# Evolutionary Games as Interacting Particle Systems

by

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ABSTRACT

This dissertation investigates the dynamics of evolutionary games based on the framework of interacting particle systems in which individuals are discrete, space is explicit, and dynamics are stochastic. Its focus is on 2-strategy games played on a $d$-dimensional integer lattice with a range of interaction $M$. An overview of related past work is given along with a summary of the dynamics in the mean-field model, which is described by the replicator equation. Then the dynamics of the interacting particle system is considered, first when individuals are updated according to the best-response update process and then the death-birth update process. Several interesting results are derived, and the differences between the interacting particle system model and the replicator dynamics are emphasized. The terms selfish and altruistic are defined according to a certain ordering of payoff parameters. In these terms, the replicator dynamics are simple: coexistence occurs if both strategies are altruistic; the selfish strategy wins if one strategy is selfish and the other is altruistic; and there is bistability if both strategies are selfish. Under the best-response update process, it is shown that there is no bistability region. Instead, in the presence of at least one selfish strategy, the most selfish strategy wins, while there is still coexistence if both strategies are altruistic. Under the death-birth update process, it is shown that regardless of the range of interactions and the dimension, regions of coexistence and bistability are both reduced. Additionally, coexistence occurs in some parameter region for large enough interaction ranges. Finally, in contrast with the replicator equation and the best-response update process, cooperators can win in the prisoner's dilemma for the death-birth process in one-dimensional nearest-neighbor interactions.

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Chapter 1

# THE FRAMEWORK OF EVOLUTIONARY GAME THEORY

## 1.1 The Origin of Evolutionary Game Theory

Game Theory originated in its modern form with the foundational work of John Von Neumann and Oskar Morgenstern (Neumann and Morgenstern 1944), which developed a mathematical framework for studying human behavior and optimal strategies in strategic games. The theory gained popularity and influence with John Nash's seminal work, which defined and proved the existence of a particular sort of equilibrium. An equilibrium of this kind would come to be known as a Nash equilibrium (Nash 1950).

In the century prior to this development of game theory, Charles Darwin introduced the notion of natural selection as the pathway of evolutionary change through descent with modification (Darwin 1876). In The Origins of Species, he described an individual's fitness as the propensity for the individual's traits to be passed into the next generation. This fitness depends, of course, on how well the individual survives and reproduces in its particular natural environment, but it also depends very importantly on how the individual interacts with others in the environment. This dependence on how individuals benefit or damage the fitness of others has led scientists and mathematicians alike to consider game theoretic pressures on evolution. In this setting, instead of trying to understand optimal strategies for rational players in a strategic game, we use game theory to understand the fixation of optimal (or suboptimal) traits in a population. John Maynard Smith and George R. Price in their pioneering 1973 article, The Logic of Animal Conflict, helped to unite evolutionary theory with the methods of game theory by defining an "evolutionarily stable strategy," or ESS (Maynard Smith and Price 1973). Evolutionary game theory originated as a framework for understanding biological evolution, but it has also been applied to economics (Friedman 1991) and other social sciences (Axelrod 1986; Nowak, Komarova, and Niyogi 2001) where traditional game theory has its roots.

A critical issue in evolutionary biology is understanding how the mechanisms at work in the evolutionary process influence the eventual outcome of evolution in a population. Evolutionary game theory approaches these questions by clearly defining how interactions between individuals will result in certain payoffs. By relating these payoffs to a Darwinian notion of fitness, evolutionary outcomes can be predicted in a variety of models. Traditionally, these models have ignored the effects of localized interactions and the process by which interactions are initiated (Lawlor and Smith 1976; Vincent and Brown 2005), as in the case of the popular replicator equation, and more recent selection-mutation models such as in Cleveland and Ackleh 2013. For a survey of deterministic, non-spatial evolutionary games, the reader is directed to Hofbauer and Sigmund 1998. However, stochastic and spatial processes have been shown to dramatically affect the evolutionary process, both in theory (Nowak, Bonhoeffer, and May 1994a; Ohtsuki and Nowak 2006; Kang and Lanchier 2012), and in observations of nature (Bolnick and Fitzpatrick 2007; Kerr et al. 2002). For a review of some spatial evolutionary games, the reader is directed to chapters 9-13 of Nowak 2006.

