Developing Elementary Teachers' Knowledge about Functions and Rate of Change through Modeling

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ABSTRACT

The purpose of this paper is to describe the development of elementary school teachers' mathematical knowledge for teaching as they participated in a Modeling Instruction environment that placed heavy emphasis on improving their subject-matter knowledge as a basis for affecting the development of their pedagogical content knowledge. We investigate the development of the teachers' content knowledge and pedagogical content knowledge by considering the results of our iterative revisions with supporting documentation of the insights we made as we refined the course to explore teachers' knowledge. We conclude that Modeling Instruction helped the teachers conceive of mathematics as a tool to explain scientific phenomena and provided the teachers with opportunities to reflect upon the process of learning mathematics, which were both foundational to the development of their subject matter knowledge and their pedagogical content knowledge.

Introduction

The purpose of this paper is to describe the development of elementary school teachers' mathematical knowledge for teaching as they participated in a Modeling Instruction environment that placed heavy emphasis on improving their subject-matter knowledge. First, we outline our approach to affecting the development of teachers' mathematical knowledge for teaching through a focus on subject matter knowledge in the context of Modeling Instruction (Jackson, Dukerich, & Hestenes, 2008). Next, we discuss design research and explain why it is an appropriate paradigm for studying teachers' mathematical knowledge for teaching while attempting to improve it using Modeling Instruction. In these sections, we describe how we enacted cycles of design research in our development of the course. We explain the course design, dwelling on modeling as the pedagogical approach to the course, the mathematical content and its sequencing, and engineering thermodynamics as an ideal context for developing essential concepts of calculus using Modeling Instruction. In our results, we attend to the development of the teachers' knowledge by considering the results of our iterative revisions of the course. We close by considering the role of Modeling Instruction for supporting (a) teachers' development of conceptual understanding of subject matter knowledge, and (b) mathematical knowledge for teaching in a way that supports teachers' conceiving of mathematics as a basis for science, and vice versa.

Theoretical Grounding and Background

Mathematical Knowledge for Teaching

Shulman (1986) proposed a theoretical framework for describing teacher knowledge in response to the then prevalent characterization of knowledge for teaching as consisting of two distinct parts: 1) Content knowledge, generally isolated from knowledge of how that content develops and what instructional supports engender that development; and 2) general knowledge of teaching and learning theoretically independent from content. Subject-matter content knowledge requires an understanding of both the substantive (i.e., propositional) and the syntactic (i.e., procedural) structure*s* of a discipline (Schwab, 1978). Pedagogical content knowledge, according to this framework, refers to the ways of presenting, organizing, and negotiating the subject for the purpose of making it meaningful and comprehensible to others.

The research program of Ball, Hill, and colleagues have examined what is entailed in the work of teaching mathematics (Ball & Bass, 2000; Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008); developed instruments to measure teachers' MKT (Ball, Hill, & Bass, 2005; Hill, Ball, et al., 2008; Hill, Schilling, & Ball, 2004); investigated the effects of teachers' MKT on student achievement (Hill, Rowan, & Ball, 2005) and the quality of their instruction (Hill, Blunk, et al., 2008). Like Shulman, the basic division among types of knowledge splits knowledge of the subject matter per se, from pedagogical content knowledge, but it is critical to note that *both* types of knowledge are recruited in instruction.

The first type, *subject matter knowledge* has two main subdivisions: specialized knowledge of content and common content knowledge. Specialized knowledge of content is mathematical knowledge specific to the practice of teaching and extends beyond generic content knowledge. Common content knowledge can be thought of the knowledge of mathematics used in teaching that is common with how it is used in other professions such as science, economics or engineering (Hill, Ball, & Schilling, 2008). Ball and colleagues also have identified "Knowledge at the Mathematical Horizon," which is an understanding of mathematical concepts that learners will encounter later in their learning and an understanding of how what students are currently being taught contributes to students' understanding of concepts "on the mathematical horizon."

(Ball, Thames, & Phelps, 2008; Bass & Ball, 2009). Together, Common Content Knowledge Specialized Content Knowledge, and Knowledge at the Mathematical Horizon constitute the subject matter knowledge teachers bring to bear upon their work. The second domain of Ball, Hill, and Colleagues' characterization of mathematical knowledge for teaching, *pedagogical content knowledge*, is comprised of three empirically distinguished subdomains. Knowledge of content and students concerns teachers' understandings of how students develop, learn, and think about particular mathematical concepts. Knowledge of content and teaching includes knowledge of teaching strategies that merges understanding of a mathematical idea with understanding pedagogical principles for teaching that idea (Ball, Thames, & Phelps, 2008). Finally, knowledge of curriculum concerns an understanding of both frameworks and standards for organizing mathematical learning experiences.

Does Subject Matter Knowledge Impact Pedagogical Content Knowledge?

Despite great interest in mathematical content knowledge and its helpfulness in illuminating the decisions teachers must make when designing and conducting instruction, research results are divided on the degree to which subject matter knowledge impacts pedagogical content knowledge and associated teaching behaviors. Hill (2005), for example, demonstrated that experiences that enable teachers to learn mathematics content through effective pedagogical techniques as part of that learning, provides teachers with opportunities to acquire knowledge in such a way that it helps improve student outcomes. In fact, the subject matter knowledge teachers learn may not be the most important aspect of their pedagogical content knowledge when it comes to implementation. Rather, the aspect of content-focused instruction that seems to be transportable to the classroom is the kinds of activities teachers *do* (Garet et al., 2011; Garet et al., 2010). In essence, the content learned in class takes form as knowledge of tasks and actions.

Whether or not teachers implement such enacted knowledge depends largely on the extent to which they are able to assimilate the content into their existing knowledge, and appropriately adapt and integrate this knowledge. But it also depends on refinement and modification of their new pedagogical content knowledge as it is enacted in the classroom (Doerr & English, 2006). Otherwise, teachers' approach to instruction may be narrow, only accepting as valid those explanations and representations they experienced as opposed to recognizing the valid aspects of students' contributions (Son & Crespo, 2009). Towards this end, it is critical for teachers' learning experiences to emphasize the flexibility and diversity of the knowledge students bring to the modeling environment, and the diversity of interpretations that can result from engagement in open-ended tasks. We explore the affordances of modeling and Modeling Instruction for this problem.

The Affordances of Modeling Instruction

Models, Modeling and Modeling Instruction

Models are conceptual structures—mental (and often physical or inscriptional) representations of real things and real phenomena. They often are defined as a simplified version of an object or phenomena, descriptive, explanatory and predictive in their use, while also possessing limitations (Etkina, Warren, & Gentile, 2006). *Modeling* as it refers to subject matter knowledge, involves building, testing and applying conceptual models of natural phenomena and is a practice that is central to learning and doing science and mathematics.

Modeling Instruction (Jackson et al., 2008) is a research-based program for middle and high school science education reform. Jackson et al. (2008) detail it in the following way:

> More specifically, teachers learn to ground their teaching in a *well- defined pedagogical framework* rather than following rules of thumb; to organize course

> content around *scientific models* as coherent units of structured knowledge; to engage students collaboratively in *making and using models* to describe, explain, predict, design, and control physical phenomena; to involve students in *using computers as scientific tools* for collecting, organizing, analyzing, visualizing, and modeling real data; to *assess student understanding* in more meaningful ways and experiment with more authentic means of assessment; to continuously improve and update instruction with new software, curriculum materials, and insights from educational research; and to work collaboratively in action research teams to mutually improve their teaching practice. (Jackson et al, 2008, p. 11)

To support these aims, Modeling Instruction entails two specific cycles of work. The first—model development—involves demonstration and class discussion to establish a common understanding of a question to be asked about nature or the world. In the second stage—model deployment—students apply their model to new situations to refine and deepen their understanding of the mathematics necessary to model that situation (Jackson et al, 2008, p. 12). This is the underlying model we employ at all levels: Build and use models of important scientific content, utilizing important mathematical and representational tools.

A key aspect of Modeling Instruction are model eliciting activities and thought revealing activities (Lesh, Hoover, Hole, Kelly, & Post, 2000; Schorr & Koellner-Clark, 2003). These tasks are a means to engage learners in purposefully developing, testing, modifying, and deploying a model to make sense of a particular context. Such conceptual systems move from those useful to describe or explain a particular phenomenon to more general models that can one can use to make sense of a class of phenomena. Therefore, instructionally, a single modeling eliciting activity is rarely adequate for students to develop a generalized model useful across a range of mathematically similar but contextually divergent phenomena (Doerr & English, 2003). The principles for model eliciting activities described by Lesh served as the framework for the design of MEAs in this study. The importance of incorporating different contexts into learning environments must be underscored here. These contexts, in a modeling environment, provide an "alternative discourse to the formal mathematical one," (Arleback, Doerr, & O'Neil, 2013, p. 334), enabling students to structure and express their growing understanding. It is only after seeing the underlying structure of a variety of contexts that the generalized model becomes a tool, deployable at will for any situation with the same underlying quantitative structure.

