting the random walk trajectory at some specific random times, we prove that, for all integers j, the random set Ω_j is almost surely infinite. Since the event that $x \in \Omega_1$ is independent of the slope of the line segment connecting x and $x + 1$, there exist infinitely many vertices in the set Ω_1 with initial opinion arbitrary close to one of the two extremist opinions, which implies that the set $\mathcal{F}_1$ also is almost surely infinite.

3. Preliminary results: dynamical properties of the profile

In this section, we prove that (2.1) implies (2.2) and that (2.3) implies (2.4) relying on new geometrical arguments. First, we let

$$\begin{aligned}\mathrm{Tr}\,(A, B, C) \ = \ \{\mu_A \cdot A + \mu_B \cdot B + \mu_C \cdot C \text{ such that } \mu_A, \mu_B, \mu_C \geq 0 \\ \text{and } \mu_A + \mu_B + \mu_C = 1\} \ \text{ for } A, B, C \in \mathbb{R}^2\end{aligned}$$

denote the triangle with corners A, B and C. In addition,

$D(A, B)$	is the straight line going through A and B,
slope $D(A, B)$	is the slope of the straight line $D(A, B)$,
$A \ll B$	means that A is below B, i.e., the second coordinate of A is smaller than or equal to the second coordinate of B,
$A \ll D(B, C)$	means that A is below $D(B, C)$, i.e., point A belongs to the lower half plane delimited by the straight line $D(B, C)$.

It will be convenient to use an idea of Harris (1972) and think of the model as being generated by the graphical representation that consists of the following independent processes: for all $x \in \mathbb{Z}$, let $N_t(x)$ be a Poisson process with intensity one, and

$$\Lambda(x) \ = \ \{t : N_{t-}(x) \neq N_t(x)\}$$

be the arrival times of this Poisson process. Then, adding one dimension for time, we define a percolation structure by drawing a double arrow between $x-1$ and x at time $t \in \Lambda(x)$ to indicate that both vertices interact, which may or may not result in an update depending on whether the vertices' opinions are compatible or not. Here and later, we use the word compatible to indicate that the opinion distance between both vertices is smaller than the confidence threshold. We start with two preliminary results that will be applied repeatedly later.

Lemma 3.1. *For all $(z, t) \in \mathbb{Z} \times \mathbb{R}$, we have*

$$\zeta_t(z) \ \in \ \mathrm{Tr}\,(\zeta_{t-}(z-1), \zeta_{t-}(z), \zeta_{t-}(z+1)).$$

Proof: We fix a time $t > 0$ and, to avoid trivialities, assume that $t \in \Lambda(z_0)$. Then, only the opinions at the vertices $z_0 - 1$ and z_0 may be updated and

$$\begin{aligned} \eta_t(z_0 - 1) + \eta_t(z_0) &= \sigma_{z_0}\, \eta_{t-}(z_0 - 1) + \sigma_{z_0}\, \eta_{t-}(z_0) \\ &= \eta_{t-}(z_0 - 1) + \eta_{t-}(z_0) \end{aligned}$$

regardless of the compatibility of the vertices, hence $\zeta_t(z) = \zeta_{t-}(z)$ for all $z \neq z_0$. In particular, the lemma holds for all $z \neq z_0$. Similarly, we have

$$\zeta_t(z_0) = \zeta_{t-}(z_0) \quad \text{whenever} \quad |\eta_{t-}(z_0 - 1) - \eta_{t-}(z_0)| \geq \epsilon.$$

It remains to deal with the (only nontrivial) case when vertices $z_0 - 1$ and z_0 are compatible at time $t-$ and interact at time t, or equivalently when

$$t \in \Lambda(z_0) \quad \text{and} \quad |\eta_{t-}(z_0 - 1) - \eta_{t-}(z_0)| < \epsilon.$$

In this case, $\zeta_t(z_0) \neq \zeta_{t-}(z_0)$ but we have

$$\begin{aligned} \xi_t(z_0) &= \xi_t(z_0 - 1) \ + \ (2\eta_t(z_0 - 1) - 1) \\ &= \xi_{t-}(z_0 - 1) \ + \ (2\sigma_{z_0}\, \eta_{t-}(z_0 - 1) - 1) \\ &= \xi_{t-}(z_0 - 1) \ + \ (2\,(1 - \mu)\, \eta_{t-}(z_0 - 1) + 2\,\mu\, \eta_{t-}(z_0) - 1). \end{aligned}$$

Decomposing $1 = \mu + (1 - 2\mu) + \mu$ and $1 - \mu = (1 - 2\mu) + \mu$, this becomes

$$\begin{aligned} \xi_t(z_0) &= \mu\, \xi_{t-}(z_0 - 1) \ + \ (1 - 2\mu)\, (\xi_{t-}(z_0 - 1) + (2\eta_{t-}(z_0 - 1) - 1)) \\ &\qquad + \mu\, (\xi_{t-}(z_0 - 1) + (2\eta_{t-}(z_0 - 1) - 1) + (2\eta_{t-}(z_0) - 1)) \\ &= \mu\, \xi_{t-}(z_0 - 1) \ + \ (1 - 2\mu)\, \xi_{t-}(z_0) \ + \ \mu\, \xi_{t-}(z_0 + 1). \end{aligned}$$

Since $\mu + (1 - 2\mu) + \mu = 1$, it follows that

$$\zeta_t(z_0) \ = \ \mu\, \zeta_{t-}(z_0 - 1) \ + \ (1 - 2\mu)\, \zeta_{t-}(z_0) \ + \ \mu\, \zeta_{t-}(z_0 + 1).$$

Since in addition $\mu \leq 1/2$, all three coefficients in this linear combination are nonnegative, from which we deduce that the point $\zeta_t(z_0)$ can be expressed as a barycenter with nonnegative weights of the three corners, therefore it belongs to the corresponding triangle. This completes the proof. $\square$

Lemma 3.2. *Let $C \subset \mathbb{R} \times \mathbb{R}$ be a convex set, $z_1 < z_2$, and assume that*

$$\zeta_s(z_1) = \zeta_0(z_1) \ \text{and} \ \zeta_s(z_2) = \zeta_0(z_2) \quad \text{for all} \quad s \leq t \quad \text{and} \quad \zeta_0((z_1, z_2)) \subset C.$$

Then, we have $\zeta_s((z_1, z_2)) \subset C$ for all $s \leq t$.

