WIGNER FUNCTION APPROACH TO OSCILLATING SOLUTIONS OF THE $1D$-QUINTIC NONLINEAR SCHRÖDINGER EQUATION

ALEX MAHALOV AND SERGEI K. SUSLOV

Abstract. We study oscillating solutions of the $1D$-quintic nonlinear Schrödinger equation with the help of Wigner's quasiprobability distribution in quantum phase space. An "absolute squeezing property", namely a periodic in time total localization of wave packets at some finite spatial points without violation of the Heisenberg uncertainty principle, is analyzed in this nonlinear model.

As is widely known, mean-field theory is very successful in description of both static and dynamic properties of Bose-Einstein condensates [1], [2]. The macroscopic wave function obeys a $3D$-cubic nonlinear Schrödinger equation. At the same time, there are several reasons to consider higher order nonlinearity in the Gross–Pitaevskii model [2]. The quintic case is of particular importance because near Feshbach resonance one may turn the scattering length to zero when the dominant interaction among atoms is due to three-body effects (see [3], [4], [5], [6], [7], [8] and the references therein; in ${}^7Li$-condensate, for example, the scattering length is reported as small as 0.01 Bohr radii [8]). Then the nonlinear term in the mean-field equation has the quintic form. Another examples include a $1D$-Bose gas in the limit of impenetrable particles [9], [10], [11] and collapse of a plane Langmuir soliton in plasma [12], [13].

A finite time blow up of solutions of the unidimensional quintic nonlinear Schrödinger equation is studied in many publications (see, for example, [13], [14], [15], [16], [17], [18], [19], [20]). This case is critical because any decrease of the power of nonlinearity results in the global existence of solutions [21], [22] (see also [10] and [23]). Related hidden symmetry, explicit oscillating and blow up solutions, the uncertainty relation and squeezing from the viewpoint of Wigner's function approach are topics discussed in this Letter.

1. Symmetry Group

The quintic derivative nonlinear Schrödinger equation in a parabolic confinement,

$$i\psi_t + \psi_{xx} - x^2\psi = ig\left(\left|\psi\right|^2\psi_x + \psi^2\psi_x^*\right) + h\left|\psi\right|^4\psi \tag{1.1}$$

(g and h are constants), is invariant under the following change of variables:

$$\psi\left(x,t\right) = \sqrt{\frac{\beta\left(0\right)}{\left|z\left(t\right)\right|}}\ e^{i\left(\alpha(t)x^2+\delta(t)x+\kappa(t)\right)}\ \chi\left(\beta\left(t\right)x+\varepsilon\left(t\right),-\gamma\left(t\right)\right) \tag{1.2}$$

Date: January 1, 2013.

1991 *Mathematics Subject Classification.* Primary 81Q05, 35C05. Secondary 42A38.

Key words and phrases. One dimensional quintic nonlinear Schrödinger equation, traveling wave and blow up solutions, Wigner function, Heisenberg uncertainty relation, Tonks–Girardeau gas of impenetrable bosons.

(the so-called Schrödinger group). We introduce $z(t) = c_1 e^{2it} + c_2 e^{-2it}$ and express everything in terms of this complex-valued function as follows:

$$\alpha(t) = i\frac{(c_1c_2)^* z^2 - c_1c_2 (z^*)^2}{2(c_1 - c_2^*)|z|^2}, \tag{1.3}$$
$$\beta(t) = \pm\frac{\sqrt{|c_1|^2 - |c_2|^2}}{2|z|^2}, \quad \gamma(t) = \frac{1}{2}\arg z,$$
$$\delta(t) = \frac{c_3 z - c_3^* z^*}{2i|z|^2}, \quad \varepsilon(t) = \pm\frac{c_3 z + c_3^* z^*}{2|z|\sqrt{|c_1|^2 - |c_2|^2}},$$
$$\kappa(t) = \frac{\left(c_3^2 z + c_3^{*\,2} z^*\right)(z - z^*)}{8i(c_1 - c_2^*)|z|^2}.$$

The complex parameters:

$$c_1 = \tfrac{1+\beta^2(0)}{2} - i\alpha(0), \quad c_2 = \tfrac{1-\beta^2(0)}{2} + i\alpha(0), \tag{1.4}$$
$$c_3 = \varepsilon(0)\beta(0) + i\delta(0)$$

are defined in terms of real initial data (we choose $\gamma(0) = \kappa(0) = 0$ for the sake of simplicity). In addition,

$$|z|^2 = |c_1|^2 + |c_2|^2 + c_1c_2^* e^{4it} + c_1^* c_2 e^{-4it}, \tag{1.5}$$
$$c_1c_2^* = \frac{1-\beta^4(0)}{4} - \alpha^2(0) - i\alpha(0).$$