The Problem and Resulting Research Questions

A variety of studies on student learning have shown that models that reflect context are critical to students' development of subject matter knowledge (Banerjee, 2010; Greca & Moreira, 2002; Jackson et al., 2008; Treagust, Chittleborough, & Mamiala, 2002). This work makes clear that one's capacity to create models of scientific phenomena, and to test those models is dependent on her/his development of mathematical ways of thinking about the phenomena, including the ability to make sense of patterns in data. However, teachers are not deeply exposed to mathematical modeling of scientific phenomena during their undergraduate careers. It has been our experience that with typically one course in mathematics at the collegiate level, and only one non-calculus-based science course, middle school teachers trained as generalist elementary education majors are often not equipped to handle the traditional mathematics curriculum.

In response to this problem, there were two major questions driving this study:

1) In what ways did teachers' mathematical knowledge for teaching about function and rate of change develop as they engaged in a course focused on modeling?

2) What was the role of modeling instruction in the development of teachers' subject matter knowledge and other aspects of their mathematical content knowledge?

In what follows, we conjecture Modeling Instruction as a fruitful means to develop teachers' subject matter knowledge and pedagogical content knowledge specifically in mathematics and science and describe ways in which we tested this hypothesis using design research.

Method: The Development of a Conjecture and Course in Which To Evaluate It

Development of a Conjecture

We conjectured that a focus on Modeling Instruction would provide at least four innovations for teachers' mathematical content knowledge. First, they could develop models to think *about* content to explain and predict phenomena (specialized content knowledge). Second, they could come to see mathematics and science as reflexive endeavors, where a model includes a scientific phenomenon and a mathematical structure to explain that phenomenon (common content knowledge). Third, teachers could develop an understanding of how students, including themselves and their classmates learn mathematics through modeling (knowledge of content and students). Fourth, they could develop an understanding of modeling instruction as a means for task design and the management of classroom discourse (knowledge of content and teaching) (Hill, Ball, & Schilling, 2008). We evaluated our conjecture in a course designed for in-service elementary teachers as they worked toward their middle grades certification. The evaluation of this conjecture involved targeting four specific types of knowledge that we describe below.

Specialized Content Knowledge. Ärlebäck et al. (2013) showed how sequences of carefully designed Model Eliciting Activities (MEAs) can contribute to the development of a conceptual system about functions and rate of change that adult learners can use to make sense of phenomena in engineering physics, including nonlinear functions. For example, they found that focus on the constancy of successive ratios for values of a dependent variable in exponential functions was a helpful heuristic for developing understanding of rate of change. Additionally, in prior research, we have found that constant successive differences is useful for developing rate of change for linear functions. By modeling rate of change across contexts that emphasized exponential functions, polynomials and power functions, we hoped to create a common underlying model of the derivative as a function that, for all values of an independent variable, and given some original bivariate function, generates an instantaneous rate of change of the dependent variable.

Common Content Knowledge. Like Ärlebäck et al. (2013), we conjectured that the sequencing of MEAs around important engineering physics contexts would provide important means for teachers to develop specialized content knowledge focusing on the mathematical models that engineers use to describe, understand, and predict these phenomena. We chose engineering thermodynamics as the primary context for modeling in that it is manifest in many specific situations ranging from simple particle collisions at the atomic level to atmospheric pressure at the earth-systems level. We built MEAs around extended laboratory activities that enabled students to physically manipulate the phenomena so as to provide additional understanding regarding just what measures changed, and why they changed in the manner they did.

Knowledge of Content and Students. We conjectured that Modeling Instruction could be an effective means by which teachers develop knowledge of their students learning about functions and rate of change since the teachers in our study would be participating in Modeling Instruction as students, and thereby would experience the conceptual challenges and accompanying affective responses their students were likely to experience while learning mathematics. That is, through Modeling Instruction teachers would have the opportunity to become intimate with the conceptual and affective obstacles students so often encounter upon trying to learn a new idea for the first time. In the process, teachers may develop a better understanding of how learners are likely to engage in instruction, what challenges they are likely to have, and what meanings they are likely to construct.

Knowledge of Content and Teaching. Knowledge of content and teaching involves knowing what activities, learning experiences, pedagogical tactics, and social and sociomathematical norms will provide students with the opportunity to understand a concept in the way the teacher intends. Modeling Instruction allows teachers to experience how mathematical modeling can be an effective means by which students can learn intended mathematics. We conjectured that if the teachers in our study were able to understand and appreciate how mathematizing scientific concepts facilitates a deeper understanding of both mathematics and science, then the teachers would be equipped to construct learning materials that purposefully engage students in modeling scientific phenomena mathematically with the objective of enhancing students' mathematics learning. In essence, teachers experience with Modeling Instruction may be instrumental in how teachers understand students' mathematical reasoning and engagement.

The Design of the Course

We designed a 15-week course focused on Pre-Calculus and Calculus content. Specifically, we emphasized the concepts of rate of change and function through a modeling approach. We created the initial framework for the course by considering the subject matter knowledge that middle school teachers need to support their students in understanding ideas like proportionality, function, and rate of change. We used previous research about students' mathematical thinking relative to the most important content knowledge for future middle school teachers. We used research about quantities (Smith III & Thompson, 2008; Thompson, 2011), variation (Clement, 1989; Silverman, 2006; Weber & Carlson, 2010), covariation (Akkoc & Tall, 2003; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Larsen, & Jacobs, 2001; Tall, McGowen, & DeMarios, 2002), and rate (Johnson, 2012; Oehrtman, Carlson, & Thompson, 2008; Tall, 1996; Thompson, 1994; Thompson & Thompson, 1994) to create a scheme of meanings that we intended the teachers develop. We constructed particular modeling tasks that would support development of those meanings for the teachers. As the teachers developed notions of mathematical ideas like function and rate of change as the basis for explaining scientific phenomena like energy transfer, we also focused on supporting them in developing pedagogical content knowledge relative to these concepts.

Motivation for Thermodynamics as a Context

While thermodynamics provides a natural entry point into the issues above, we also hypothesized that most of the teachers had everyday experience with heat and temperature, energy, pressure and work, whether informal or formal. Thermodynamics is an area of science where several key variables, such as entropy, cannot be measured directly; one can only measure them by examining related variables and determining relationships between them. We emphasized convection and conductance as key ideas in thermodynamics because their accessibility provides teachers with the opportunity to talk deeply about rate of change. Moreover, the mathematics is absolutely critical for understanding the science—thermodynamics cannot be understood at even a basic level without it. In general, we wanted to emphasize the deep connections between mathematics and science, and help the teachers understand how one can better understand science by representing scientific phenomena mathematically.

Participants

The focus teachers in this study were elementary certified teachers and were enrolled in the modeling course that occurred at a university in the Southwestern United States. The teachers enrolled in the course as part of their master's degree coursework in STEM modeling. Because we were only able to work with the teachers that enrolled, the cohorts should not be considered *generalizable* to STEM teachers in general, though we think the teachers' experiences we describe in the results likely could apply to most teachers in a similar setting. The course lasted for fifteen weeks and there were two iterations (cohorts) corresponding to the year in which the teachers enrolled in the master's degree program and the instructors were the same for both iterations. There was a third cohort of participants, but they were not part of the systematic study as the first two were and are not reported here. Of the teachers, four had a calculus course early in their collegiate experience, but the rest had no mathematics beyond basic College Mathematics and Mathematics Methods for Elementary Teachers. All but three were female, and all were teaching elementary grades subjects at the time they participated in the course.

Role of Model Eliciting Activities

For the reported course, we organized the content into modules based on our characterization of the knowledge important for mathematics teaching in the middle grades:

1. Proportional Reasoning: Direct, Inverse, and Joint Variation
2. Average Rate of Change: $\Delta y/\Delta x$, Analyzing Functions Qualitatively, Rational Expressions
3. Families of Functions: Linear, Polynomial, Exponential/Logarithmic/Power
4. Derivatives as local average rate of change
5. Derivatives at a point; Derivatives as Functions; Limits

In each class session, we presented the teachers with a new thermodynamics problem to model. Following class, they were required to complete a formal lab write-up of their model. Their analyses addressed the following points:

1. A description of the relationship they attempted to model
2. A description of the methods they used to model the relationship
3. The problem solution presented in a general form (e.g., a rule, procedure, or symbolization, which was often a combination of formulae, graphs, and data tables).
4. An analysis of why the problem solution was correct.

Problems emphasized convection rates, conductance/insulative properties of materials and radiation rates. Teachers were actively engaged in collecting, organizing, and analyzing data to construct relationships between quantities' measures. The teachers collaborated in planning and conducting experiments, collecting and analyzing data to answer or clarify the question. Teachers were required to present and justify their conclusions in oral and/or written form, including a formulation of their own models for the phenomena in question and an evaluation of the set of models by comparison with data. These activities allowed us to both generate new hypotheses about the developing content knowledge of the teachers, and evaluate previous hypotheses (see examples in Appendix A and B).