Proof: As in the proof of Lemma 3.1, times at which the process is not updated can be ignored, therefore it suffices to prove the following: whenever

$$s \in \Lambda(z_0) \cap (0, t) \quad \text{and} \quad |\eta_{s-}(z_0 - 1) - \eta_{s-}(z_0)| < \epsilon$$

we have the implication

$$\zeta_{s-}((z_1, z_2)) \subset C \quad \text{implies} \quad \zeta_s((z_1, z_2)) \subset C.$$

Note that z_0 cannot be equal to z_1 or z_2 since the profile at these two vertices is not updated before time t by assumption, while the case when $z_0 \notin (z_1, z_2)$ does not affect the configuration of the profile in the convex set. To prove the implication in the case when $z_0 \in (z_1, z_2)$, note that

$$\zeta_s(z) = \zeta_{s-}(z) \in C \quad \text{for all } z \in (z_1, z_2), \ z \neq z_0$$

while to deal with vertex $z = z_0$, we use

$$\zeta_{s-}(z_0 - 1),\, \zeta_{s-}(z_0),\, \zeta_{s-}(z_0 + 1) \in C$$

the convexity of C, and Lemma 3.1 to deduce that

$$\zeta_s(z_0) \in \mathrm{Tr}\,(\zeta_{s-}(z_0 - 1), \zeta_{s-}(z_0), \zeta_{s-}(z_0 + 1)) \subset C.$$

This completes the proof of the lemma. $\square$

Proposition 3.3. (2.1) $\Rightarrow$ (2.2), *i.e., whenever*

$$\zeta_0((-\infty, x]) \subset \mathrm{West}\,(\zeta_0(x), \theta) \quad \textit{and} \quad \zeta_t(x) = \zeta_0(x) \quad \textit{for all } t > 0$$

the inclusion remains true at all times:

$$\zeta_t((-\infty, x]) \subset \mathrm{West}\,(\zeta_0(x), \theta) \textit{ for all } \ t > 0.$$

Proof: This follows from Lemma 3.2 since West $(\zeta_0(x), \theta)$ is convex. $\square$

Proposition 3.4. (2.3) $\Rightarrow$ (2.4), *i.e., whenever*

$$\zeta_0(\mathbb{Z}) \subset \mathrm{West}\,(\zeta_0(x), \theta) \cup \mathrm{East}\,(\zeta_0(x), \theta)$$

we have, uniformly over all realizations of the Harris' graphical representation,

$$\zeta_t(\mathbb{Z}) \subset \mathrm{West}\,(\zeta_t(x), 3\theta) \cup \mathrm{East}\,(\zeta_t(x), 3\theta) \quad \textit{for all } t > 0.$$

As mentioned in Section 2, the proof of Proposition 3.4 is more challenging because it is no longer assumed that the profile at the common corner of the left and right triangles is static. In particular, the profile is no longer contained in the union of the left and right triangles. The first step is to show that the profile cannot simultaneously intersect the upper triangle and the lower triangle, which again relies on Lemma 3.1. This is done in the following lemma.

Lemma 3.5. *Assume* (2.3)*. Then, for all* $t \geq 0$,

$$\zeta_t(\mathbb{Z}) \cap \mathrm{South}\,(\zeta_0(x), \theta) = \varnothing \quad \textit{or} \quad \zeta_t(\mathbb{Z}) \cap \mathrm{North}\,(\zeta_0(x), \theta) = \varnothing. \tag{3.1}$$

Proof: The key is to prove conjointly (3.1) and the fact that the part of the profile that has moved outside the left and right triangles is connex. To make this statement rigorous, we introduce the collections of subsets

$$\begin{aligned} C_x &= \{(z_1, z_2) \cap \mathbb{Z} : z_1 \leq x \leq z_2\} \\ \Upsilon_{\mathrm{North}}\,(t) &= \{z \in \mathbb{Z} : \zeta_t(z) \in \mathrm{North}\,(\zeta_0(x), \theta)\}. \end{aligned}$$

In particular, $\varnothing \in C_x$. Then, we will prove that either

$$\zeta_t(\mathbb{Z}) \cap \mathrm{South}\,(\zeta_0(x), \theta) = \varnothing \quad \text{and} \quad \Upsilon_{\mathrm{North}}\,(t) \in C_x \tag{3.2}$$

holds or the dual property obtained by exchanging North and South holds. Before proving that the dynamics preserve (3.2) or its dual property, we observe that both equalities in (3.1) hold simultaneously at time 0 while we also have

$$\Upsilon_{\mathrm{North}}\,(0) = \Upsilon_{\mathrm{South}}\,(0) = \varnothing \in C_x$$

therefore both (3.2) and its dual property hold at time 0. Let $t \in \Lambda(z_0)$ and assume that (3.2) holds at time $t-$ just before the process is updated. First, whenever

$$\zeta_{t-}(z_0 - 1),\, \zeta_{t-}(z_0),\, \zeta_{t-}(z_0 + 1) \in \mathrm{North}\,(\zeta_0(x), \theta)$$

the updated point $\zeta_t(z_0)$ also belongs to the upper triangle according to Lemma 3.1 since the upper triangle is convex, hence property (3.2) holds trivially. The same holds if all three points belong to the left triangle or all three points belong to the right triangle. Second, whenever

$$\zeta_{t-}(z_0 - 1) \in \mathrm{West}\,(\zeta_0(x), \theta) \quad \text{and} \quad \zeta_{t-}(z_0 + 1) \in \mathrm{North}\,(\zeta_0(x), \theta)$$

the intermediate point $\zeta_{t-}(z_0)$ belongs to either the left triangle or the upper triangle. Since the union of these two triangles is convex, the first equality in (3.2) holds at time t. Also, since the connexity property holds at time $t-$

$$z_0 \ = \ \min \Upsilon_{\mathrm{North}}(t-) \quad \text{or} \quad z_0 \ = \ \min \Upsilon_{\mathrm{North}}(t-) - 1$$

hence $\Upsilon_{\mathrm{North}}(t) \in C_x$ that $z_0 \in \Upsilon_{\mathrm{North}}(t)$ or not, which proves that (3.2) holds at time t. The same holds if the two neighbors of the point to be updated belong to the upper triangle and the right triangle, respectively, at time $t-$. To conclude, we examine the remaining case when

$$\begin{aligned} \zeta_{t-}(z_0 - 1) \ &\in \ \mathrm{West}(\zeta_0(x), \theta) \\ \zeta_{t-}(z_0) \ &\in \ \mathrm{North}(\zeta_0(x), \theta) \\ \zeta_{t-}(z_0 + 1) \ &\in \ \mathrm{East}(\zeta_0(x), \theta) \end{aligned}$$

which imposes $z_0 = x$. The connexity property is useful in this case. If the updated point remains in the upper triangle then (3.2) holds trivially at time t, whereas if the updated point jumps outside the upper triangle one has

$$\Upsilon_{\mathrm{North}}(t) \ = \ \varnothing \quad \text{and} \quad \Upsilon_{\mathrm{South}}(t) \ \in \ \{\varnothing, \{x\}\}$$

since $\Upsilon_{\mathrm{North}}(t-) = \{x\}$ by connexity. This implies that

$$\zeta_t(\mathbb{Z}) \cap \mathrm{North}(\zeta_0(x), \theta) \ = \ \varnothing \quad \text{and} \quad \Upsilon_{\mathrm{South}}(t) \ \in \ \{\varnothing, \{x\}\} \ \subset \ C_x$$

which gives the dual property of (3.2). This completes the proof. $\square$

Returning to the proof of Proposition 3.4, assume first that $\zeta_t(x) = \zeta_0(x)$. In this case, the proof of Lemma 3.5 implies that the profile at time t is contained in the union of the left and right triangles therefore, following the lines of the proof of Proposition 3.3, we deduce that

$$\begin{aligned} \zeta_t(\mathbb{Z}) \quad &\subset \quad \mathrm{West}(\zeta_t(x), \theta) \cup \mathrm{East}(\zeta_t(x), \theta) \\ &\subset \quad \mathrm{West}(\zeta_t(x), 3\theta) \cup \mathrm{East}(\zeta_t(x), 3\theta) \quad \text{for all } t > 0. \end{aligned}$$