The Schrödinger group was originally introduced by Niederer [24] as the maximum kinematical invariance group for the linear harmonic oscillator when $g = h = 0$ (and for the free particle [25]). We complement these results by identifying the nonlinear terms that are invariant under the action of this group. (The real form of transformation (1.2) and visualization of the corresponding oscillating solutions for the linear harmonic oscillator can be found in [26] and [27].) Our goal is to describe a class of oscillating solutions to the nonlinear equation (1.1) with the aid of Wigner quasiprobability distribution. It is worth noting that formulas (1.2)–(1.4) allow one to construct a six-parameter family of time-periodic oscillating solutions to equation (1.1) from any known solution.

2. Explicit Traveling Wave and Blow Up Solutions

Although explicit solutions to nonlinear Schrödinger equation (1.1) and their experimental observations are not readily available in the literature (see, for example, [28] and the references therein for $g = 0$), Bose condensation and/or nonlinear effects in "non-Kerr materials", e.g. optical fibers beyond the cubic nonlinearity, may provide important examples.

2.1. **Traveling Waves.** The $1D$-quintic nonlinear Schrödinger equation without potential in dimensionless units,

$$iA_t + A_{xx} \pm \frac{3}{4}|A|^4 A = 0, \tag{2.1}$$

has the following explicit solutions adapted from [29] (we use the notation and terminology from [29] and [30]; see also [31] and [32]).

Pulses:

$$A\left(x,t\right)=e^{i\phi}\left[\frac{k}{\cosh k\left(x-vt\right)}\right]^{1/2}\exp i\frac{2vx+\left(k^{2}-v^{2}\right)t}{4} \tag{2.2}$$

(ϕ, v and k are real parameters, the upper sign of the nonlinear term should be taken in (2.1); see also [17] and the references therein). We have

$$\int_{-\infty}^{\infty}\left|A\left(x,t\right)\right|^{2}\ dx=\pi \tag{2.3}$$

and the corresponding plane wave expansion,

$$A\left(x,t\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipx}B\left(p,t\right)\ dp, \tag{2.4}$$

can be found in terms of gamma functions:

$$\begin{aligned} B\left(p,t\right)&=\frac{e^{i\phi}}{2\pi\sqrt{k}}\exp i\left(\frac{v^{2}+k^{2}}{4}-pv\right)t \\ &\times\Gamma\left(\frac{1}{4}+\frac{i}{2k}\left(p-\frac{v}{2}\right)\right)\Gamma\left(\frac{1}{4}-\frac{i}{2k}\left(p-\frac{v}{2}\right)\right) \end{aligned} \tag{2.5}$$

with the aid of a special case of integral (A.1).

It is worth noting that

$$\overline{x}=\frac{\langle x\rangle}{\langle 1\rangle}=vt,\quad \overline{p}=\frac{\langle p\rangle}{\langle 1\rangle}=\frac{v}{2}=\text{constant} \tag{2.6}$$

and

$$\left(\delta x\right)^{2}=\overline{x^{2}}-\overline{x}^{2}=\frac{\pi^{2}}{\left(2k\right)^{2}},\quad \left(\delta p\right)^{2}=\overline{p^{2}}-\overline{p}^{2}=\frac{k^{2}}{8} \tag{2.7}$$

with

$$\left(\delta p\right)^{2}\left(\delta x\right)^{2}=\frac{\pi^{2}}{32}>\frac{1}{4} \tag{2.8}$$

for the traveling wave solution (2.2) by direct integral evaluations. The energy functional is given by

$$E=\overline{p^{2}}-\frac{1}{4}\overline{\left|\psi\right|^{4}}=\frac{v^{2}}{4}\geq 0 \tag{2.9}$$

and its positivity provides a sufficient condition for developing of a blow up in this critical case, namely a singularity such that the wave amplitude tends to infinity in a finite time [13], [16], [21], [33], [34]. Indeed, pulses (2.2) are unstable. A six-parameter family of blow up solutions is explicitly constructed below (2.11).