The Role of Design Research

Cyclic Design of Course

We treated the design of the course as a cyclic process. It was cyclic in two ways. First, prior to each class session we prepared a set of hypotheses about how the teachers' subject matter knowledge and pedagogical content knowledge were developing given our observations and reflections from the previous class session. We then designed the activities for that day, sometimes keeping existing plans and sometimes modifying them to evaluate these hypotheses. Thus, on a daily basis we were engaged in modifying the course activities and discussions in a way that would allow us to gain insight into our research questions (see Foci of Design Cycles section). Second, at the beginning and end of the full course, we prepared and then evaluated hypotheses about the teachers' subject matter knowledge and pedagogical content knowledge. In this phase we considered how the course as whole allowed us to characterize teachers' knowledge, and the role of Modeling Instruction in its development.

Design Research as an Analytical Lens

We evaluated the use of Modeling Instruction and its usefulness for the teachers' development of mathematical knowledge for teaching using a design research approach. Lamberg and Middleton (2009) described how design theory can provide a rigorous paradigm for disciplined inquiry in learning and instruction, and a pragmatic framework for defensible change in curriculum and instruction. Their description of design theory builds on the pioneering work of scholars at the Freudenthal Institute, who characterized "developmental research" (Gravemeijer, 1994; Gravemeijer & Cobb, 2006) and Wittmann (1995), who expanded on notions of "design science". Briefly, Lamberg and Middleton's approach outlines seven "phases" through which a research program progresses, beginning with more grounded methods, where the researcher attempts to understand an under-researched area of inquiry, consolidating evidence about learning and practice, until a local theory of change can be generated that seems to explain and predict positive movement along a trajectory of change. Through iterative testing and revision, tasks are made more effective, while at the same time, the theory is refined, refuted, and made more useful and explanatory (Middleton, Gorard, Taylor, & Bannan-Ritland, 2008). Design research thus has two primary outcomes: A theory of change that describes how learning progresses across a defined set of topics, and a mechanism for at least partly driving that change—curriculum materials or other supports that have been refined over time to be optimally (or at least *reasonably*) effective. In our case, we used design theory to test a hypothesis: that the teachers' mathematical knowledge for teaching relative to function and rate of change would develop, and that Modeling Instruction could, at least in part, account for that development.

Foci of Design Cycles

The research team consisted of four mathematics educators with engineering, science, and mathematics backgrounds, and an expert in teacher professional development. The research/teaching team met each week following the class sessions. We reviewed student work, planned subsequent sessions, and revised tasks and assignments based on the discourse engendered in the daily sessions through the design theory lens. Sources of data included their homework, modeling problems, and their posts to the class discussion group. Our review sessions had two major foci related to our research questions:

1) We made inferences about the difficulties teachers had in their conceptual understanding of function and rate of change, as well as their pedagogical content knowledge related to these ideas. During the course, we made our inferences from qualitative data that consisted of teachers' responses to applied activities and their reflections on those activities as a way to create conceptual learning opportunities for students. In response to our inferences, we developed new tasks to help them bridge their difficulties both about subject matter knowledge and the use of that knowledge in the development of pedagogical content knowledge.
2) We made inferences about the role of Modeling Instruction in the development or lack thereof in the teachers' knowledge about function and rate of change. Specifically, we

considered the teachers' modeling activities and homework in conjunction with their verbal reflections to understand in what ways a focus on the development of models supported the development of teachers' knowledge.

As a result of these sessions, we were able to create a record of how we developed our inferences and in what ways our insights into the research questions may have changed over time.

Data Source 1: Structure, Role and Coding of Teacher Interviews

Interviews with both individual and small groups of teachers were a key aspect of our gathering data to inform hypotheses during our review sessions. We selected a group of four (first cohort) and three (second cohort) teachers who were willing and had availability to participate in subsequent interviews. Each group contained teachers who were strong, average and weak mathematically based on an initial pre-test of content knowledge at the onset of the term (see PCA assessment below). The groups of teachers participated in interviews each week. These interviews occurred either during a break in class activity or immediately after a class session each week. The purpose of each interview was to gain insight into teachers' views of their own progress in understanding the mathematical content in the course as well as how they were translating their knowledge of the mathematics to their thinking about teaching. The interviewer created a set of notes identifying important aspects of the conversation as it unfolded and a summary of the issues that arose in a summary immediately after the completion of the interview.

Once we collected the data, the coding process centered on creating and then simplifying categories about teachers' knowledge of mathematical content and their pedagogical content knowledge. This coding occurred via a combination of open and axial coding and constant comparative analysis to evaluate the use of those codes (Strauss & Corbin, 1998). The coding of video data served two purposes. First, it was a way to create objective counts of various coded instances. Second, it allowed for identification of all instances falling under a particular code, which supported further analysis of behaviors marked with a specific code. The analysis of the data led to three things: a log of the hypotheses developed about teachers' knowledge in the interviews, a tracking of the researcher's inferences made during the interviews, and a set of codes over the same time span. This data corpus provided a means to describe patterns by tracking the use of particular codes during the interviews and analyses, which became a key component of the design decisions for the course.

Data Source 2: Pre-Calculus Assessment

The Pre-Calculus Assessment (PCA) is a 25-item multiple-choice instrument addressing students' understanding of the fundamental concepts of pre-calculus, in particular the notions of function and rate of change. The assessment addresses students' ability to view a function as a process, to engage in covariational reasoning, to engage in proportional reasoning and to think about basic notions of rate of change (Carlson, Oehrtman, & Engelke, 2010) which were the foci of the learning goals we developed in the construction of the course. In terms of content, the PCA assesses students' understanding of linear, exponential, rational, and other types of nonlinear functions. These items directly assess the concepts and skills we addressed in the modeling instruction course, but were not set within the context of thermodynamics. Teachers in both cohorts took the Pre-Calculus Assessment in the first week of their course, at the end of the course. Thus, it served as a means to complement our qualitative insights gained during the iterative design cycles.

Data Source 3: Classroom Observations

We selected a subset of seven teachers in Cohort 1 for classroom observations in the spring semester following their engagement in the modeling mathematics course. We conducted three observations with each teacher to establish a sense of teachers' typical classroom practice. We utilized the Reformed Teaching Observation Protocol (RTOP) (Piburn et al., 2000; Sawada et al., 2002) and the Modeling Implementation Rubric (MIR) (Megowan-Romanowicz, 2013) to code teachers' practice on the degree to which they embodied reform-oriented practices (RTOP) and elements of modeling instruction (MIR).

Reformed Teaching Observation Protocol

The RTOP is a twenty-five-item classroom observation protocol that was developed to assess the degree to which classroom practices embody each of five factors related to pedagogy conforming to "reform-oriented" practice. The authors of RTOP designed it to be utilized during the teaching of a regular lesson. Each item is rated on a five-point scale based on the degree to which the indicated behavior (e.g., "In this lesson, student exploration preceded formal presentation") is observed during the class period. The items are organized into the following five subscales, consisting of five items each:

1. Lesson Design and Implementation
2. Content: Propositional Knowledge
3. Content: Procedural Knowledge
4. Classroom Culture: Communicative Interactions
5. Classroom Culture: Student/Teacher Relationships.

Five subscale scores (each can range from 0 to 20) and a total score (can range from 0 to 100) are calculated by summing the applicable ratings[1].

Modeling Implementation Rubric

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[1] The complete instrument is available at: http://physicsed.buffalostate.edu/AZTEC/RTOP/RTOP_full/using_RTOP_2.html.

The Modeling Implementation Rubric (MIR) is a twenty-three-item classroom observation protocol developed to assess the design and management of a learning environment and activities that reflect specifically the objectives of the modeling method of instruction. Specifically, it targets teachers' ability to organize course content around scientific models as coherent units of structured knowledge; to engage students collaboratively in making and using models to describe, explain, predict, design, and control physical phenomena; to involve students in using computers as scientific tools for collecting, organizing, analyzing, visualizing, and modeling real data; and to assess student understanding in more meaningful ways and experiment with more authentic means of assessment. Each rubric item is rated on a five-point scale of 0 to 4, based on the degree to which the indicated behavior (e.g., "The teacher encourages students to lead classroom discourse") is observed during a class period. The items are organized into the following four subscales which corresponding to Lesh's work on modeling (Carlson, Larsen, & Lesh, 2003; Lesh et al., 2000; Lesh & Yoon, 2007).

1. Elements – essential activities of modeling (six items)
2. Operations – teaching practices that facilitate modeling (six items)
3. Relations – interactions among individuals in a modeling classroom (six items)
4. Rules – ways in which modeling instruction is done (five items).

Elements consists of items like "students linking procedural knowledge to the conceptual model, reasoning from a physical interpretation of data or information." *Operations* consists of items such as "teacher's role in managing classroom interactions is to frame and reframe classroom discourse so that students are working in a frame that fosters the development of conceptual coherence." *Relations* are comprised of items focused on individuals' interactions like "Students listen carefully to each other, build on each others' thoughts, and offer positive feedback to collaborators when appropriate." Finally, *rules* consist of items like "Conceptual models are represented in a variety of ways." Four subscale scores (scales 1 to 3 can range from 0 to 24; scale 4 can range from 0 to 20) and a total score (can range from 0 to 92) are calculated by adding the applicable ratings.