To deal with the case $\zeta_t(x) \neq \zeta_0(x)$, we may assume that $\xi_t(x) > \xi_0(x)$ without loss of generality in view of the obvious symmetry of the problem. The idea is to break down the evolution of the process at some random times going backwards in time, and then deduce relevant properties that hold between these consecutive random times going forward in time. Let

$$s(x) \ = \ \sup\{s < t : \xi_{s-}(x) < \xi_0(x) \le \xi_s(x)\},$$

be the last time before time t the profile at x jumps above $\zeta_0(x)$,

$$\begin{aligned} m(x) \quad &= \quad \max\{\xi_s(x) : s \in (s(x), t)\} \\ t(x) \quad &= \quad \inf\{s \in (s(x), t) : \xi_s(x) = m(x)\} \end{aligned}$$

and denote by $D(x)$ the straight line

$$D(x) \ = \ D(\zeta_{t(x)}(x-1), \zeta_{t(x)}(x+1)).$$

In a similar way, we define recursively for $j \ge 1$

$$\begin{aligned} s(x+j) \ = \ \sup\{s < t(x+j-1) \text{ such that} \\ \xi_{s-}(x+j) < \xi_{t(x+j-1)}(x+j) \le \xi_s(x+j)\}, \end{aligned}$$

the last time before $t(x+j-1)$ the profile at $x+j$ jumps above $D(x+j-1)$,

$$\begin{array}{rcl} m(x+j) & = & \max\{\xi_s(x+j) : s \in (s(x+j), t(x+j-1))\} \\ t(x+j) & = & \inf\{s \in (s(x+j), t(x+j-1)) : \xi_s(x+j) = m(x+j)\} \end{array}$$

and denote by $D(x+j)$ the straight line

$$D(x+j) \ = \ D(\zeta_{t(x+j)}(x+j-1), \zeta_{t(x+j)}(x+j+1)).$$

Also, let $D'(x)$ be the straight line parallel to $D(x)$ going through $\zeta_{t(x)}(x)$. Finally, we introduce the intersection points

$$A \ = \ D'(x) \cap D(\zeta_0(x), \theta) \quad \text{and} \quad A_j \ = \ D'(x) \cap D((x+j, 0), \infty)$$

for all $j \in \mathbb{Z}$, where a line with infinite slope means a vertical line, and let

$$\begin{array}{rcl} m & = & \max\{j \geq 0 : A_j \notin \text{East}\,(\zeta_0(x), \theta)\} \\ B & = & D((x+m+1, 0), \infty) \cap D(\zeta_0(x), -\theta). \end{array}$$

The proof of Proposition 3.4 relies on Lemmas 3.6–3.10.

Lemma 3.6. *For* $0 \leq j \leq m-1$, $\text{slope}\,(D(x+j)) \leq \text{slope}\,(D(x+j+1))$.

Proof: Since $\xi_{t(x)}(x) > \xi_0(x)$, point $\zeta_s(x)$ jumps up at time $s = t(x)$ so

$$\zeta_{t(x+1)}(x) \ \ll \ \zeta_{t(x)}(x) = (x, m(x)) \ \ll \ D(x) \tag{3.3}$$

according to Lemma 3.1. Also, by definition of $s(x+1)$ and $t(x+1)$,

$$\zeta_{t(x+1)}(x+1) = (x+1, m(x+1)) \ \gg \ \zeta_{t(x)}(x+1) \ \in \ D(x). \tag{3.4}$$

In case $m = 0$, we are done. Otherwise,

$$A_1 \notin \text{East}\,(\zeta_0(x), \theta) \ \text{ therefore } \ \xi_{t(x+1)}(x+1) \geq \xi_{t(x)}(x+1) > \xi_0(x+1)$$

which implies that $\zeta_s(x+1)$ jumps up at time $s = t(x+1)$. In particular,

$$\zeta_{t(x+1)}(x+1) \ \ll \ D(x+1) \tag{3.5}$$

according again to Lemma 3.1. We deduce that

$$\text{slope}\,(D(x)) \ \leq \ \text{slope}\,(D(\zeta_{t(x+1)}(x), \zeta_{t(x+1)}(x+1))) \ \leq \ \text{slope}\,(D(x+1))$$

where the first inequality follows from (3.3) and (3.4), and the second inequality follows from (3.5). This proves the result at step $j = 0$. In case $m = 1$, the proof is complete, otherwise the exact same reasoning as above gives the result at the next step. The lemma follows from a simple induction. $\square$

Lemma 3.7. *For all* $j = 1, 2, \ldots, m$,

$$\zeta_s(x+j) \ \gg \ D(x) \ \text{ for all } \ s(x+j) \leq s \leq t(x+j-1).$$

Proof: Since $\zeta_{t(x)}(x+1) \in D(x)$, the definition of $s(x+1)$ implies that

$$\zeta_s(x+1) \ \gg \ D(x) \ \text{ for all } \ s(x+1) \leq s \leq t(x),$$

which proves the result at step $j = 1$. Now, fix $j < m$ and assume that

$$\zeta_s(x+j) \ \gg \ D(x) \ \text{ for all } \ s(x+j) \leq s \leq t(x+j-1). \tag{3.6}$$

Using as previously that $\zeta_{t(x+j)}(x+j+1) \in D(x+j)$, we obtain

$$\zeta_s(x+j+1) \ \gg \ D(x+j) \ \text{ for all } \ s(x+j+1) \leq s \leq t(x+j). \tag{3.7}$$

Also, from (3.6) and the fact that $\zeta_s(x+j)$ jumps up at time $s = t(x+j)$,

$$\zeta_{t(x+j)}(x+j) \ \gg \ D(x) \quad \text{and} \quad \zeta_{t(x+j)}(x+j) \ \ll \ D(x+j). \tag{3.8}$$

Using (3.7)-(3.8) and the fact that

$$\text{slope}\,(D(x+j)) \;\geq\; \text{slope}\,(D(x))$$

according to Lemma 3.6, we deduce that

$$\zeta_s(x+j+1) \;\gg\; D(x) \quad \text{for all} \quad s(x+j+1) \leq s \leq t(x+j).$$

The lemma follows by induction. □

Lemma 3.8. *The integer m is finite.*

Proof: First, we observe that

$$x_- := \sup\{z < x : N_t(z) = 0\} > -\infty$$
$$x_+ := \inf\{z > x : N_t(z) = 0\} < +\infty.$$

In addition, Lemma 3.2 implies that for all $s \leq t$

$$\zeta_s(z) \;\in\; \text{West}\,(\zeta_0(x), \theta) \,\cup\, \text{East}\,(\zeta_0(x), \theta) \quad \text{for all } z \notin (x_-, x_+).$$

This and Lemma 3.7 indicate that $x + m < x_+ < \infty$. □

Lemma 3.9. *For all $0 \leq i \leq j \leq m$,*

$$\zeta_s(x+j) \;\gg\; D(A_i, B) \quad \textit{for all} \quad t(x+i) \leq s \leq t(x+i-1)$$

where we assume that $t(x-1) = \sup\{s \leq t : \zeta_s(x) \neq m(x)\}$.