Sources and sinks:

$$\begin{aligned} A\left(x,t\right)&=\left[\frac{\cosh\left(\sqrt{3}r\left(x-vt\right)\right)\pm 1}{\cosh\left(\sqrt{3}r\left(x-vt\right)\right)\mp 2}\right]^{1/2} \\ &\times e^{i\phi}r^{1/2}\exp i\left(\frac{vx}{2}-\left(v^{2}+3r^{2}\right)\frac{t}{4}\right) \end{aligned} \tag{2.10}$$

(ϕ, v and r are real parameters; see also [10]). Equation (2.1) has also a class of (double) periodic solutions in terms of elliptic functions [31], [32] (see also [35] and [36]).

2.2. **Blow Up Solutions.** A direct action of the Schrödinger group [25], [37] on (2.2) produces a six-parameter family of square integrable solutions:

$$\psi(x,t) = \sqrt{\frac{\beta(0)}{1+4\alpha(0)t}} \tag{2.11}$$
$$\times \exp i\left(\frac{\alpha(0)x^2+\delta(0)x-\delta^2(0)t}{1+4\alpha(0)t}+\kappa(0)\right)$$
$$\times A\left(\beta(0)\frac{x-2\delta(0)t}{1+4\alpha(0)t}+\varepsilon(0),\ \frac{\beta^2(0)t}{1+4\alpha(0)t}-\gamma(0)\right).$$

Here, one can choose $v=0$ without loss of generality. The so-called one-parameter subgroup of expansions [37], when $\beta(0)=1$ and $\delta(0)=\varepsilon(0)=\kappa(0)=0$, is discussed in [34] (see also [14], [38], [39], [40] and the references therein regarding these symmetry transformations). The corresponding expectation values are given by

$$\overline{x} = 2\left(\delta(0)-\frac{2\alpha(0)\varepsilon(0)}{\beta(0)}\right)t-\frac{\varepsilon(0)}{\beta(0)}, \tag{2.12}$$

$$\overline{p} = \delta(0)-\frac{2\alpha(0)\varepsilon(0)}{\beta(0)} = \text{constant} \tag{2.13}$$

and the variances are

$$(\delta x)^2 = \frac{\pi^2}{(2k)^2}\left(\frac{1+4\alpha(0)t}{\beta(0)}\right)^2, \tag{2.14}$$
$$(\delta p)^2 = \frac{\pi^2}{(2k)^2}\left(\frac{2\alpha(0)}{\beta(0)}\right)^2+\frac{k^2}{8}\left(\frac{\beta(0)}{1+4\alpha(0)t}\right)^2$$

with the uncertainty relation

$$(\delta p)^2(\delta x)^2 = \frac{\pi^2}{32}+\frac{\pi^4}{4k^4}\left(\alpha(0)\frac{1+4\alpha(0)t}{\beta(0)}\right)^2 \geq \frac{\pi^2}{32} > \frac{1}{4}. \tag{2.15}$$

We choose $v=0$ in (2.2) without loss of generality because the general action of the Schrödinger group already includes the Galilean transformation [25], [24]. (The real-valued initial data for the corresponding Riccati-type system are taken; see [26] and [37] for more details.)

Evidently, all of these solutions blow up at the point $x_0=-\delta(0)/2\alpha(0)$ in finite time, when $t\to t_0=-1/4\alpha(0)$ and $\alpha(0)\neq 0$. At this moment in time, the wave packet becomes totally localized with $\delta x=0$ and $\delta p=\infty$, when the uncertainty relation attains its minimum value $\pi^2/32$. The energy functional and virial theorem have the following explicit forms

$$E = \frac{\pi^2}{(2k)^2}\left[\frac{2\alpha(0)}{\beta(0)}\right]^2+\left[\delta(0)-\frac{2\alpha(0)\varepsilon(0)}{\beta(0)}\right]^2 \geq 0,$$
$$\frac{d^2}{dt^2}\overline{x^2} = 8E \tag{2.16}$$

on our solutions (2.11), respectively, and equation (2.13) shows the momentum conservation.

In this Letter, we would like to emphasize that the blow up pulses (2.11) can be effectively studied in quantum phase space. The corresponding Wigner function [41], [42] is easily evaluated from definition (3.11) with the help of integral (A.1). Computational details are left to the reader.

Although Wigner's function approach is a standard tool in quantum optics (e. g. [43], [44], [45], [46]), the use of this powerful method is very limited in the available literature on nonlinear Schrödinger equations.