The validity and reliability of both the MIR and the RTOP were established with systematic training of the observers. The training followed the principles of the RTOP training manual, with specific adaptations made for the MIR. In both cases, observers were introduced to explanation of items' intent, provided brief examples, and were then asked to rate videos of middle school STEM classes using the RTOP and the MIR. Ratings were compared and the number of items in agreement between the coders was recorded. Discussion of differences in ratings focused on behaviors exhibited in the videos, and asked observers to explain their reasons for rating them in the manner they did. Through discussion and clarification of item intent, observers achieved 95% inter-rater reliability for both the RTOP and the MIR.

Coordination of Data Sources and Justification of Mixed Methodology

As we have described above, this study involved multiple data sources and evaluation of them at various points in time. The purpose of using these multiple sources of data was to triangulate our conclusions about the research questions, which were:

1) In what ways did teachers' mathematical knowledge for teaching about function and rate of change develop as they engaged in a course focused on modeling?
2) What was the role of modeling instruction in the development of teachers' subject matter knowledge and other aspects of their mathematical content knowledge?

The first phase of our analysis came from our weekly meetings, evaluations of student work, and discussions with teachers in both cohorts. Our analysis of the week-to-week activities, our impressions of the teachers' learning and progress, and the adjustments made to the course based on these sources of data were qualitative in nature and allowed us to generate initial hypotheses about teachers' mathematical knowledge and the role of modeling instruction in its development. The second phase of our analysis involved quantitative measures of teachers' mathematical knowledge (PCA) and their teaching (RTOP and MIR). The purpose of the second phase was to triangulate and evaluate our conclusions from the first phase of the analysis. The second phase was not hypothesis generating. Instead, we used the quantitative measures to evaluate the viability of our conclusions from the qualitative analysis. Thus, while the RTOP and MIR do not necessarily provide distinct information for analysis, they were a key part of triangulation. In the results section, we first describe the qualitative analyses and then consider in what ways the quantitative measures supported (or did not support) the conclusions from the qualitative phase.

Results

Our presentation of results provides insight into the phases of course design, our inferences about teachers' subject matter knowledge of function and rate of change, and our inferences about teachers' pedagogical content knowledge relative to these fundamental concepts. In the presentation of our results, we also communicate our evolving hypotheses about the role of Modeling Instruction in the development of teachers' knowledge. We have identified two teachers from each of the groups to illustrate our major findings. We argue that they are "representative" of the teachers in each cohort based on a) their pre and post PCA scores, b) their MIR evaluations and c) the instructors' evaluations of their weekly written work. Moreover, our evaluations of the teachers and their thinking suggested that their responses tended to be indicative of the other students in the course and the students in the interviews. Thus, we find the teachers to be representative of the other teachers in the study based on both our qualitative and quantitative assessments of their content knowledge, their mathematical knowledge for teaching, and their week-to-week work in the course. At the same time, it is important to note they cannot perfectly represent every teacher. However, we believe our conversations with them reflect the most important themes of the study, which is why we focus on them in the results.

Iteration 1: Initial Fits and Starts

In the first iteration of the course, we set out develop a mathematically rich curriculum focused on thermodynamics, and eventually we were able to talk about difficult mathematical concepts to explain and model scientific phenomena. However, the level of resistance and frustration the teachers exhibited in the first three weeks of the course was palpable. As we worked to engage the teachers in modeling activities to motivate that mathematics behind the models, it became clear that the basic mathematics we had assumed as background knowledge was not present. We determined this from reading teachers' homework reflections and early course evaluations. They noted that they were highly frustrated or confused about concepts like algebraic field properties and representation of proportional quantities using fractions. For instance, when we engaged the teachers in an activity about conduction designed to motivate a focus on rate of change, they were confused as to why they needed to determine a change in two different quantities in order to measure slope. The discussion between Brad and Shana in Excerpt 1 is based on this issue and conduction activity.

Excerpt 1 – Brad and Shana discuss their initial confusion and frustration (I = Instructor, S = Shana, B = Brad).

I: I noticed on that last activity we did about conduction that you became really frustrated when talked about rate of change. Could you clue me in as to why?

S: It is not too hard. I simply do not get the mathematics at all and it is driving me insane. How am I supposed to do these advanced things, like rate of change, when I am struggling to do the basics? Also, how do you expect me to think about how my students might do this stuff when I am super confused about it myself?

B: We do not know enough of the mathematics to use it in a way for our students. For instance, I got caught up on even thinking about conducting and a graph to represent it, much less rate of change. Then when the assignments are asking us to think about the implications for student learning, I think others see it as a hopeless endeavor.

Given our data and feedback from the teachers, an example of which appears in in Excerpt 1, we concluded there were two major obstacles to the teachers' developing the understandings we hoped to support. First, a significant barrier we faced was the confluence of factors associated with teachers' knowledge and how these factors could be made compatible and even complementary. Second, there was a real disconnect between the mathematics and science for the teachers. For instance, many of the teachers noted that it was difficult to see why measuring convection and conduction necessitated describing ideas like rate of change and function. Thus, during our initial analysis, we believed that our attempt to support the teachers in achieving these understandings was actually counterproductive and required a step back to more basic mathematical content. We anticipated that this step back would allow the teachers to better understand how the scientific modeling motivated the mathematics we intended to discuss.

Iteration 1: Teachers' Subject Matter Knowledge

We hypothesized that teachers' subject matter knowledge of variable, function, field properties and their relation to the manipulative skills of algebra, basic concepts of exponents, factors and multiples, and proportional reasoning was generally poor and uneven across the first cohort. For instance, we found that many of the teachers had difficulty solving simple algebraic problems, working with rules of exponents and thinking about inverse and direct proportions. Excerpt 2 highlights a conversation between Shana and Brad that demonstrates their underdeveloped subject matter knowledge as they reflected on the Cooling Curves activity which required the students to use notions of function and the rates of change of those functions.

Excerpt 2 – Brad and Shana discuss their use of functions and graphs.

I: So we have focused pretty extensively on representing on the Cooling Curves activity (Appendix A). In what ways did you see function used in that activity?

S: I saw a function used when we created a graph. A function is similar to have a picture of a situation so you can understand what is going on. For example, you could have temperature and time on a graph. I was not totally clear on how this modeling approach helps here. I am just thinking I am wrong all the time.

I: Brad, what do you think?

B: I go along with Shana here, I get really frustrated because I do not know what I know. I think I see a function as relating two things that vary. Like, in the Cooling Curves activity (Appendix A). I still do not see why the value of *e* is important to this function, and why we don't just use the graph. I guess, like Shana, I just don't think I am right.

As a result of conversations, such as those in Excerpt 2, we conjectured that nearly all of the teachers possessed poor mathematical self-efficacy when engaging in mathematics[2]. As a result, teachers' confidence in their conjectures and insight appeared to be tentative; their ability to

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[2] Our discussion about self-efficacy resulted from our perceptions and inferences, not a particular instrument.

judge their learning and understanding (meta-cognition) was underdeveloped. For instance, Brad and Shana discussed at length their negative feelings toward doing mathematics that resulted from their initial impressions of mathematics as memorized calculations. Thus, it seemed they had never had an opportunity to think about mathematics as a problem solving tool. At the same time, we noted that the teachers were still not focusing on using mathematics to explain the science they were experiencing (energy transfer) whereas we had originally hypothesized modeling would allow them to make this connection.

We believed the reason the teachers did not perceive a linkage between mathematics and modeling energy transfer was that they did not have the basic mathematics we had originally assumed, and thus were not able to quantify and explain a system with those basic mathematical understandings in mind. Thus, we anticipated a shift in the level of content would be crucial for allowing Modeling Instruction to achieve its intended effect. We took time to focus on the most basic ideas of quantities, algebraic manipulative skills and their conceptual basis in the field properties, recognizing the form of familiar functions (linear, polynomial, exponential). Once we helped the teachers construct a foundation of basic mathematical concepts, we used rate of change in the context of modeling problems as an integrative concept to help students look at different regions within the same curve, and the same regions across different curves, to examine the behavior of these functions, and their similarities and differences. Our analyses of conversations like Excerpt 3 led us to believe that once we met the teachers where they were mathematically, they were able to conceive of the usefulness of models as a way to explain the science with mathematical ideas, even when the mathematical understandings they possessed remained basic.

Excerpt 3 – Brad and Shana discuss their perception of rate of change.

I: I'd like to revisit a discussion we had a few weeks ago about the Cooling Curves (Appendix A) activity and thermal conductivity lab (Appendix B) when you were quite frustrated. How do you look at rate of change now?

S: I think I had no idea what was happening mathematically so I never could understand your talking about using rate of change to represent what was happening. Now I am better able to think about energy as a rate, kind of like a function in and of itself.

I: Brad, what do you think?

B: I think we spoke so much about models that I did not understand what was meant by it. Now I think about my model of heat transfer as dependent on rate of change and functions, those two ideas help explain a way to think about heat transfer. Like we have talked about, a way to link the science and mathematics.