Proof: The result is proved by induction going forward in time starting with $i = m$ which is possible because $m < \infty$ by Lemma 3.8. According to Lemma 3.7,

$$\begin{aligned} \zeta_s(x+m) \;\gg\; A_m \quad \text{for all} \quad s \in {} & [t(x+m), t(x+m-1)] \\ & \subset [s(x+m), t(x+m-1)], \end{aligned}$$

which implies the result at step $i = m$. Now, fix $i > 0$ and assume that

$$\zeta_s(x+j) \;\gg\; D(A_i, B) \quad \text{for all} \quad t(x+i) \leq s \leq t(x+i-1)$$

for all $j = i, i+1, \ldots, m$. This implies that

$$\zeta_{t(x+i-1)}(x+j) \;\gg\; D(A_{i-1}, B) \quad \text{for all} \quad j = i, i+1, \ldots, m. \tag{3.9}$$

In other respects, according to Lemma 3.7,

$$\zeta_s(x+i-1) \;\gg\; A_{i-1} \quad \text{for all} \quad t(x+i-1) \leq s \leq t(x+i-2), \tag{3.10}$$

while, according to Lemma 3.5,

$$\zeta_s(x+m+1) \;\gg\; B \quad \text{for all} \quad t(x+i-1) \leq s \leq t(x+i-2). \tag{3.11}$$

Combining (3.9)-(3.11), and applying Lemma 3.2, we obtain

$$\zeta_s(x+j) \;\gg\; D(A_{i-1}, B) \quad \text{for all} \quad t(x+i-1) \leq s \leq t(x+i-2),$$

for all $j = i-1, i, \ldots, m$, which proves the result at step $i - 1$. □

Lemma 3.10. *The slope of $D(A_0, B)$ is larger than -3θ.*

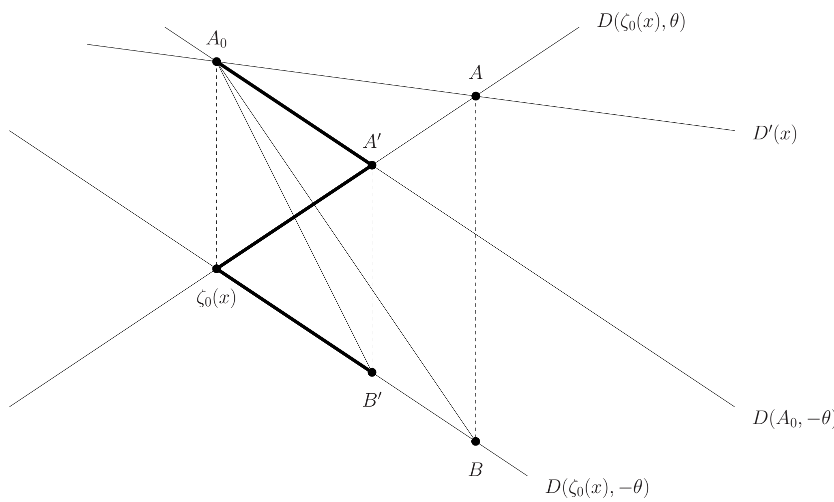

Figure 3.2. Picture related to the proof of Lemma 3.10.

Proof: Since $m < \infty$, the line $D'(x)$ intersects $D(\zeta_0(x), \theta)$ above $\zeta_0(x)$. Using obvious symmetry, we deduce that both

$$D'(x) \cap D(\zeta_0(x), \theta) \gg \zeta_0(x) \quad \text{and} \quad D'(x) \cap D(\zeta_0(x), -\theta) \gg \zeta_0(x)$$

therefore the absolute value of the slope of the straight line $D(x)$ is strictly less than θ. The rest of the proof is based on a simple geometric construction, which is depicted in Figure 3.2. First, we recall that

$$\begin{aligned} A &= D'(x) \cap D(\zeta_0(x), \theta) \\ B &= D((x + m + 1, 0), \infty) \cap D(\zeta_0(x), -\theta) \end{aligned}$$

and introduce the other intersection points

$$A' = D(A_0, -\theta) \cap D(\zeta_0(x), \theta) \quad \text{and} \quad B' = D(A', \infty) \cap D(\zeta_0(x), -\theta).$$

Since the absolute value of the slope of $D(x)$ is bounded by θ,

$$A' \in [\zeta_0(x), A] \quad \text{and so} \quad B' \in [\zeta_0(x), B].$$

This implies that the absolute value of the slope of $D(A_0, B)$ is smaller than the absolute value of the slope of $D(A_0, B')$ which is 3θ since

$$(\zeta_0(x), A_0, A', B') \text{ is a trapezoid}$$

and the straight lines $D(B', \zeta_0(x))$, $D(\zeta_0(x), A')$, $D(A', A_0)$ have slope $\pm\theta$. □

Taking $i = 0$ in Lemma 3.9 gives

$$\zeta_s(x + j) \gg D(A_0, B) \quad \text{for all} \ \ t(x) \le s \le t(x - 1) \text{ and } j = 0, 1, \ldots, m.$$

In fact, this holds for all $j \ge 0$ since by Lemma 3.5 the profile does not intersect the lower triangle. In particular, a new application of Lemma 3.2 implies that

$$\zeta_t(x + j) \gg D(\zeta_t(x), B) \quad \text{for all} \ \ j \ge 0. \tag{3.12}$$

Now, according to Lemma 3.10, we also have

$$\text{slope}\,(D(\zeta_t(x), B)) \ \ge \ \text{slope}\,(D(A_0, B)) \ \ge \ -3\theta \ \text{ since } \ \zeta_t(x) \ll A_0. \tag{3.13}$$

Combining (3.12)-(3.13), and using obvious symmetry looking at the left side of the profile rather than the right side, we deduce that

$$\zeta_t(\mathbb{Z}) \subset \text{West}\,(\zeta_t(x), 3\theta) \cup \text{East}\,(\zeta_t(x), 3\theta).$$

This completes the proof of the proposition.

4. Proof of Proposition 1.1

Relying on the results of Section 3, we now prove convergence to a consensus when the confidence threshold is larger than one half and (1.3) holds. The proof relies on Proposition 3.4 which is used to show the existence of infinitely many vertices with opinions arbitrarily close to the centrist opinion at all times. In particular, when $\epsilon > 1/2$, these vertices are compatible with their neighbors at all times and act as sources from which an almost centrist opinion spreads.

Lemma 4.1. *Assume that $\epsilon > 1/2$ and (1.3) holds. Then*

$$\lim_{t\to\infty} P\,(-\epsilon < \eta_t(x) - \eta_t(x+1) < \epsilon) \;=\; 1 \quad \textit{for all } x \in \mathbb{Z}.$$

Proof: First, we let $x \in \mathbb{Z}$, fix $\theta > 0$ such that $4\theta < \min(2\epsilon - 1, \epsilon)$, and define

$$x_- \;=\; \sup\,\{z \in \Omega_0 : z < x\} \quad \text{and} \quad x_+ \;=\; \inf\,\{z \in \Omega_0 : z > x\}.$$

Condition (1.3) implies that

$$P\,(x_- > -\infty) \;=\; P\,(x_+ < +\infty) \;=\; 1 \quad \text{therefore} \quad x_+ - x_- := K < \infty \;\; \text{a.s.}$$