### 1.2 Two Strategy Symmetric Games

The most popularly studied games, and the ones studied in this work, are symmetric games with two available strategies. A symmetric game is one in which each player has the same set of strategies available to them, and the payoffs depend only on the strategies being played, not on which players are using the strategy. In this case, the payoffs can be summarized by a two-by-two payoff matrix $A = (a_{ij})$ where $a_{ij}$ is the payoff received by a player playing strategy $i$ when playing against someone playing strategy $j$, with $i, j \in \{1, 2\}$. The payoff matrix, therefore, has four parameters. The ordering of these parameters characterizes the game. We assume without loss of generality that strategy 1 is the strategy that satisfies $a_{21} > a_{12}$. There are $4! = 24$ orderings of the parameters, but there is not yet any distinction between strategies 1 and 2. Taking account for this symmetry, there are 12 distinct games. We now introduce the most popular two strategy symmetric games. Figure 1 accompanies the descriptions below to give an overview of how the strategies for each game fit into our evolutionary game models. Figure 2 exhibits how each game fits into a phase portrait of payoff parameters in the $a_{11}$, $a_{22}$–plane.

| Game | Strategy 1 | Strategy 2 |
|---|---|---|
| stag hunt | hare-hunter | stag-hunter |
| harmony game | cooperate | defect |
| deadlock game | cooperate | defect |
| prisoner's dilemma | cooperate | defect |
| hawk-dove game | dove | hawk |
| battle of the sexes | partner's preference | own preference |
| leader game | follower | leader |

Figure 1: *A list of some two-player two-strategy games, the strategies involved, and the ordering of the strategies that ensures the* $a_{21} > a_{12}$ *constraint is satisfied.*

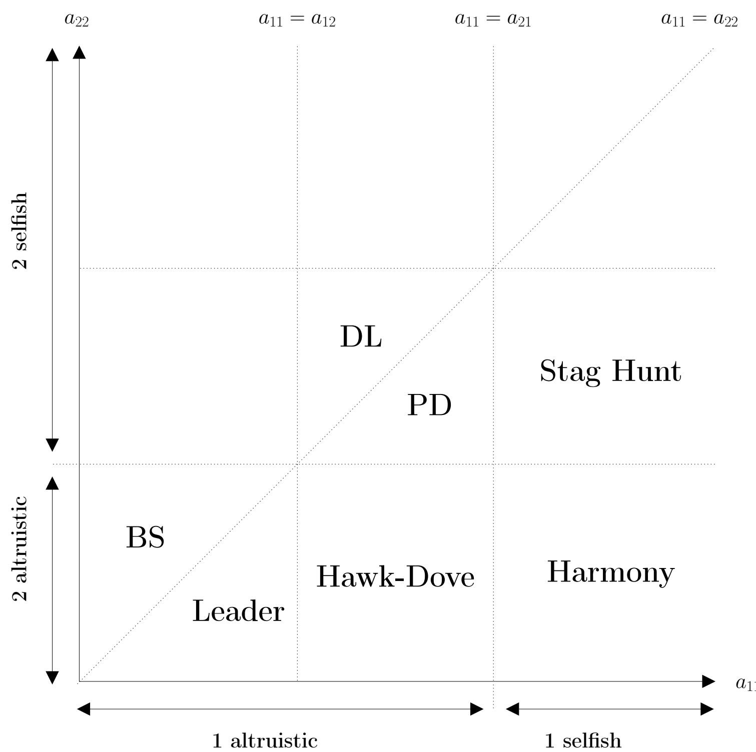



Figure 2: *A phase portrait demonstrating the parameter region associated to each two-player, two-strategy game. BS = Battle of the Sexes. DL = Deadlock. PD = Prisoner's Dilemma.*