Brad and Shana's conversation in Excerpt 3, among others, led us to believe two things: (1) the scope of mathematics we hoped to cover in the course would need to be curtailed to make the modeling activities meaningful to the teachers, and (2) curtailing the scope of the mathematics while focusing on modeling would allow the teachers to better understand the linkages between the mathematical ideas and the scientific models they were developing for energy transfer.

Iteration 1: Teachers' Development of Pedagogical Content Knowledge

Because of the resistance and frustration we encountered from the teachers, we decided to target the development of pedagogical content knowledge after the teachers became more comfortable with the content, rather than trying to develop subject matter knowledge and pedagogical content knowledge in rapid succession. Thus, after five weeks of focus on content alone, we began to ask the teachers to think about how their students might come to develop knowledge about functions and rate of change. Brad and Shana's discussion about the Cooling

Curves activity (Appendix A) suggests that while they had made shifts in their subject matter knowledge, they had difficulty reflecting on those experiences as a basis for thinking about instruction of their own students.

Excerpt 4 – Brad and Shana discuss their students' use of functions.

I: So we just completed the Cooling Curves activity (Appendix A). Our discussion was about what understandings were crucial for your students to have about function and rate and how they might come to develop those.

S: As I said in the big group conversation I cannot imagine what I want students to know or how they might know it since I am unsure about my own content expertise. Like a function and rate of change, they seem far outside my ability level right now.

I: Brad, how might you respond in light of Shana's words?

B: I get it. I know I want my students to understand that a function is a relationship between things and a rate measures changes in the function. But like in our discussion, I am unsure why we need to connect the mathematics and science here to do these things. I think they could understand the science without it.

Excerpt 4, and other conversations based on the modeling activities, led us to develop two hypotheses regarding teachers' developing pedagogical content knowledge: First, the teachers were not reflecting on their own learning experiences in a way that allowed them to understand how these experiences contributed to the development of their subject matter knowledge. Second, even when the teachers recognized the usefulness of mathematics for explaining science, they were unsure it would be valuable for their students' learning. Indeed, there initially was palpable resistance from the teachers about the usefulness of modeling as a principle for their classroom instruction. A number of the teachers noted that modeling stood apart from the typical memorization approach used in mathematics and that the "new" approach made them nervous about how to work with students.

In response, we structured the last three weeks of the course to emphasize activities that supported teachers in (a) thinking about the difficulties they had experienced in learning the mathematics as a means to structure their students' learning in modeling activities and (b) explicitly attending to the relationship between mathematics and science in their modeling problems and labs. During our conversations with the teachers and in their final reflections on the course, it became clear that they began to describe science and mathematics as reflexive endeavors in which one (i.e. mathematics) area allowed for explanation of another (i.e. energy transfer). For example, consider Brad and Shana's discussion in Excerpt 5 which took place after their work with the Cooling Curves and Conductivity Lab. Their discussion revealed to us that the connection between mathematics and science was something they would try to help their students come to understand.

Excerpt 5 – Brad and Shana discuss their view of mathematics as a basis for explaining science with their students.

I: We just had an interesting discussion about the Cooling Curves (Appendix A) and Conductivity labs (Appendix B), could both of you try to sum up the discussion and offer your perspective?
S: I was pretty floored because I realized how much my views had changed. These modeling stuff we've done helped me see how to help students use mathematics not in and of itself, but to explain odd phenomena that they do not understand.
I: Brad, what do you think?
B Because I experienced it myself, and now after reflecting a lot I can see what types of roadblocks my students might need to experience to see this connection of using functions and rates to describe scientific observations.
S: I think I originally said science could happen without the math. Not anymore, or not how I envision my students coming to see it. I think this displays a natural way in which my students might come to think about math and science as interrelated.

Nearly every teacher in Cohort 1 mentioned that the modeling activities provided a way for them to think about teaching their students in a way that allowed mathematics to be useful for explaining scientific phenomena. At least five of the teachers mentioned that a focus on models helped them see (for themselves and their teaching of students) mathematics as an important part of understanding energy transfer.

Reflections and Directions: Iteration 1 to Iteration 2

Our discussion around Excerpts 1-5 explains the hypotheses we developed as we engaged in teaching Iteration 1 of the course. After the first iteration was over, the research team completed a reflection on our original research questions prior to teaching the same course to a second cohort. Our reflections on these questions served as the basis for developing activities in the second iteration.

Table 1

Conclusions from Iteration One

Research Question	Example Conclusions from Iteration 1
In what ways did teachers' subject matter knowledge and pedagogical content knowledge about function and rate of change develop as they engaged in a course focused on modeling?	-Began to think about function as a relationship between quantities - Conceived of rate of change as a function - Conceived of energy transfer as a rate of change - Developed intuition about how to help students see the necessity of mathematics in science - Subject matter knowledge remained at a low level relative to our expectations
What was the role of modeling instruction in the development of teachers' subject matter knowledge and pedagogical content knowledge?	- Allowed the teachers to conceive of mathematics and science as reflexive endeavors - When students did not have the appropriate mathematics available, it constrained the utility of particular modeling activities - Teachers' focus on modeling helped them reflect on important understandings about function and rate of change and how their students might come to develop those understandings

As a result of the first iteration, we made a number of adjustments to the course that we anticipated would help us test our hypotheses while also better supporting teachers' knowledge development. First, we curtailed the amount of subject matter covered to focus almost entirely on ideas related function and rate of change in a way that would be propitious for using these ideas to model energy transfer. Second, we emphasized from the outset ways in which the teachers could use their own learning experiences to construct effective learning opportunities for their students.

Iteration 2: Teachers' Subject Matter Knowledge

The teachers in the second cohort were more comfortable in working with function and rate of change than the first cohort. By comfortable, we mean that they were more willing to share their thinking about these ideas and were more confident in their claims about modeling situations with functions and derivatives. In general, their facility with the mathematics and willingness to share their thinking resulted in more critical and high-level conversations about the mathematical ideas of function and rate of change. We use Excerpt 6 as an example where two science teachers, Gina and Robin, discuss their meaning for function and rate.

Excerpt 6 – Gina and Robin discuss their understanding of function and rate of change. (I = Instructor, G = Gina, R = Robin).

I: We just began, but could each of you articulate what you understand about function and rate of change?
G: I see a function as a relationship linking two things, for example, velocity and time or weight and time. The link is the crucial part. A rate of change describes that function, qualitatively, like velocity with respect to time would tell you acceleration. In the

Cooling Curves (Appendix A), we are talking about temperature linked with time and worked on finding the function.

I: Robin, what do you think?

R: I also perceive it as a relationship, but a fixed one. Like the output is twice as big as the input always for a linear function. A rate of change measures that function and is another function itself. Like we could graph the rate of change of the cooling curve we found in that activity.

Most of the teachers in Cohort 2 were initially unconvinced that the mathematics was necessary to articulate a model of energy transfer in their modeling labs. Instead, they argued that mathematics and science were separate topics that did not illuminate each other. It was only after they engaged in the full sequence of modeling labs, along with targeted questions from our research team, that they began to conceive of the mathematics and science as reflexive. In Excerpt 7, Gina and Robin discuss their ah-ha moments related to this reflexivity.

Excerpt 7 – Gina and Robin discussion their understanding rate of change in representing energy transfer and function as a rate.

I: We've done a number of labs now throughout the term that have talked about energy and rate of change. Both of you seemed to have keyed in on that idea.

R: Well, I had a big ah-ha moment. We had been talking about rates of change, heat, conductivity, and others, but I finally realized that energy is represented with a rate of change itself because heat is a change. I kind of sat there dumbfounded, as it was really the first time I have ever thought about needing math to do the science.

I: Robin, I see you are nodding?

G: Yes, what really shook me was hearing all this talk about STEM in our schools and finally seeing an example of the interrelated stuff. In this case, the science really needs the mathematics, and then as a result, you can better understand and think about energy and measure it, graph it, and more.

The discussion in Excerpt 7 suggests that Gina and Robin shifted from thinking about science as disconnected to mathematics as a tool to represent and model phenomena like energy. The excerpt also demonstrates that Gina (and other teachers) began to conceive of STEM as a representation of the relationship between mathematics, science, engineering and technology. For Cohort 2, the modeling activities appeared to engender a way to think about mathematical models as a basis for understanding energy transfer for the teachers.

Iteration 2: Teachers' Development of Pedagogical Content Knowledge

In Cohort 2, we focused on the development of pedagogical content knowledge from the first week of the course. The teachers appeared to struggle with how to lead conversations and how to design activities that would help the students develop desired subject matter knowledge. This was particularly problematic as they began to think about how to design and implement their own lessons. In Excerpt 8, Gina and Robin discuss their difficulty in allowing their experiences developing subject matter knowledge in the context of Modeling Instruction to inform their teaching practices.

Excerpt 8 – Gina and Robin discuss how they anticipate their students coming to understand rate of change.

G: So I see as foundational that my students understand that a rate measures how a function is changing and that it requires measuring changes in variables, like time and temperature, for example. But I really am struggling with how they might be able to come to do that. Like what kind of activities "work", and what do I do?