Now, take a denominated vertex $z \in \mathbb{Z}$ and assume that vertices z and $z-1$ are compatible. Then, the opinion at vertex z after n interactions is given by

$$\begin{aligned}(\sigma_z^n \eta_t)(z) \;&=\; \mu\,(\sigma_z^{n-1}\eta_t)(z-1) \;+\; (1-\mu)\,(\sigma_z^{n-1}\eta_t)(z) \\ &=\; \mu\,(\sigma_z^{n-1}\eta_t)(z-1) \;+\; \mu(1-\mu)\,(\sigma_z^{n-2}\eta_t)(z-1) \\ &\qquad\qquad\qquad\qquad + \;(1-\mu)^2\,(\sigma_z^{n-2}\eta_t)(z) \\ &=\; \mu\,(\sigma_z^{n-1}\eta_t)(z-1) \;+\; \mu(1-\mu)\,(\sigma_z^{n-2}\eta_t)(z-1) \;+\; \cdots \\ &\qquad\qquad + \;\mu(1-\mu)^{n-1}\,\eta_t(z-1) \;+\; (1-\mu)^n\,\eta_t(z).\end{aligned} \tag{4.1}$$

In other respects, since $x_- \in \Omega_0$, according to Proposition 3.4, we have

$$\begin{aligned}2 \times |\eta_t(x_-) - \eta_t(x_-+1)| \;&=\; |(2\eta_t(x_-) - 1) - (2\eta_t(x_-+1) - 1)| \\ &\leq\; 1 + 3\theta \;<\; 2\epsilon\end{aligned} \tag{4.2}$$

so vertices x_- and $x_- + 1$ are compatible at all times. In addition,

$$|\,2\,(\sigma_{x_-+1}^n\eta_t)(x_-) - 1| \;<\; 3\theta \quad \text{for all } n \geq 0$$

again by Proposition 3.4. Therefore, applying (4.1) to vertex $z = x_- + 1$, we deduce that there exists an integer $N \geq 0$ fixed from now on such that

$$\begin{aligned}|2\,(\sigma_{x_-+1}^N\eta_t)(x_-+1) - 1| \;&\leq\; (1 - (1-\mu)^N) \times 3\theta \;+\; 2 \times (1-\mu)^N \\ &\leq\; (3 + K^{-1})\,\theta.\end{aligned} \tag{4.3}$$

We say that pattern 1 occurs from time t to time $t+s$ whenever the graphical representation of the process restricted to the interval $[x_-, x_+]$ consists, between these two times, of a succession of exactly N interactions between vertex x_- and vertex $x_- + 1$, and no other interaction in the interval. Then, writing $\tau_1 = \sigma_{x_-+1}^N$, whenever pattern 1 occurs between time t and time $t+s$, it follows from the evolution rules of the process and inequality (4.3) that

$$\begin{array}{rcl} |2\eta_{t+s}(x_- + 1) - 1| & = & |2\,(\tau_1\,\eta_t)(x_- + 1) - 1| \ \leq \ (3 + K^{-1})\,\theta \\ \eta_{t+s}(z) & = & \eta_t(z) \quad \text{for all } z \in (x_- + 1, x_+). \end{array}$$

This, together with the argument used to obtain (4.2), indicates that, each time pattern 1 occurs, the opinion at $x_- + 1$ is compatible with the opinion at $x_- + 2$ and close to the centrist opinion. Now, we say that pattern 2 occurs in some time interval whenever we see N times in a row the succession of pattern 1 and an interaction between vertices $x_- + 1$ and $x_- + 2$, and then pattern 1 once more. More generally, patterns are defined recursively with pattern k being N times in a row the succession of pattern $k-1$ and an interaction between vertex $x_- + k - 1$ and vertex $x_- + k$, and then pattern $k-1$ once more. Since K is almost surely finite, pattern K occurs infinitely often. In particular, defining recursively

$$\tau_1 \ = \ \sigma_{x_-+1}^N \quad \text{and} \quad \tau_k \ = \ \tau_{k-1} \circ (\sigma_{x_-+k} \circ \tau_{k-1})^N \quad \text{for } k = 2, 3, \ldots, K$$

and applying successively the argument used for (4.2) show that

$$|2\eta_{t+s}(x_- + k) - 1| \ = \ |2\,(\tau_K\,\eta_t)(x_- + k) - 1| \ \leq \ (3 + k \times K^{-1})\,\theta \ \leq \ 4\theta$$

for all $k = 1, 2, \ldots, K$ and some $t + s < \infty$. Therefore, for all $z_1, z_2 \in [x_-, x_+]$,

$$\begin{array}{rcl} 2 \times |\eta_{t+s}(z_1) - \eta_{t+s}(z_2)| & = & |(2\eta_{t+s}(z_1) - 1) - (2\eta_{t+s}(z_2) - 1)| \\ & \leq & 8\theta \ < \ 2\epsilon. \end{array} \tag{4.4}$$

In view of the evolution rules of the system, (4.4) remains true after time $t+s$. In particular, the result follows by taking $z_1 = x$ and $z_2 = x + 1$. □

5. Proof of Proposition 1.2

In this section, we rely on Proposition 3.4 to show the lack of convergence to a consensus whenever condition (1.4) holds and the confidence threshold is smaller than one half. The idea is to rely on the existence of infinitely many vertices with initial opinion arbitrary close to one of the two extremist opinions and whose neighbors' opinions are close to the centrist opinion at all times. When the confidence threshold $\epsilon < 1/2$, such vertices are never updated.

Lemma 5.1. *Assume that $\epsilon < 1/2$ and (1.4) holds. Then*

$$P\left(\text{card}\,\{x \in \mathbb{Z} : |\eta_t(x) - \eta_t(x+1)| > \epsilon \textit{ for all } t \geq 0\} = \infty\right) \ = \ 1.$$

Proof: First, we fix $\theta > 0$ such that $2\theta < 1 - 2\epsilon$, and recall that

$$\mathcal{F}_1 \ = \ \{x \in \mathbb{Z} : x \in \Omega_1 \text{ and } |2\eta_0(x) - 1| > 1 - \theta\}.$$

See Figure 5.3 for a picture. Since the set $\mathcal{F}_1$ is almost surely infinite according to (1.4), it suffices to show that, for all x in this set and all $t \geq 0$,

$$|\eta_t(x) - \eta_t(x+1)| \ > \ \epsilon \quad \text{and} \quad |\eta_t(x) - \eta_t(x-1)| \ > \ \epsilon. \tag{5.1}$$

Note that (5.1) follows from the claim: for all $x \in \mathcal{F}_1$ and all times $t \geq 0$,

$$\begin{array}{l} |\xi_t(x+1) - \xi_t(x)| \ > \ 1 - \theta \quad \text{and} \\ \qquad \zeta_t(\mathbb{Z}) \ \subset \ \text{West}\,(\zeta_0(x), \theta) \ \cup \ \text{East}\,(\zeta_0(x+1), \theta). \end{array} \tag{5.2}$$