Blow up of solutions to the unidimensional quintic nonlinear Schrödinger equation without potential is a classical topic (e. g. [13], [14], [15], [16], [17], [18], [19], [20] and the references therein) because any decrease of the power of nonlinearity results in the global existence of solutions [13], [21], [22] (see also Refs. [10] and [23] for a renormalization approach). We elaborate on connections with the nontrivial symmetry of this nonlinear PDE, when the singularity is developing in a finite time by variation of solutions that decay sufficiently fast at infinity.

3. Oscillating Nonlinear Wave Packets

The quintic nonlinear Schrödinger equation in a parabolic confinement,

$$i\psi_t + \psi_{xx} - x^2\psi \pm \frac{3}{4} |\psi|^4 \psi = 0, \tag{3.1}$$

describes a mean-field model of strongly interacting $1D$-Bose gases for the practically important case of a harmonic trap [3], [6], [10], [18], [23], [47], [48], and, in particular the so-called *Tonks–Girardeau* gas of impenetrable bosons [9], [11]; see [49] and [50] for experimental observations. (The time-independent version of the quintic nonlinear Schrödinger equation has been rigorously derived from the many-body problem [51]; see also [52] for a rigorous derivation of the Gross-Pitaevskii energy functional.)

3.1. **Special Case.** By the gauge transformation (e. g. [26], [37], [38], [40] and the references therein for the linear problem, the quintic nonlinearity is also invariant under this transformation [18], [34]), equation (3.1) has the following solution:

$$\psi(x,t) = \frac{e^{-(i/2)x^2 \tan 2t}}{\sqrt{\cos 2t}} A\left(\frac{x}{\cos 2t}, \frac{\tan 2t}{2}\right), \tag{3.2}$$

where $A(x,t)$ is any solution of (2.1), in particular, the pulses and sources (2.2) and (2.10).

Oscillating pulses:

$$\begin{aligned} \psi(x,t) = {} & \sqrt{\frac{2k}{\cos 2t}} \operatorname{sech}^{1/2}\left(\frac{2k}{\cos 2t}(x - v\sin 2t)\right) \\ & \times e^{i\phi} \exp i \frac{2vx + (k^2 - v^2 - x^2)\sin 2t}{2\cos 2t} \end{aligned} \tag{3.3}$$

(ϕ, v and k are real parameters, the upper sign should be taken in the nonlinear term). They are square integrable at all times:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} |\psi(x,t)|^2 \ dx = 1 \tag{3.4}$$

and

$$\begin{aligned} & \frac{1}{\pi}\int_{-\infty}^{\infty} |x\psi(x,t)|^2 \ dx \\ & = \frac{\pi^2}{(4k)^2}\cos^2 2t + v^2 \sin^2 2t, \end{aligned} \tag{3.5}$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left|\psi_{x}\left(x,t\right)\right|^{2}\ dx \tag{3.6}$$
$$=\frac{\pi^{2}}{\left(4k\right)^{2}}\sin^{2}2t+v^{2}\cos^{2}2t+\frac{k^{2}}{2\cos^{2}2t}.$$

The expectation values and variances of the position x and momentum $p=i^{-1}\partial/\partial x$ operators are given by

$$\overline{x}=\frac{\langle x\rangle}{\langle 1\rangle}=v\sin 2t,\qquad \overline{p}=\frac{\langle p\rangle}{\langle 1\rangle}=v\cos 2t \tag{3.7}$$

and

$$\left(\delta x\right)^{2}=\overline{x^{2}}-\overline{x}^{2}=\frac{\pi^{2}}{\left(4k\right)^{2}}\cos^{2}2t, \tag{3.8}$$
$$\left(\delta p\right)^{2}=\overline{p^{2}}-\overline{p}^{2}=\frac{\pi^{2}}{\left(4k\right)^{2}}\sin^{2}2t+\frac{k^{2}}{2\cos^{2}2t},$$

respectively. The energy functional takes the form

$$E=\overline{H}=\overline{p^{2}}+\overline{x^{2}}-\frac{1}{4}\overline{\left|\psi\right|^{4}}=\frac{\pi^{2}}{\left(4k\right)^{2}}+v^{2}>0 \tag{3.9}$$

by a direct evaluation.