R: The first thing I notice is that students need to know what to measure and why. For instance, in the Cooling Curves, we had to decide what to measure and what changes to pay attention to. We did that from knowing we wanted to talk about heat transfer. Like you said, the activity and the questions I ask are really important but I am not sure how to ask them.

Gina and Robin's struggles reflected those of Cohort 2 as a whole. Most of the teachers could identify a way of thinking they intended their students to develop and they espoused a desire to connect the mathematics and science for the students. However, they were unsure how the students might develop that understanding and or how they could develop activities to support students in doing so. It appeared to us that while the teachers were intensively engaged in and making sense of modeling problems, they had not internalized modeling as an instructional orientation.

In response, we had the teachers "reverse engineer" a modeling problem. This involved a weekly activity in which the teachers began with an understanding they intended a student to develop, and ended with a modeling situation that could engender that understanding. As a whole, the teachers found they could often "target" multiple ways of thinking for students from a single modeling problem, and were able to design creative instructional sequences to do so. As an example, the teachers worked for two weeks on helping students come to see a rate of change of a function as a function itself. Their goal was to help their students come to think about rate of change as a function that measured how fast one quantity was changing with respect to another quantity. After some initial frustration, they realized that many of the previous modeling problems they had done, including Cooling Curves, were flexible enough to allow multiple types of experiments that would lead students to think about functions and their rates. Indeed, some of the students commented that most of the mathematics they wanted students to learn could be done in the context of just one of the modeling problems we included in the course. Moreover, this “reverse engineering” helped the teachers think about the links between mathematics and science. In Excerpt 9, Gina and Robin discuss how the reverse engineering activities helped them think about how students link mathematics and science.

Excerpt 9 – Gina and Robin discuss the relationship between mathematics and science and its implications for teaching.

G: I loved the reverse engineering stuff. For energy, and other ideas, you can develop a better understanding of the science by working from a model with math. Then that model with math helps you make ideas to test about the science. This is an obstacle I think students are ready to encounter but they don’t get the right situations to do it.

R: I expect that the students we teach can experience these ah-ha moments too, but they need to see why math and science are linked, not just be told it like we have always been. Like I could never have seen function and rate of change as legit problem solving tools for science before this class.

Reflections and Directions on Iteration 2

After Iteration 2, our research team again considered our original research questions and what insight we had gained into them as a result of the second design phase.

Table 2

Conclusions from Iteration Two

Research Question	Example Conclusions from Iteration 2
In what ways did teachers' subject matter and pedagogical content knowledge about function and rate of change develop as they engaged in a course focused on modeling?	-Subject matter knowledge became slightly more sophisticated. -Pedagogical content knowledge appeared to increase a great deal, particularly with regard to rate of change and its applications. -Developed an ability to reverse engineer a problem situation and appropriate instruction around that problem.
What was the role of modeling instruction in the development of teachers' subject matter knowledge and pedagogical content knowledge?	- As in Cohort 1, allowed the teachers to conceive of mathematics and science as reflexive endeavors - Internalizing modeling as an instructional philosophy engendered teachers' ability to identify appropriate learning goals and activities to support achieving those goals.

Evaluating Qualitative Hypotheses: Statistical Tests of Teachers' Subject Matter Knowledge Gains

Our qualitative analyses suggested that while teachers had experienced initial problems thinking about content and reflecting on it in a productive way, they had made progress in thinking about ideas like function and rate of change. We sought to evaluate this belief using the Pre-Calculus Assessment as a quantitative assessment of that knowledge. This section considers the gains the teachers made as evidenced by their Pre-Calculus Assessment scores and complements our qualitative interpretation of the ways in which teachers' subject matter knowledge had changed. A repeated measures analysis of variance, with teachers modeled as random effects was employed to test the hypothesis that the modeling course significantly increased teachers' knowledge of function and rate of change. The potential effect of Cohort was then tested to determine if there were significant within-Cohort differences. Table 3 shows that teachers showed significant gains. Despite these differences, each cohort exhibited significant learning gains (about 5 points on average). The overall model was a good fit to the data. Partial eta^2, the proportion of variance explained by each source of variation in the model was 86% for Gain, and 35% for Cohort.

Table 3

Repeated Measures Analysis of Variance of Teacher Learning Gains.

Source	**SS**	**df**	**MS**	**F**	**p**	**Partial Eta^2**
Pre-Post Gain	267.252	1	267.252	150.029	.000	.857
Gain X Cohort	23.895	2	11.948	6.707	.005	.349
Error	44.533	22	1.781			
Total	335.680	25				

Having established that the gains in student learning were significant, we compared teachers' learning gains against published gains for similar courses, and from one study of student learning utilizing similar teaching methods (Carlson et al., 2010). All comparison groups utilized the assessment as an outcome measure. Table 4 shows that, on average, teachers gained 4.57 points with a standard deviation of 2.25 points. This is an effect size of 2.03 standard deviation units.

Table 4

Analysis of PCA Scores Across Courses Using PCA as an Outcome Measure.

	N	Pre	Post	Gain	t
Typical College Algebra. Pre-Post	36	7.9(2.4)	8.7(3.7)	0.8	8.86, p<.05
College Algebra Post	550		6.8(3.20)		8.27, p<.05
Pre-Calculus Post	902		10.2(4.1)		1.99, p<.05
Modeling Pre-Post	20	7.71(5.61)	12.29(5.56)	4.57(2.25)	

Note: Alpha for each comparison is .05.

Note: University level college algebra and pre-calculus scores (some pre-post and some only post) provide a comparison for teachers' subject matter knowledge.

This analysis shows that a modeling approach to inservice professional development improved teachers' performance to a level on par with Carlson & Oehrtman's students (CAR), taught using similar methods and content emphases. Additionally our teachers achieved an average score of 12.29 on the post-test. This is significant in that Carlson and Oehrtman have shown that students with PCA scores greater than 12 have a pass rate in Calculus about double over students who score less than 12. While the gains the teachers' made in the modeling course was significant, their subject matter knowledge was not far beyond that of a pre-calculus student. Thus, the subject matter knowledge for many remained at a low level.

Implementation of Modeling Instruction

Following the three observations per classroom of Cohort 1 and Cohort 2 teachers, Reformed

Teaching Observation Protocol scores and Modeling Implementation Rubric scores were determined. The mean total RTOP score was moderately high at 77.43 (SD = 13.75), with a minimum and maximum score of 60 (about typical, compared to validation studies) and 94 (very high compared to validation studies), respectively (Piburn et al., 2000). Subscale mean scores ranged from 12.86 (SD = 3.72) for the Content: Procedural Knowledge scale to 18.71 (SD = 1.70) for the Classroom Culture: Student/Teacher Relationships scale; see Figure 1.

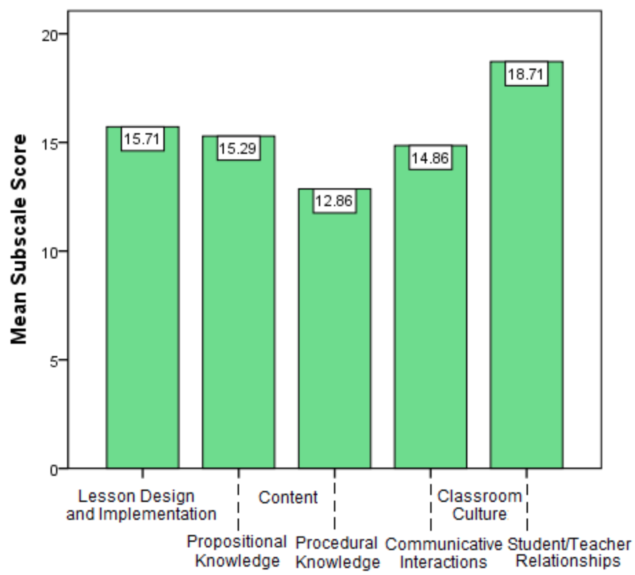

Figure 1. Means of RTOP subscales.

Results from each of the subscales show reasonably effective implementation of reform-oriented classroom norms. Items across Lesson Design and Implementation (3.24), effective emphasis on Propositional Knowledge (3.06), and Communicative Interactions (2.97) were all rated on average, high (~3 out of a total 4 points per item). Student/Teacher Relationships, a measure of the degree to which the teacher served as a listener, valued the input of students, and encouraged students to generate alternative solutions, was rated extremely high, relative to other reform-oriented practices (3.74 out of 4 possible). These findings indicate that the quality of content, and the degree to which teachers emphasized conceptual understanding and student communication of their understanding were well implemented in the studied teachers. Moreover, as stated earlier, our concern that teachers, in learning new content themselves might fall into the trap of interpreting student work narrowly according to the ways in which the teachers personally would have represented and communicated their thinking, was largely alleviated with the evidence from the Student/Teacher Relationships subscale.

However, teachers' practice was not rated highly for items relating to students' critically assessing their developed procedures, reflection on their own learning, and application of a variety of means of representation (Procedural Knowledge Subscale, M=2.57).