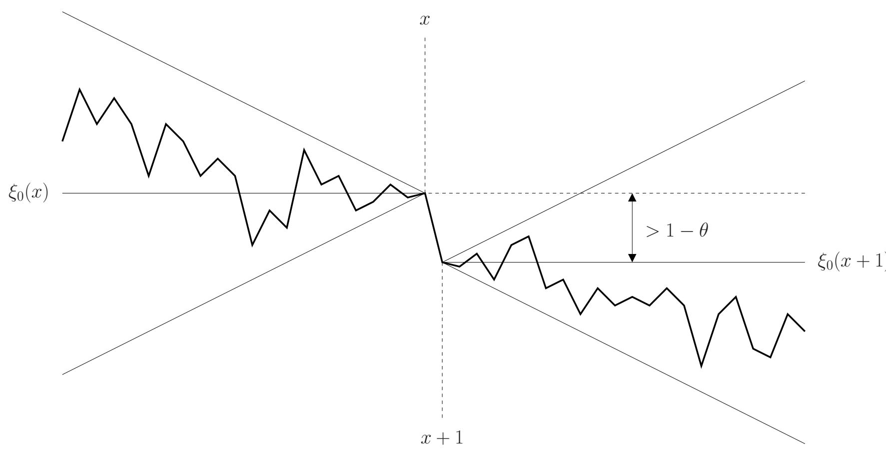

FIGURE 5.3. Picture of the event $x \in \mathcal{F}_1$

Indeed, (5.2) implies that $|2\eta_t(x \pm 1) - 1| < \theta$, so by the triangle inequality

$$\begin{array}{rcl} 2 \times |\eta_t(x) - \eta_t(x \pm 1)| & = & |(2\eta_t(x) - 1) - (2\eta_t(x \pm 1) - 1)| \\ & = & |(\xi_t(x+1) - \xi_t(x)) - (2\eta_t(x \pm 1) - 1)| \\ & \geq & ||\xi_t(x+1) - \xi_t(x)| - |2\eta_t(x \pm 1) - 1|| \\ & > & |1 - \theta - \theta| \ = \ 1 - 2\theta \ > \ 2\epsilon \end{array} \tag{5.3}$$

which gives (5.1). Now, let $s \in \Lambda(z_0)$ and assume that (5.2) holds at $t = s-$.

- If $z_0 < x$ then, according to Proposition 3.3, we have
$$\zeta_s((-\infty, x]) \subset \text{West}\,(\zeta_0(x), \theta)$$
therefore (5.2) holds at time $t = s$.
- If $z_0 = x$ then property (5.2) holds trivially at time $t = s$ since there is no update due to an incompatibility at time $s-$ between vertex $x - 1$ and vertex x as shown in (5.3).

By symmetry, (5.2) holds at time $t = s$ when $z_0 \geq x + 1$. Since by definition the property also holds initially for all vertices $x \in \mathcal{F}_1$, the proof is complete. $\square$

6. Proof of Proposition 1.3

To complete the proof of our main result, Theorem 1.4, it remains to establish Proposition 1.3. In particular, throughout this section, it is assumed that the initial opinions are independent and uniformly distributed over the interval $[0, 1]$. Recall from Section 2 that, to prove (1.3) and (1.4), it is convenient to think of the spatial structure $\mathbb{Z}$ as a doubly-infinite temporal structure, in which case the initial profile can be seen as the concatenation of the realizations of two symmetric random walks, one evolving forward in time and the other one evolving backwards in time. Both random walks start at zero at time 0 and have increments which are uniformly distributed over the interval $[-1, 1]$. The first step is to prove that the future of the initial profile starting at the current time x is contained in the corresponding right triangle with positive probability. This is done in the following lemma.

Lemma 6.1. *For all $x \in \mathbb{Z}$ and all $\theta > 0$,*

$$P\,(\zeta_0([x,\infty)) \subset \text{East}\,(\zeta_0(x),\theta)) \;=\; c(\theta) \;>\; 0.$$

Proof: Since $\{\xi_0(z) : z \geq x\}$ is a martingale with $|\xi_0(z+1) - \xi_0(z)| < c_z = 2$,

$$\begin{aligned}
&P\,(\zeta_0([x+v,\infty)) \not\subset \text{East}\,(\zeta_0(x),\theta)) \\
&\quad= P\,(\zeta_0(z) \notin \text{East}\,(\zeta_0(x),\theta) \text{ for some } z \geq x+v) \\
&\quad= P\,(\xi_0(z) \notin (\xi_0(x) - \theta\,(z-x), \xi_0(x) + \theta\,(z-x)) \text{ for some } z \geq x+v) \\
&\quad\leq 2 \times P\,(\xi_0(z) > \xi_0(x) + \theta\,(z-x) \text{ for some } z \geq x+v) \\
&\quad\leq 2 \times P\,(\xi_0(z) > \theta z \text{ for some } z \geq v) \\
&\quad\leq 2 \times \textstyle\sum_{z \geq v}\ \exp(-(\theta z)^2\,(2\,c_1^2 + \cdots + 2\,c_z^2)^{-1}) \\
&\quad= 2 \times \textstyle\sum_{z \geq v}\ \exp(-\theta^2 z/8)
\end{aligned}$$

where the last inequality follows by applying the Azuma-Hoeffding inequality. In particular, there exists a finite integer $v = v(\theta)$ fixed from now on such that

$$P\,(\zeta_0([x+v,\infty)) \not\subset \text{East}\,(\zeta_0(x),\theta)) \;\leq\; 1/2.$$

By conditioning on the first v increments being small, we deduce

$$\begin{aligned}
&P\,(\zeta_0([x,\infty)) \subset \text{East}\,(\zeta_0(x),\theta)) \;=\; P\,(\zeta_0(z) \in \text{East}\,(\zeta_0(x),\theta) \text{ for all } z \geq x) \\
&\quad\geq P\,(\zeta_0(z) \in \text{East}\,(\zeta_0(x),\theta) \text{ for all } z \geq x \mid \\
&\qquad\qquad\qquad 2\eta_0(x+z) - 1 \in (-\theta,\theta) \text{ for } z = 1,\ldots,v) \\
&\qquad\qquad \times\ P\,(2\eta_0(x+z) - 1 \in (-\theta,\theta) \text{ for } z = 1,\ldots,v) \\
&\quad\geq \theta^v\ P\,(\zeta_0(z) \in \text{East}\,(\zeta_0(x),\theta) \text{ for all } z \geq x+v) \;\geq\; \theta^v\,(1-1/2) \;>\; 0.
\end{aligned}$$

This completes the proof. $\square$

Similarly, the past of the initial profile starting at time x is contained in the corresponding left triangle with positive probability, as show in the following lemma.

Lemma 6.2. *For all $x \in \mathbb{Z}$ and all $\theta > 0$,*

$$P\,(\zeta_0((-\infty,x]) \subset \text{West}\,(\zeta_0(x),\theta)) \;=\; c(\theta) \;>\; 0.$$

Proof: Using obvious symmetry, we have

$$P\,(\zeta_0((-\infty,x]) \subset \text{West}\,(\zeta_0(x),\theta)) \;=\; P\,(\zeta_0([x,\infty)) \subset \text{East}\,(\zeta_0(x),\theta)).$$

The result then follows from Lemma 6.1. $\square$

From Lemmas 6.1–6.2 and the fact that the events

$$\begin{aligned}
\zeta_0((-\infty,x]) \;&\subset\; \text{West}\,(\zeta_0(x),\theta) \\
\zeta_0([x+j,\infty)) \;&\subset\; \text{East}\,(\zeta_0(x+j),\theta)
\end{aligned} \tag{6.1}$$

are independent, it follows that the set Ω_j is almost surely nonempty. To prove that this set is almost surely infinite, the main problem is that events in (6.1) for different values of x are not independent, and the basic idea of the proof is to show that a bad event at x has no effect on the occurrence of the event that $y \in \Omega_j$ provided both times are far enough from each other. This idea is expressed rigorously in terms of almost surely finite random times. For an illustration of the random times introduced in Lemmas 6.3–6.4 below, we refer the reader to Figure 6.4.