A remarkable feature of the oscillating solution (3.3) is that the corresponding probability density converges, say as a sequence, periodically in time, to the Dirac delta function at the turning points: $\left|\psi\left(x,t\right)\right|^{2}\rightarrow\pi\delta\left(x\mp v\right)$ as $t\rightarrow\pm\pi/4$ etc., when an "absolute squeezing" and/or total localization, namely $\min\delta x=0$, occurs with $\max\delta p=\infty$. The fundamental Heisenberg uncertainty principle holds

$$\left(\delta p\right)^{2}\left(\delta x\right)^{2}=\frac{\pi^{2}}{32}\left(1+\frac{\pi^{2}}{32k^{4}}\sin^{2}4t\right)\geq\frac{\pi^{2}}{32}>\frac{1}{4} \tag{3.10}$$

at all times. (It is worth noting that $\pi^{2}/8\approx 1.2337$. The minimum-uncertainty squeezed states for a linear harmonic oscillator, when the absolute minimum of the product $1/4$ can be achieved, are constructed in [53].)

The Wigner quasiprobability distribution in phase space [41], [42] is a standard way to study the squeezed states of light in quantum optics (see [43], [44], [45], [46], [53] and the references therein). We apply a similar approach to blow up solutions of the quintic nonlinear Schrödinger equations. The corresponding Wigner function:

$$W\left(x,p,t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi^{\ast}\left(x+y/2,t\right)\psi\left(x-y/2,t\right)e^{ipy}\ dy, \tag{3.11}$$

can be evaluated in terms of hypergeometric function:

$$W\left(x,p,t\right) \tag{3.12}$$
$$=\operatorname{sech}\,\omega\ {}_{2}F_{1}\left(\begin{array}{c}1/2+i\omega,1/2-i\omega\\ 1\end{array};-\sinh^{2}\vartheta\right)$$

with the aid of integral (A.1). Here

$$\omega=\frac{1}{2k}\left(p\cos 2t+x\sin 2t-v\right), \tag{3.13}$$

$$\vartheta = \frac{2k}{\cos 2t} \left(x - v \sin 2t \right).$$

It is known that

$$\left| \psi \left(x, t \right) \right|^2 = \int_{-\infty}^{\infty} W \left(x, p, t \right) \ dp, \tag{3.14}$$

where Wigner's function remains finite in the entire phase space at all times by the Cauchy–Schwarz inequality. In the linear case of a quadratic system, the graph of Wigner function simply rotates in phase plane without changing its shape (see, for example, [46] and [53]). The time-evolution in the nonlinear case is more complicated. Examples are presented in Figures 1 and 2.