Individual RTOP items with the highest mean ratings were items 8 ("The teacher had a solid grasp of the subject matter content inherent in the lesson"), 23 ("In general the teacher was patient with students"), and 24 ("The teacher acted as a resource person, working to support and enhance student investigations"), which were rated at M=4 across all MNS teachers. The individual RTOP items with the lowest mean ratings (M = 2.14; SD = 0.69 and M = 2.14; SD = 1.22, respectively) were items 9 ("Elements of abstraction [i.e., symbolic representations, theory building] were encouraged when it was important to do so") and 12 ("Students made predictions, estimations and/or hypotheses and devised means for testing them.")

Put together, these findings, coupled with field notes from observations indicates that teachers' application of modeling-based activities were somewhat restricted to those aspects of the tasks that were similar to those they learned in class. For instance, some teachers implemented new tasks they had developed themselves, but emphasized the representations and methods utilized in class, while others implemented the same MEAs, modified as they saw fit for their elementary and middle school students, emphasizing the same methods, models, and vocabulary. This finding supports those of Garrett, et al., (2010), who argues that the learning teachers achieve depends on the types of tasks in which they engaged while in content-focused professional development experiences.

Table 5

Pearson correlation between RTOP ratings and MIR ratings.

MIR Subscale	***N* Items**	**Mean (*SD*)**	**Percentage Score**
Elements	6	15.14 (4.91)	63.10%
Operations	6	17.29 (3.86)	72.02%
Relations	6	16.14 (3.34)	67.26%
Rules	5	10.71 (1.89)	53.57%
Total MIR Score	23	59.29 (13.46)	64.44%

Conclusions

Research Question 1: Development of Teacher Knowledge (Subject Matter Knowledge and Pedagogical Content Knowledge)

We originally hypothesized that by integrating Modeling Instruction with teachers' content learning experiences we could impact their ability to understand subject matter, in particular function and rate of change, in a way that enabled teachers to create conceptual learning opportunities for students.

Subject Matter Knowledge

The scope of the course was such that we were able to observe a significant (both statistical and practical) shift in the teachers' subject matter knowledge related to function and rate of change, which are the core ideas of thermodynamics. The teachers became accustomed to thinking about function as an invariant relationship between quantities and rate of change as a measure of that relationship. By thinking of function and rate of change in this manner, the teachers were able to construct experiments and evaluate data with an understanding that they were searching for patterns in the rates of change. By searching for patterns in the rate of change they were able to determine what type of function (or specific function) could reasonably represent the phenomena at hand. Thus, they not only experienced shifts in content knowledge, but they also experienced a change in how they conceptualized the usefulness of those ideas.

We also find it important to note that the two cohorts discussed in the results initially had different comfort levels with mathematics, as indicated by both our in person observations of the teachers in the first cohort and their slightly lower PCA scores compared to the teachers in the second cohort. A natural question then emerges: was the achievement level of the second cohort due to the effects of the course or cohort two's initially stronger mathematical ability? We think that while the second cohort was more prepared mathematically in areas like basic algebra, they, like the first cohort, had mostly naïve notions of function and rate of change. Additionally, neither cohort previously had the occasion to think about function and rates of change as tools for modeling scientific phenomena. Because both cohorts had roughly the same background knowledge about function and rate of change, we do not think cohort two began with a significant advantage compared to cohort one. Instead, we think that the strength of cohort two came from refined lessons and engaging activities that focused the teachers on modeling that would not have been possible without the first iteration of the course. Thus, while we do not have a direct means of measuring this claim given our data, it seems likely the success of cohort two was primarily due to the nature of the course.

Pedagogical Content Knowledge

In addition to the shifts in mathematical knowledge, we also observed how teachers' increasingly complex reflections on and use of subject matter knowledge about function and rate of change allowed them to modify tasks from the modeling course in such a way that they were able to implement these tasks in an elementary classroom. This implementation was non-trivial. Most of the mathematical ideas discussed in the course were at a high school or undergraduate level, particularly function and rate of change. Initially, the teachers struggled to see how these "hard ideas" could be useful to elementary students. However, they began to see that they could modify the tasks from the course in such a way that they became interesting and useful for students who did not yet have sophisticated notions of mathematics. To do so, they asked the students to create basic models of situations and phenomena and having them make conjectures about how certain aspects of these scenarios were related to each other. For instance, many of the teachers modified the modeling activities so that their younger students would focus on the meaning of "hotter" and "cooler" and what quantities in a situation could make an object (i.e. cup of coffee) hotter or cooler. They engaged students in basic data collection and asked them how the data was related to heating up or cooling down. In short, the teachers engaged their elementary students in activities where they were able to describe what was occurring, focus on important quantities that could affect what was occurring, but stopped short of asking them to use more formal mathematics. In the cases of both subject matter knowledge and pedagogical content knowledge, we noted that almost every teacher became attuned to the reflexive relationship between "doing mathematics" and "doing science". Indeed, they not only conceived of this reflexivity in their own subject matter knowledge, but also strived to develop learning opportunities for their students to help them conceive of that same reflexivity.

Research Question 2: The Role of Modeling Instruction

Our iterations of design and analyses using both qualitative and quantitative measures suggest that the shifts the teachers experienced might not have occurred without an explicit focus on modeling physical phenomena. We believe this because even as the teachers were able to fluently describe function as an invariant relationship between quantities and rate of change as a measure of that relationship. We think that in both cases this understanding emerged as they developed increasingly viable models of the phenomena in the course. Once the teachers had constructed a model for these phenomena, they began to grapple with how to quantify these phenomena, which often involved overcoming some of their initial confusion with basic numerical operations and representations.

As the teachers engaged in creating models for the phenomena in the modeling labs, they began to conceive of the mathematics as a tool so that function and rate of change became necessary to explaining and predicting relationships between quantities in energy transfer. We argue that their path to conceiving of mathematics as a tool for doing science was a product of the Modeling Instruction that strongly influenced the development of their pedagogical content knowledge. Our analyses suggest that two things drove the development of pedagogical content knowledge relative to function and rate of change. First, teachers developed an understanding of how students, including themselves and their classmates learn mathematics through modeling. Second, they developed an understanding of modeling instruction as a means for task design and the management of classroom discourse. Thus, we argue that the focus on Modeling Instruction affected the teachers' *knowledge of content and students* and *knowledge of content and teaching*, two key aspects of pedagogical content knowledge.

Discussion

Contribution to Research

Teacher professional development has traditionally emphasized the conveyance of subject matter knowledge and research-based pedagogical practices that provide opportunities for students to understand the content in the way the teacher intends. Subject matter knowledge and pedagogical practices are often emphasized sequentially but very rarely simultaneously. Moreover, the pedagogical practices emphasized in professional development are scarcely ever fashioned by the subject matter, and as a result, are too general to facilitate effective teaching of particular mathematics concepts. Pedagogical principles do not constitute the pedagogical knowledge needed to provide opportunities for students to construct a mature understanding of specific mathematical ideas. What is needed is pedagogical knowledge that derives from and is constituted by the teachers' mathematical understanding—pedagogical *content* knowledge. We have advanced Modeling Instruction as a vehicle for developing teachers' subject matter knowledge while providing teachers with opportunities to reflect upon this learning process in ways that support the development of pedagogical content knowledge. We have shown that through Modeling Instruction, pedagogical content knowledge can develop as a consequence of emerging subject matter knowledge; and that the pedagogical content knowledge constructed is necessarily sufficiently detailed and robust to promote the conceptual teaching of specific mathematics concepts since it was fashioned by it.

Contributions to Modeling and the Integration of Mathematics and Science

Through participating in Modeling Instruction, the teachers in our study came to recognize the utility of modeling scientific phenomena for advancing students' understanding of fundamental mathematics concepts such as function and rate of change. However, prior to this realization, they came to appreciate the way in which mathematics can enrich the study of science by examining the quantities inherent in a scientific situation in a way governed by the sense-making norms of mathematics. Modeling Instruction afforded the teachers the opportunity to conceive of mathematics and science in this way. Indeed, we think that one of the most significant shifts for teachers was their conceiving of mathematics and science as complementary tools for making sense of observations. Our findings also suggest that it is crucial for modeling courses to provide teachers with authentic experiences in which they observe phenomenon they initially do not understand (such as convection), but their experience allows them use mathematics as a tool to explain or predict the behavior of that phenomenon.