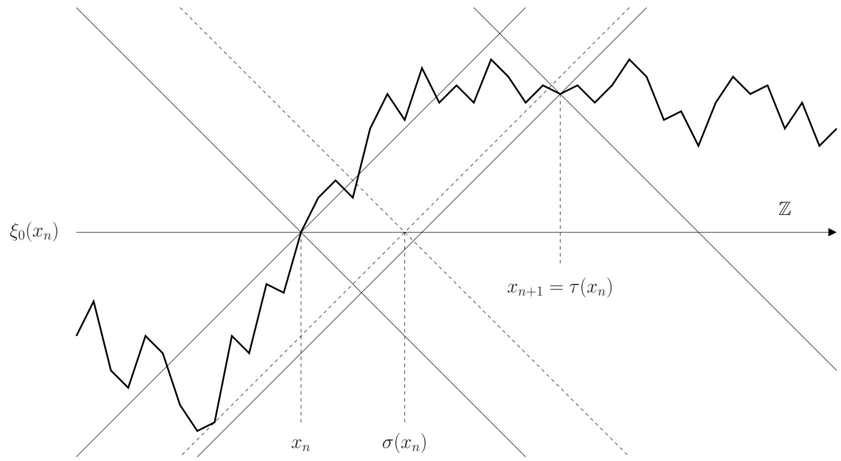

FIGURE 6.4. Random times $\sigma(x)$ and $\tau(x)$

Lemma 6.3. *For all $x \in \mathbb{Z}$ and all $\theta > 0$,*

$$\sigma(x) \ := \ \inf \{z \geq x : \zeta_0((-\infty, x]) \subset \text{West}\,((z, \xi_0(x)), \theta)\} \ < \ \infty \quad a.s.$$

Proof: Using the same argument as in the proof of Lemma 6.1, we obtain

$$\begin{aligned} P\,(\sigma(x) > x + v) \ &= \ P\,(\zeta_0((-\infty, x]) \not\subset \text{West}\,((x + v, \xi_0(x)), \theta)) \\ &\leq \ P\,(\zeta_0([x + v, \infty)) \not\subset \text{East}\,(\zeta_0(x), \theta)) \\ &\leq \ 2 \times \textstyle\sum_{z \geq v} \exp(-\theta^2 z/8) \\ &= \ 2 \times (1 - \exp(-\theta^2/8))^{-1} \ \exp(-\theta^2 v/8). \end{aligned}$$

In particular, $\sigma(x)$ is almost surely finite. $\square$

Lemma 6.4. *For all $x \in \mathbb{Z}$ and all $\theta > 0$,*

$$\tau(x) \ := \ \inf \{z \geq \sigma(x) : \zeta_0([z, \infty)) \subset \text{East}\,((\sigma(x), \xi_0(x)), \theta)\} \ < \ \infty \quad a.s.$$

Proof: First, we observe that for all $v \geq 0$

$$\begin{aligned} &\text{East}\,(\zeta_0(x), \theta/2) \cap \big\{(x + 2v, \infty) \times \mathbb{R}\big\} \\ &\qquad \subset \ \text{East}\,((x + v, \xi_0(x)), \theta) \cap \big\{(x + 2v, \infty) \times \mathbb{R}\big\}. \end{aligned}$$

In particular, using again the Azuma-Hoeffding inequality, we obtain

$$\begin{aligned} &P\,(\tau(x) > x + 2v \mid \sigma(x) \leq x + v) \\ &\quad \leq \ P\,(\zeta_0([x + 2v, \infty)) \not\subset \text{East}\,((x + v, \xi_0(x)), \theta)) \\ &\quad \leq \ P\,(\zeta_0([x + 2v, \infty)) \not\subset \text{East}\,(\zeta_0(x), \theta/2)) \\ &\quad \leq \ 2 \times \textstyle\sum_{z \geq 2v} \ \exp(-\theta^2 z/32) \\ &\quad = \ 2 \times (1 - \exp(-\theta^2/32))^{-1} \ \exp(-\theta^2 v/16). \end{aligned}$$

This, together with the proof of Lemma 6.3, implies that

$$\begin{aligned} P\,(\tau(x) > x + 2v) \ &\leq \ P\,(\tau(x) > x + 2v \text{ and } \sigma(x) \leq x + v) \ + \ P\,(\sigma(x) > x + v) \\ &\leq \ 2 \times (1 - \exp(-\theta^2/32))^{-1} \ \exp(-\theta^2 v/16) \\ &\qquad\qquad + 2 \times (1 - \exp(-\theta^2/8))^{-1} \ \exp(-\theta^2 v/8). \end{aligned}$$

This completes the proof. □

With Lemmas 6.1–6.4 in hands, we are now ready to prove that the set Ω_0 is infinite for almost all realizations of the initial profile. For simplicity, we give the details of the proof only for this set and will explain in Lemma 6.7 how to deduce the result for the set Ω_j with $j > 0$. First, we introduce the Bernoulli random variables

$$\begin{aligned} h(x,+) \ &= \ \mathbf{1}\,\{\zeta_0([x,+\infty)) \subset \text{East}\,(\zeta_0(x),\theta)\} \\ h(x,-) \ &= \ \mathbf{1}\,\{\zeta_0((-\infty,x]) \subset \text{West}\,(\zeta_0(x),\theta)\} \end{aligned}$$

and observe that we have the equivalence

$$x \ \in \ \Omega_0 \quad \text{if and only if} \quad h(x,+) = h(x,-) = 1.$$

Lemma 6.5. *For all $x \in \mathbb{Z}$, we have*

$$P\,(h(\tau(x),-) = 1 \mid x \notin \Omega_0) \ \geq \ c(\theta) \ > \ 0.$$

Proof: By definition of $\sigma(x)$ in Lemma 6.3,

$$\zeta_0((-\infty,x]) \subset \ \text{West}\,((\sigma(x),\xi_0(x)),\theta)$$

from which it follows that, for all $(z,r) \in \text{East}\,((\sigma(x),\xi_0(x)),\theta)$

$$\zeta_0((-\infty,x]) \subset \ \text{West}\,((\sigma(x),\xi_0(x)),\theta) \subset \ \text{West}\,((z,r),\theta).$$

Since in addition $(\tau(x),\xi_0(\tau(x))) \in \text{East}\,((\sigma(x),\xi_0(x)),\theta)$, we deduce that

$$\zeta_0((-\infty,x]) \subset \ \text{West}\,((\tau(x),\xi_0(\tau(x))),\theta).$$

Therefore, recalling the definition of $h(\tau(x),-)$, we conclude that

$$\begin{aligned} &P\,(h(\tau(x),-) = 1 \mid x \notin \Omega_0) \\ &\quad = \ P\,(\zeta_0((-\infty,\tau(x)]) \subset \ \text{West}\,((\tau(x),\xi_0(\tau(x))),\theta) \mid \\ &\qquad\qquad x \notin \Omega_0 \text{ and } \zeta_0((-\infty,x]) \subset \text{West}\,((\tau(x),\xi_0(\tau(x))),\theta)) \\ &\quad \geq \ \inf_{z\in\mathbb{Z}} \ P\,(\zeta_0((-\infty,z]) \subset \ \text{West}\,(\zeta_0(z),\theta)) \ = \ c(\theta) \ > \ 0 \end{aligned}$$

where the last equality follows from Lemma 6.2. □

Lemma 6.6. *For all $x \in \mathbb{Z}$, we have*

$$P\,(h(\tau(x),+) = 1 \mid h(\tau(x),-) = 1 \textit{ and } x \notin \Omega_0) \ \geq \ c(\theta) \ > \ 0.$$