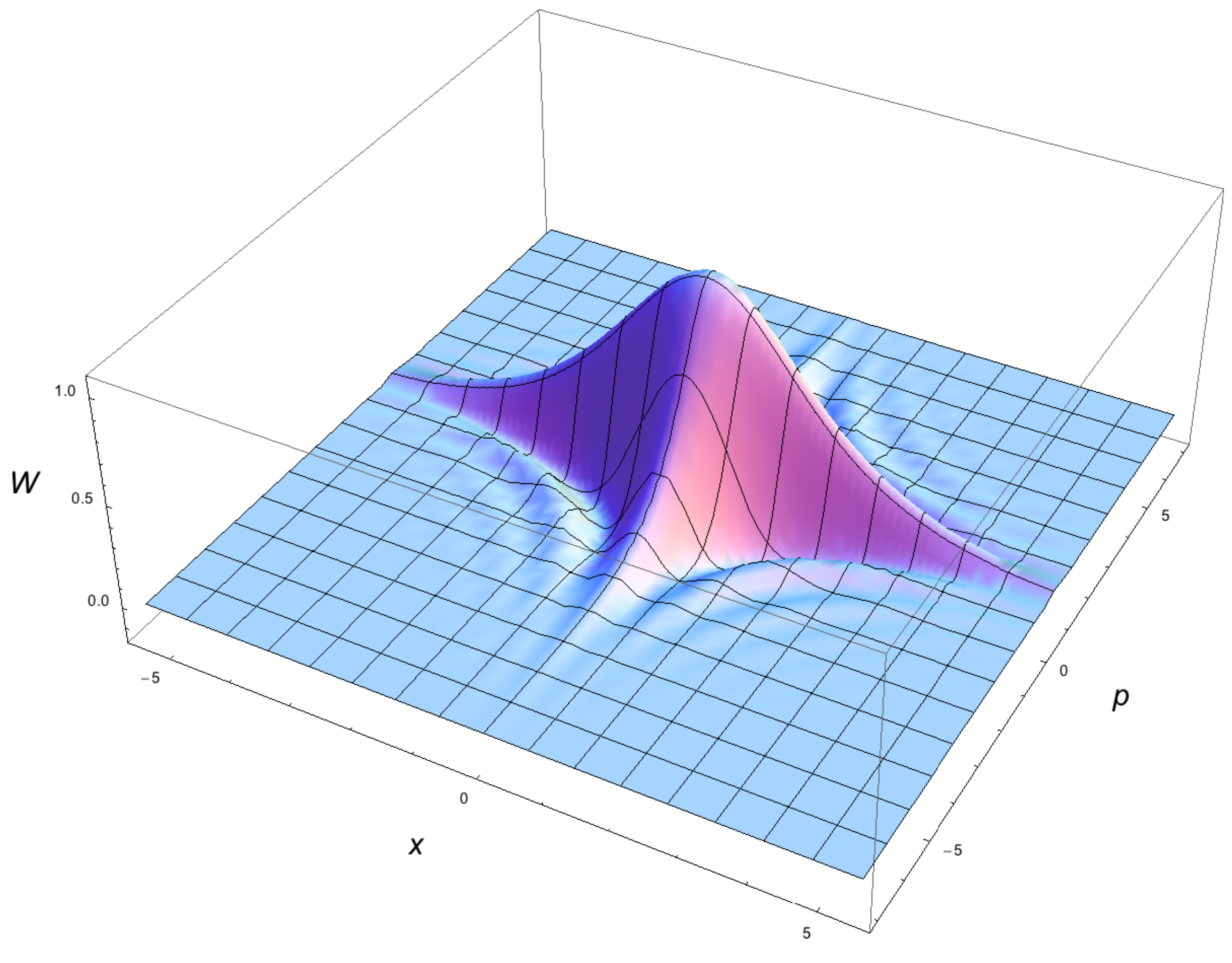

FIGURE 1. The Wigner function $W(x, p, t)$ given by formula (3.12) with $v = 1$, $k = 1/2$, and $t = 0$.

This example reveals a surprising result that a medium described by the quintic nonlinear Schrödinger equation (3.1) may allow, in principle, to measure the coordinate of a "particle" with any accuracy, below the so-called vacuum noise level and without violation of the Heisenberg uncertainty relation. The latter is a major obstacle, for example, in the direct detection of gravitational waves [54], [55].