Prisoner's Dilemma

The prisoner's dilemma is the most popularly studied symmetric two person game, and it represents social dilemmas faced in many real-world situations. The dilemma is typically described by a narrative in which two prisoners, say Alice and Bill, are being held for trial. They have each been brought in on a minor charge, but are suspected of committing a greater crime. The detective offers each, individually, a deal. If Alice will testify against Bill for the greater crime, then Alice will be acquitted of the lesser charge. The same deal is offered to Bill for testifying against Alice. Alice and Bill have a dilemma. If they cooperate with one another (neither testifies), then they each get the light sentence for the minor crime they have been arrested for. This light sentence is referred to as the reward payoff, $R$, a reward for mutual cooperation. But Alice serves her own best interest by testifying against Bill. As long as Bill does not also testify, then Alice has reduced her sentence. This reduced sentence is referred to as the temptation payoff, $T$. Bill's sentence in this case is the worst possible and is referred to as the sucker's payoff, $S$. He does not get the benefit from having testified against Alice, so he receives a sentence for both crimes. Bill clearly wants to avoid this situation, so he is inclined to testify against Alice as well. In the case that both testify, they both receive the sentence for the greater crime, but are released from the lesser sentence. We refer to this as the punishment payoff, $P$, a punishment for mutual defection. The payoffs are ordered as $T > R > P > S$. If we consider cooperation to be strategy 1 and defection to be strategy 2, then this payoff ordering is $a_{21} > a_{11} > a_{22} > a_{12}$. The only Nash equilibrium for this game is defection, but defection is clearly not the socially optimal strategy. In a non-spatial model of evolutionary games, the defectors always win, but we will show that for one of the spatial game models, cooperators will win in a subset of the prisoner's dilemma region.

Deadlock Game

The Deadlock game is identical to the Prisoner's Dilemma except that the payoff to mutual defectors is greater than the payoff to mutual cooperators. Thus, the payoffs are ordered as $T > P > R > S$, and there is no longer a dilemma; cooperation always leads to a lower payoff than defection regardless of whether the cooperation is mutual or not. As might be expected, for the Deadlock game, defectors win in both the non-spatial and spatial models examined in the following chapters.

Harmony Game

The Harmony game is another game of cooperators and defectors. Suppose that two players share common property, which is threatened to be damaged by an imminent storm. If the two work together (cooperate), they can secure the property and avoid any damage. This results in the greatest payoff $a_{11}$. An individual may choose not to work (defect) because the work is difficult, but then the property will only be partially secured and it will suffer some damage, which both individuals would equally like to avoid. The defector would receive a higher payoff than the cooperator, because both experienced the damage, but the defector at least didn't have to work. Thus, $a_{21} > a_{12}$. If both individuals defect by choosing not to work, then the property is a total loss, resulting in the smallest payoff $a_{22}$. So the payoff ordering for the Harmony Game is $a_{11} > a_{21} > a_{12} > a_{22}$.

### Hawk-Dove Game

Another popular symmetric two-person, two-strategy game is the Hawk-Dove game, first formulated in Maynard Smith and Price 1973. In this game, the two players are either playing as an aggressive hawk or a passive dove when they encounter each other at a food source, which they would both like to have. If both players are doves, they will share the food peaceably and each get half, a payoff of $V/2$. Calling the doves strategy 1, this means $a_{11} = V/2$. If both are hawks, they fight until one sustains an injury. The uninjured one gets the food, a payoff of $V$, and the injured one gets nothing but an injury, a payoff of $-C$. Assuming each hawk is evenly matched, the expected payoff to a hawk is $(V - C)/2 = a_{22}$ when encountering another hawk. The benefit to being a hawk comes when meeting a dove. The dove chooses not to fight, a payoff of $0 = a_{12}$, and the hawk gets the entire payoff $V = a_{21}$. If $V > C$, this is equivalent to a prisoner's dilemma. It is only referred to as a hawk-dove game if $V < C$, in which case the payoffs are ordered as $a_{21} > a_{11} > a_{12} > a_{22}$. Any game with this payoff ordering is considered a hawk-dove game. This game has also been formulated using different narratives, and may be referred to as the game of chicken or the snowdrift game according to the story in mind. In the non-spatial replicator equation, a hawk-dove game always results in coexistence, but in the spatial games it is possible for either hawks or doves to win.