Proof: Recalling the definition of $\tau(x)$, we have

$$\zeta_0([\tau(x),\infty)) \subset \ \text{East}\,((\sigma(x),\xi_0(x)),\theta).$$

Moreover, $h(\tau(x),-)$ only depends on the value of the initial opinions located strictly to the left of $\tau(x)$ whereas $h(\tau(x),+)$ only depends on the initial opinions located to the right of vertex $\tau(x)$. Also, the occurrence of $h(x,+) = 0$ is due to parts of the profile located between vertex x and vertex

$$\inf\,\{z \geq x : \zeta_0([z,\infty)) \subset \ \text{East}\,(\zeta_0(x),\theta)\}$$

which is smaller than $\tau(x)$. It follows that

$$\begin{aligned} & P\,(h(\tau(x),+) = 1 \mid h(\tau(x),-) = 1 \text{ and } x \notin \Omega_0) \\ & \quad = \; P\,(\zeta_0([\tau(x),\infty)) \subset \text{East}\,(\zeta_0(\tau(x)),\theta) \mid \\ & \qquad\qquad \zeta_0([\tau(x),\infty)) \subset \text{East}\,((\sigma(x),\xi_0(x)),\theta)) \\ & \quad \geq \; \inf_{z\in\mathbb{Z}} \; P\,(\zeta_0((-\infty,z]) \subset \text{East}\,(\zeta_0(z),\theta)) \; = \; c(\theta) \; > \; 0 \end{aligned}$$

where the last equality follows from Lemma 6.1. $\square$

Lemma 6.7. *For all $\theta > 0$, we have*

$$P\,(\text{card}\,(\Omega_0 \cap \mathbb{Z}_+) = \infty) \; = \; P\,(\text{card}\,(\Omega_0 \cap \mathbb{Z}_-) = \infty) \; = \; 1.$$

Proof: According to Lemmas 6.5 and 6.6, for all $x \in \mathbb{Z}$,

$$\begin{aligned} P\,(\tau(x) \in \Omega_0 \mid x \notin \Omega_0) \; & = \; P\,(h(\tau(x),-) = h(\tau(x),+) = 1 \mid x \notin \Omega_0) \\ & = \; P\,(h(\tau(x),+) = 1 \mid h(\tau(x),-) = 1 \text{ and } x \notin \Omega_0) \\ & \qquad \times \, P\,(h(\tau(x),-) = 1 \mid x \notin \Omega_0) \; \geq \; c(\theta)^2 \; > \; 0. \end{aligned} \tag{6.2}$$

To deduce that, with probability one, $\Omega_0 \cap \mathbb{Z}_+$ is infinite, we introduce the sequence starting at some $x_0 \in \mathbb{Z}$ and defined recursively by the relationship

$$x_{n+1} \; := \; \max\,(\tau(x_n), x_n + 1) \quad \text{for all } n \geq 0.$$

Note that $x_n < x_{n+1} < \infty$ for all $n \geq 0$ by Lemmas 6.3 and 6.4. Moreover, by the definition of the random times $\sigma(x)$ and $\tau(x)$ in these lemmas, we have

$$x_n \notin \Omega_0 \quad \text{if and only if} \quad \tau(x_n) \geq x_n + 1 \quad \text{if and only if} \quad x_{n+1} = \tau(x_n).$$

In particular, applying inequality (6.2) to vertex $x = x_n$, we have

$$P\,(x_{n+1} \in \Omega_0 \mid x_n \notin \Omega_0) \; = \; P\,(\tau(x_n) \in \Omega_0 \mid x_n \notin \Omega_0) \; \geq \; c(\theta)^2 \; > \; 0,$$

hence, with probability one,

$$\text{card}\,(\Omega_0 \cap \mathbb{Z}_+) \; \geq \; \text{card}\,\{n \geq 0 : x_n \in \Omega_0\} \; = \; \infty.$$

The second part of the lemma follows from obvious symmetry. $\square$

Lemma 6.8. *For all integers $j > 0$ and for all $\theta > 0$, we have*

$$P\,(\text{card}\,(\Omega_j \cap \mathbb{Z}_+) = \infty) \; = \; P\,(\text{card}\,(\Omega_j \cap \mathbb{Z}_-) = \infty) \; = \; 1.$$

Proof: This is similar to the proofs of Lemmas 6.3–6.7 using the random times

$$\begin{aligned} \sigma_j(x) \; & = \; \inf\,\{z \geq x + j : \zeta_0((-\infty,x]) \subset \text{West}\,((z,\xi_0(x)),\theta)\} \\ \tau_j(x) \; & = \; \inf\,\{z \geq \sigma_j(x) \text{ such that} \\ & \qquad \zeta_0([z,\infty)) \subset \text{East}\,((\sigma_j(x),\xi_0(x)),\theta) \cap \text{East}\,(\zeta_0(x+j),\theta)\} \end{aligned}$$

in place of $\sigma(x)$ and $\tau(x)$. Note that, $\sigma_0(x) = \sigma(x)$ and $\tau_0(x) = \tau(x)$ since

$$\text{East}\,((\sigma(x),\xi_0(x)),\theta) \, \cap \, \text{East}\,(\zeta_0(x),\theta) \; = \; \text{East}\,((\sigma(x),\xi_0(x)),\theta).$$

The proofs of Lemmas 6.3–6.4 easily extend to show that

$$P\,(\sigma_j(x) < \infty) \; = \; P\,(\tau_j(x) < \infty) \; = \; 1 \quad \text{for all } x \in \mathbb{Z}.$$

Then, defining recursively $x_{n+1} = \tau_j(x_n)$ starting from an arbitrary $x_0 \in \mathbb{Z}$ again induces an increasing sequence of integers since the random times are finite and

$$\tau_j(x) \; \geq \; \sigma_j(x) \; \geq \; x + j \; > \; x \quad \text{for all } j > 0.$$

Finally, the arguments in the proofs of Lemmas 6.5–6.7 imply that

$$P\,(x_{n+1} \in \Omega_j \mid x_n \notin \Omega_j) \;=\; P\,(\tau_j(x_n) \in \Omega_j \mid x_n \notin \Omega_j) \;\geq\; c(\theta)^2 \;>\; 0,$$

which shows that $\Omega_j \cap \mathbb{Z}_+$ is almost surely infinite. The complete result again follows relying on the obvious symmetry properties of the initial profile. □

Lemma 6.9. *For all $\theta > 0$, we have*

$$P\,(\mathrm{card}\,(\mathcal{F}_1 \cap \mathbb{Z}_+) = \infty) \;=\; P\,(\mathrm{card}\,(\mathcal{F}_1 \cap \mathbb{Z}_-) = \infty) \;=\; 1.$$

Proof: This follows from Lemma 6.8 with $j = 1$ since the event $x \in \Omega_1$ is independent of the slope of the line segment between vertices x and $x + 1$. □

Acknowledgment. The author was supported by NSF Grant DMS-10-05282 and also thanks an anonymous referee for her/his useful comments.

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