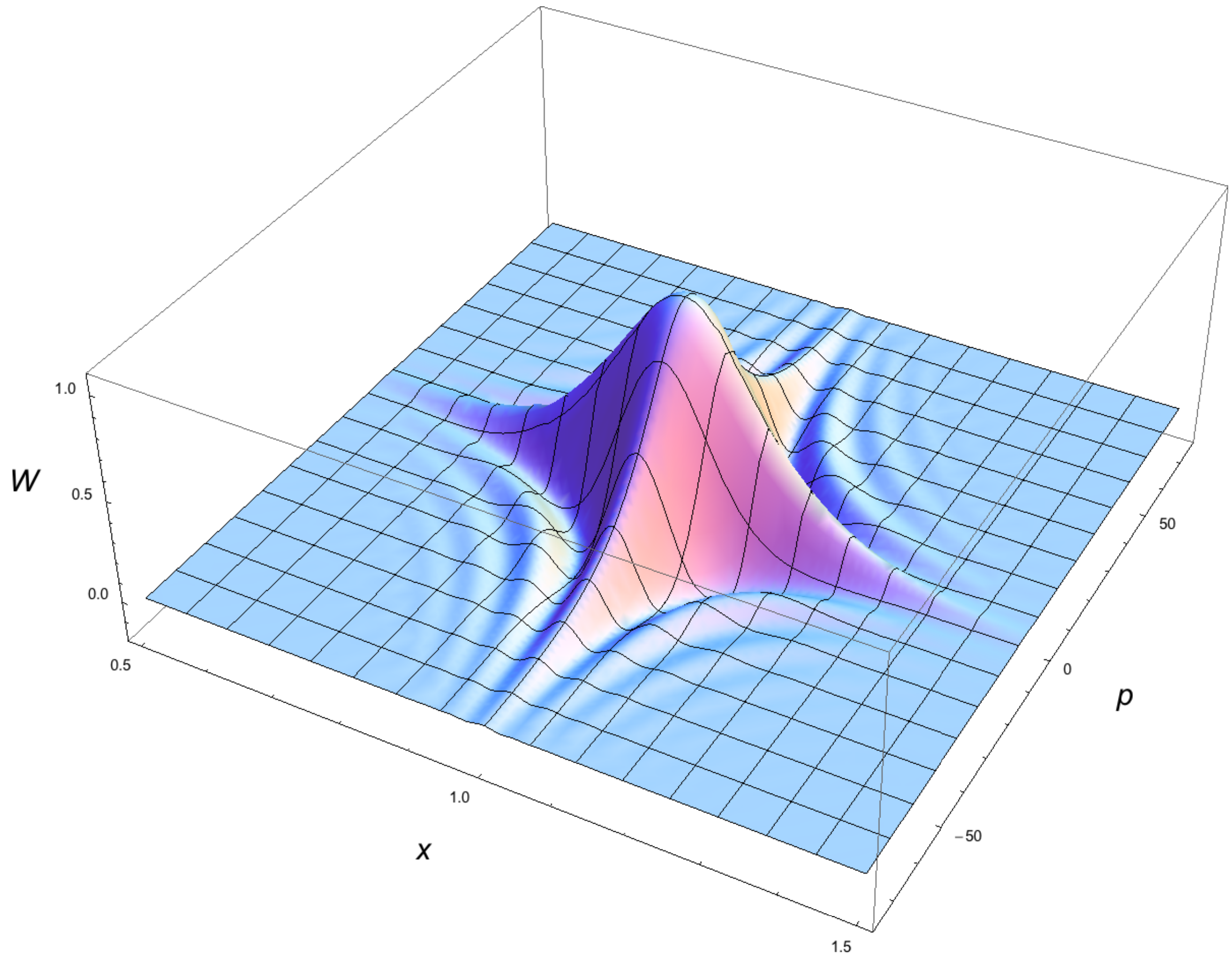

FIGURE 2. The Wigner function $W(x,p,t)$ given by formula (3.12) with $v=1$, $k=1/2$, and $t=0.75\approx\pi/4\approx 0.785$.

Oscillating sources and sinks:

$$\psi(x,t)=e^{i\phi}\sqrt{\frac{2r}{3^{1/2}\cos 2t}} \tag{3.15}$$
$$\times\left[1-\frac{3}{\cosh\left(\dfrac{2r}{\cos 2t}(x-v\sin 2t)\right)+2}\right]^{1/2}$$
$$\times\exp i\frac{2vx-(v^2+r^2+x^2)\sin 2t}{2\cos 2t}$$

(ϕ, v and r are real parameters, we have chosen the lower sign of the nonlinear term in (3.1)). Their detailed investigation will be given elsewhere.

3.2. **Extension.** In the general case, the action of Schrödinger group, say in our complex form (1.2)–(1.4), on (3.3) and/or (3.15) produces a six-parameter family of new oscillating solutions of equation (3.1). For example, the following extension of (3.3) holds:

$$\psi(x,t) = e^{i\phi}\sqrt{\frac{2k\beta(0)}{2\alpha(0)\sin 2t+\cos 2t}} \tag{3.16}$$
$$\times\ \operatorname{sech}^{1/2}2k\left(\beta(0)\frac{x-\delta(0)\sin 2t}{2\alpha(0)\sin 2t+\cos 2t}+\varepsilon(0)\right)$$
$$\times\exp i\frac{(2\alpha(0)\cos 2t-\sin 2t)\,x^2+\delta(0)\,(2x-\delta(0)\sin 2t)}{2\,(2\alpha(0)\sin 2t+\cos 2t)}$$
$$\times\exp i\left[\beta(0)\,\frac{k^2\beta(0)\sin 2t}{2\,(2\alpha(0)\sin 2t+\cos 2t)}\right],$$

which presents the most general solution of this kind. (We assume that $\gamma(0)=\kappa(0)=0$ for the sake of simplicity; see [37] for more details. Although the breather/pulsing solution, when $\alpha(0)=\delta(0)=\varepsilon(0)=0$ and $\beta(0)=1$, was already found in Ref. [18], our discussion of the uncertainty relation and evaluation of the Wigner function seems to be missing in the available literature.) The blow up occur periodically in time at the points

$$x_0 = \pm\frac{\delta(0)}{\sqrt{4\alpha^2(0)+1}},\qquad \cot 2t = -2\alpha(0). \tag{3.17}$$