### Stag Hunt Game

The stag hunt game is a fifth popular symmetric two-person, two-strategy game. The stag hunt is a dilemma between safety and social cooperation. In it a pair of hunters must choose to either hunt a stag or a hare. They would each prefer to fell a stag, but it will require both cooperating in order to do so. The hare can be caught without cooperation. If one leaves his home with stag-hunting gear and the other leaves his home with hare-hunting gear, the stag-hunter will catch neither and will starve, receiving the lowest payoff $L$, while the hare-hunter will eat the non-optimal, but adequate hare, a payoff $H$. If both leave their homes with stag-hunting gear, they will each eat the stag and receive the optimal payoff $S$. Calling strategy 1 the strategy of stag-hunting, then $a_{11} = S > a_{21} = H = a_{22} > a_{12} = L$. More generally, a game with the payoff ordering $a_{11} > a_{21} > a_{22} > a_{12}$ is also considered a stag hunt game. In replicator dynamics, the stag hunt game is a region of bistability: either all hunters will become stag-hunters or all hunters will become hare-hunters. Which strategy wins depends on the initial density of the strategy. However, in the spatial game models, this is not typically the case. For instance, in best response dynamics, it is true that all hunters will become either stag-hunters or all hunters will become hare-hunters, but the strategy that dominates is determined strictly by the payoffs and is independent of initial densities.

Battle of the Sexes

The battle of the sexes is a game in which the two players, say Alice and Bill, are imagined to be planning for a date. Alice would prefer to attend the opera, and Bill would prefer to attend the football game, but each prefers most of all to be with the other. This creates a situation in which if they both act altruistically (Alice choosing to attend the football game and Bill choosing to attend the opera), then they end up neither being together nor being at the event they were interested in. Letting strategy 1 be the strategy of altruism, $a_{11}$ is the lowest payoff available. On the other hand, if they are both selfish and attend their own preferred event, hoping that their partner will do the same, then they end up with a slightly higher payoff for attending an interesting event but being alone, $a_{22}$. The optimal payoff is awarded if one chooses the selfish strategy and the other chooses the altruistic strategy. In this case, they end up at the event together. The selfish player gets the highest payoff $a_{21}$ and the altruistic player gets the next highest payoff $a_{12}$. Thus the battle of the sexes game is seen as the game with payoff ordering $a_{21} > a_{12} > a_{22} > a_{11}$. The battle of the sexes is another game for which both strategies always coexist for the non-spatial game, but for some of the spatial game models, the region is divided between areas of coexistence and areas where the selfish strategy wins (driving out the altruistic strategy and making social optimality impossible).

Leader Game

The Leader game named in Rapoport 1967 is a game in which the two players are assumed to meet and then choose a leader, typically without communication. A standard framing of the game is to assume that each player drives a car and both arrive at opposite ends of a one-lane bridge. They must then determine who will go first, but without the presence of a standard rule for determining right-of-way. Each driver may choose one of two strategies: either to concede the right of way or to assert their right of way. The best situation is for one driver to assert and the other to concede. In this case, the assertive driver (leader) receives the highest available payoff, L, for the leader. The conceding driver (follower) receives the next highest available payoff, F, for the follower. A less optimal payoff, D for delay, is given if both drivers concede the right of way, while the worst payoff, A, is given in the socially awkward case that both drivers assert their right of way. With concession being called strategy 1 and assertion being strategy 2, this results in the payoff ordering $a_{21} > a_{12} > a_{11} > a_{22}$. In the non-spatial (well-mixed) model of evolutionary games, this game results in coexistence between the two strategies. However, in some of the spatial models studied here, it can be shown that for different parameter choices, the population may approach a state of all-leaders or a state of all-followers.

An alternative view of this parameter region is to frame it as a division of labor game. As such, we consider two players to be working together in a productive process. Each player may choose either to focus on sustenance (i.e. farming) or to focus on innovation (i.e. developing new farm tools). The task of sustenance is less desirable than the task of innovation, but both players benefit most from having one sustainer (strategy 1) and one innovator (strategy 2). If both players are innovators, then there is no one to provide the sustenance, resulting in the lowest payoff $a_{22}$. If both players are sustainers, then they are both content but not working optimally. This results in the next lowest payoff, $a_{11}$. Since sustenance is less desirable than innovation, but division of labor is preferred over the alternative, then $a_{21} > a_{12} > a_{11} > a_{22}$.