Future Directions for Work

While there are many ways in which this study might be modified or reconsidered in the future, we want to point out two that we feel are especially important. The first issue is considering how the teachers being from an elementary school setting may have had an effect on the results of our study. Specifically, what special circumstances may have emerged because we were teaching secondary mathematics content to elementary school teachers? We suspect that teachers from the middle and high school level would have encountered less initial difficulty with the mathematics, but would have had a similar experience with learning to model, as they likely would not have had occasion to do so prior to the course. However, it would be useful in a subsequent study to focus on teachers with more mathematics specific background to provide insight into this issue. The second issue is that while this paper demonstrates that Modeling Instruction can serve as domain in which teachers' subject matter knowledge and pedagogical content knowledge can develop in tandem, it was beyond the scope of this paper to provide a detailed cognitive account of specifically how teachers' experiences with Modeling Instruction facilitate the development of their pedagogical content knowledge. We have shown that reflection upon learning experiences plays a vital role in the construction of pedagogical content knowledge and that participating in Modeling Instruction gave teachers conceptual learning experiences to reflect upon, but we did not investigate what constitutes productive reflection or examine how this reflection interacts with the teachers' evolving subject matter knowledge. Specifically how teachers' evolving subject matter knowledge and reflection upon the construction of that knowledge contribute to developing their pedagogical content knowledge is an important area for further research. Additionally, characterizing the instructional moves that promote productive reflection in the context of Modeling Instruction is another area that would benefit from further investigation. It is important to note that we see this study as a step in a sequence of investigations focused on the affordances of Modeling Instruction. This is particularly true for elementary contexts, in which modeling is a means for creating integrated STEM activities which are becoming ever more crucial in light of the new mathematics and science standards and the recommendations based upon them.

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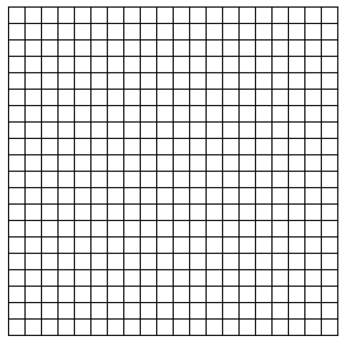
APPENDIX A: COOLING CURVES

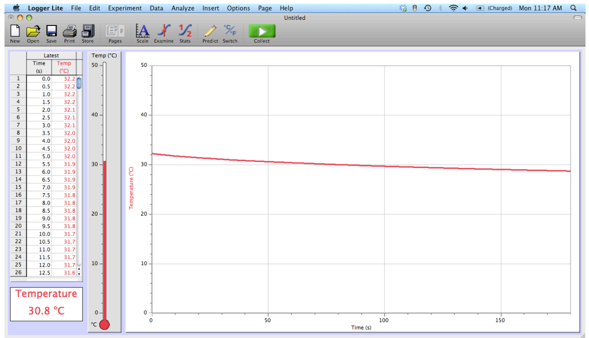
Activity 1: Cooling Curves for Hot Liquids

Do you remember the news article about the woman who sued McDonalds when the cup of hot coffee she had just bought scalded her? It turns out that she bought the cup in the drive through window, and put the cup between her legs as she drove off. She won the lawsuit, but it was eventually overturned. Nevertheless, McDonalds does not serve hot coffee in its drive through windows. They let it cool down to a safe but unpalatable temperature. In this lab, we will examine the shape of the function that describes how liquids cool over time. Pretend you are an engineer working for the McDonald's corporation.

1. First, sketch the graph of your coffee cooling, using what you know about liquids and temperature.



2. We will use Labquest mini and Logger Lite on your computer to collect data on how liquids cool and let the program run while we are exploring some other situations[*].
    a. Find two temperature probes;
    b. Plug one temperature probe into Channel 1 on the LabQuest mini, plug the second temperature probe into Chanel 2, and plug the LabQuest mini into your computer;
    c. Open LoggerLite and work through the menus, setting up the probes and testing each for a short time to see if it works properly (putting it under your arm works great);
    d. Now set up the program to run an experiment where you will immerse the probes into a cup of hot liquid for ½ hour, recording the temperature at 30-second intervals.
    e. I will give you a cup of hot water in which you should immerse your temperature probe.
    f. Let the experiment run for 30 minutes.



3. Now lets examine the plot of the data and describe the behavior of the cooling curve. Compare your actual data with the graph you sketched above.
    a. What aspects of your sketch did you anticipate correctly?
    b. What is different about the actual data plotted when compared with your original sketch?
    c. What kind of function do you think will model this kind of behavior? Linear? Polynomial? Exponential? Power? Examine your data and explain your reasoning.
4. What would happen to the curve if you used cups of different material? What would happen if you put a lid on your cup? Describe the key features of your graph and the effect of these new experimental conditions.

APPENDIX B: THERMAL CONDUCTIVITY

Activity 2: Thermal Conductivity

Have you every roasted marshmallows over a campfire? Or, have you stirred a really hot pot of soup with a metal spoon? If you have, you probably have a pretty good idea about *conduction*. When you first put a long piece of steel into a fire, the steel is cold. But over time, the end that is in the fire heats up, and the thermal energy of the fire slowly travels up the steel to your hand. Likewise, when you stir a hot pot of soup, the molecules in the spoon collide with the molecules whizzing around in the pot, and the spoon begins to heat up, starting at then submersed end, moving slowly up the handle to your hand.

What affects the rate at which different materials conduct thermal energy? For metals, the electrical conductivity is closely related to thermal conductivity, because electrons in the outer orbitals of atoms, which conduct electricity, also conduct thermal energy. But for other substances, the kind of molecules that are formed, their crystalline structure, and their macroscopic structure interact to determine its thermal conductivity. Below is a table of common substances and their thermal conductivity:

Thermal Conductivity for Materials

Material	**k (W/m^2 K)**	**material**	**k (W/m^2 K)**
air, sea level	0.025	neoprene	0.15–0.45
air, 10,000 m	0.020	particle board	0.15
aluminum	237	paper	0.04–0.09
asphalt	0.15–0.52	plaster	0.15–0.27
brass (273 K)	120	plywood	0.11
brick	0.18	polyester	0.05
bronze (273 K)	110	polystyrene foam	0.03–0.05
carbon, diamond	895	polyurethane foam	0.02–0.03
carbon, graphite	1950	snow (< 273 K)	0.16
carpet	0.03–0.08	steel, plain (273 K)	45–65
concrete	0.05–1.50	steel, stainless (273 K)	14
copper	401	straw	0.05

Notice that the units for conductivity are in Watts per square meter per degree Kelvin (W/m^2 K). Watts (kgm^2/s^3) is a unit of power, or energy per second. It is equivalent to Joules (kgm^2/s^2) per second. So we can think of it as a rate of energy flow per unit time.

When would it be a good thing to think about the thermal conductivity of materials? Conductivity is related to the concepts of *insulation* (a really bad conductor is an insulator. Look at carpet in the table above), *resistance* (the tendency of some materials to withstand energy flow) and *thermal transmittance* (the total thermal energy transferred in a system via conduction, convection, and radiation). In this Lab, you will explore the relationship between thermal conductivity for two substances, paper, a good insulator, and aluminum foil, a good conductor.

Materials:

1. Insulated cups (Styrofoam is a good choice for this one. If you are environmentally conscious, you can use insulated cups of any kind, so long as they are not too big in terms of the amount of fluid they can hold).
2. Stack of paper. Any kind of paper is good, so long as your stack consists of the same kind of paper.
3. Sheet of aluminum foil.
4. Scissors to cut paper and foil
5. Tape to seal the cups
6. CBL
7. TI83 or TI84 graphing calculator with Physics program or EasyData.
8. 2 temperature probes.

In the class, each group will be in charge of modeling two conditions: 1) the thermal conductivity of Paper of a given thickness, and 2) the thermal conductivity of Aluminum Foil of the same thickness as the Paper condition. Some discussion may be in order to determine how we can figure out the thickness of our two conditions. Each group will choose a different thickness of paper/foil to test. In this manner, the class as a whole will collect data modeling 1) the effect of thickness of a material on its conductivity, and 2) the effect of material on conductivity.

Procedure:

1. Each group should cut a cover for their insulated cups out of the paper and aluminum foil. The Area of the covers must be identical, and the thickness must be identical.
2. Set up your CBL with both temperature probes, to collect temperature data every 30 seconds for 1 hour. Make sure the system works prior to moving on to step 3.
3. Pour boiling water into the cups right up to the brim, but do not allow the meniscus to extend over the top of the cup. It is probably best to leave 4-5 mm gap between the water and the top of the cup, but no more.
4. Place your paper and foil covers over the cups quickly and seal them with tape
5. Tape the temperature probes securely on your lids.
6. Start collecting the data.
7. After one hour is up, your group should begin to examine the rate of change of temperature over time for your two conditions.

Prepare your boards to show clearly how the rate of change between your two conditions is different (if it is, in fact, different). Also, show how that rate changes over time in some visual or numeric fashion.

The board meeting should compare results across groups and across conditions.

1. Describe what quantities you are measuring as you use your measurement tools. What changes in those quantities are you measuring, and how?

1. Describe any relationships you believe exist between the quantities you are measuring? Why do you think these relationships exist?

3. Individually prepare your lab write up to show what the overall effects of material and thickness have over thermal conductivity.
4. Since conductive thermal energy transfer follows this pattern,

$$\frac{dT}{dt} = \frac{kA(T_{hot} - T_{cold})}{d}$$

Where *k* is the thermal conductivity of your barrier, *A* is the surface area exposed to the hot water, and *d* is the thickness of the barrier.

- Use a regression method (using your calculator) to create a function to predict the Temperature over time for your data in each of the conditions: Paper versus aluminum.
- What type of regression do you think will fit the data best? Why?
- How well does your predicted curve match your data you collected?