Indeed, the expectation values and variances are given by

$$\overline{x} = \left(\delta(0)-\frac{2\alpha(0)\,\varepsilon(0)}{\beta(0)}\right)\sin 2t-\frac{\varepsilon(0)}{\beta(0)}\cos 2t, \tag{3.18}$$
$$\overline{p} = \left(\delta(0)-\frac{2\alpha(0)\,\varepsilon(0)}{\beta(0)}\right)\cos 2t+\frac{\varepsilon(0)}{\beta(0)}\sin 2t \tag{3.19}$$

and

$$(\delta x)^2 = \frac{\pi^2}{(4k)^2}\left(\frac{2\alpha(0)\sin 2t+\cos 2t}{\beta(0)}\right)^2, \tag{3.20}$$
$$(\delta p)^2 = \frac{\pi^2}{(4k)^2}\left(\frac{2\alpha(0)\cos 2t-\sin 2t}{\beta(0)}\right)^2 + \frac{k^2}{2}\left(\frac{\beta(0)}{2\alpha(0)\sin 2t+\cos 2t}\right)^2, \tag{3.21}$$

respectively. The uncertainty relation takes the form

$$(\delta p)^2(\delta x)^2 = \frac{\pi^2}{32}+\frac{\pi^4}{(4k)^4} \times\frac{(2\alpha(0)\cos 2t-\sin 2t)^2\,(2\alpha(0)\sin 2t+\cos 2t)^2}{\beta^4(0)} \geq\frac{\pi^2}{32}>\frac{1}{4} \tag{3.22}$$

and its minimum,

$$(\delta p)^2(\delta x)^2 = \frac{\pi^2}{32}, \tag{3.23}$$

occurs when $\tan 2t = 2\alpha(0)$ and

$$\max(\delta x)^2 = \frac{\pi^4}{(4k)^4}\beta^{-2}(0), \quad \min(\delta p)^2 = \frac{k^2}{2}\beta^2(0)$$

or when $\cot 2t = -2\alpha(0)$ and $\min(\delta x)^2 = 0$, $\max(\delta p)^2 = \infty$. The energy functional is equal to

$$\begin{aligned} E &= \overline{p^2} + \overline{x^2} - \frac{1}{4}\overline{|\psi|^4} \\ &= \frac{\pi^2}{(4k)^2}\frac{4\alpha^2(0)+1}{\beta^2(0)} \\ &\quad + \left(\delta(0) - \frac{2\alpha(0)\varepsilon(0)}{\beta(0)}\right)^2 + \frac{\varepsilon^2(0)}{\beta^2(0)} > 0. \end{aligned} \tag{3.24}$$

The corresponding Wigner function is given by our formula (3.12) with the following values of parameters:

$$\begin{aligned} \omega &= \frac{1}{2k\beta(0)} \\ &\times \left[(p - 2\alpha(0)x)\cos 2t + (2\alpha(0)p + x)\sin 2t - \delta(0)\right], \\ \vartheta &= 2k\left(\beta(0)\frac{x - \delta(0)\sin 2t}{2\alpha(0)\sin 2t + \cos 2t} + \varepsilon(0)\right). \end{aligned} \tag{3.25}$$

(We put $v = 0$ in (3.16)–(3.25) without loss of generality.)

Acknowledgments. The authors are grateful to Robert Conte for valuable discussions. We thank Kamal Barley and Oleksandr Pavlyk for assistance with graphics enhancement. This research was partially supported by an AFOSR grant FA9550-11-1-0220.

Appendix A. Integral Evaluation

The following integral,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{i\omega s}\, ds}{\sqrt{\cosh s + \cosh c}} &= \frac{\sqrt{2}\pi}{\cosh \pi\omega} \\ &\times {}_2F_1\left(\begin{matrix} \frac{1}{2} + i\omega, \ \frac{1}{2} - i\omega \\ 1 \end{matrix}; -\sinh^2\frac{c}{2}\right), \quad \left|\sinh\frac{c}{2}\right| < 1, \end{aligned} \tag{A.1}$$

can be derived as a special case of integral representation (2) on page 82 of Ref. [56]. The hypergeometric function is related to the Legendre associated functions, which are a special case of Jacobi functions, see [57], [58], [59]:

$$P_{1/2-i\omega}(\cosh c) = {}_2F_1\left(\begin{matrix} \frac{1}{2} + i\omega, \ \frac{1}{2} - i\omega \\ 1 \end{matrix}; -\sinh^2\frac{c}{2}\right) \tag{A.2}$$

and Mehler conical functions.

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School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ 85287–1804, U.S.A.

E-mail address: `mahalov@asu.edu`

School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ 85287–1904, U.S.A.

E-mail address: `sks@asu.edu`

URL: `http://hahn.la.asu.edu/~suslov/index.html`