## 1.3 The Replicator Equation

A typical method for studying evolutionary dynamics in well-mixed populations is to use the replicator equation, which was introduced in Taylor and Jonker 1978 and studied further and named in Schuster and Sigmund 1983. Since our proposed study focuses on evolutionary games with two possible strategies, we will focus on that setting when describing the replicator equation. In this approach, the payoff a player receives from each game they play is taken to represent the player's fitness, and is frequency-dependent. Letting $u_i$ be the proportion of players using strategy $i$, the payoff of each type $i$ player is given according to the payoff matrix by

$$\phi_i(u_1, u_2) = a_{i1}u_1 + a_{i2}u_2 \qquad \text{for } i = 1, 2.$$

Treating this payoff as the growth rate for the set of players of strategy $i$, the replicator equation describes the evolution as

$$\dot{u}_i = u_i \left(\phi_i(u_1, u_2) - [\phi_1(u_1, u_2)u_1 + \phi_2(u_1, u_2)u_2]\right).$$

Applying the expression for $\phi_i$ and using the fact that $u_2 = 1 - u_1$, we arrive at the replicator equation describing the proportion of type 1 players:

$$\begin{aligned}
\dot{u}_1 &= u_1 \left(\phi_1(u_1, u_2) - \phi_2(u_1, u_2) - [\phi_1(u_1, u_2) - \phi_2(u_1, u_2)]\, u_1\right) \\
&= u_1(1 - u_1) \left[\phi_1(u_1, u_2) - \phi_2(u_1, u_2)\right] \\
&= u_1(1 - u_1) \left[(a_{11} - a_{21})u_1 + (a_{12} - a_{22})u_2\right] \\
&= u_1(1 - u_1) \left[(a_{11} - a_{21} - a_{12} + a_{22})u_1 - (a_{22} - a_{12})\right].
\end{aligned}$$

We introduce the notation

$$a_1 := a_{11} - a_{21} \qquad \text{and} \qquad a_2 := a_{22} - a_{12}$$

and some clarifying terminology by saying that strategy $i$ is:

- *altruistic* if $a_i < 0$ in the sense that a player with strategy $i$ confers a greater payoff to players of the opposite strategy than to players of her own strategy,
- *selfish* if $a_i > 0$ in the sense that a player with strategy $i$ confers a greater payoff to players of her own strategy than to players of the opposite strategy.

In these terms, the replicator equation for the proportion of type 1 players becomes

$$\dot{u}_1 = u_1(1 - u_1) \left[(a_1 + a_2)u_1 - (a_2)\right], \tag{1.1}$$

which has three fixed points:

$$e_1 := 1, \qquad e_2 := 0, \qquad \text{and} \qquad e_{12} = a_2(a_1 + a_2)^{-1}.$$

Some basic analysis shows that the limiting behavior of $u_1$ depends only on the signs of $a_1$ and $a_2$, and can be summarized by the following theorem:

**Theorem 1** For the replicator equation 1.1, with

$$e_1 := 1, \qquad e_2 := 0, \qquad \text{and} \qquad e_{12} = a_2(a_1 + a_2)^{-1},$$

the limiting behavior of $u_1$ is as follows:

*a*) If both strategies are selfish $(a_1, a_2 > 0)$, then $e_{12}$ is unstable and $u_1$ converges to $e_1$ or $e_2$, depending on whether $u_1$ is initially larger or smaller than $e_{12}$. Thus, the system is bistable.

*b*) If both strategies are altruistic $(a_1, a_2 < 0)$, then $e_{12}$ is globally stable, so $u_1$ converges to $e_{12}$ and the system exhibits coexistence.

*c*) When one strategy is selfish and the other altruistic, say $a_i > 0 > a_j$, then the selfish strategy, $i$, wins in the sense that $u_1$ converges to $e_i$.

In terms of evolutionarily stable strategies, strategy $i$ is an ESS if and only if strategy $i$ is selfish.

The replicator equation described above is also the mean-field model associated with the spatial game studied in Lanchier 2015 and summarized in appendix A, where payoffs affect birth and death rates. This connection can be made rigorous by considering an interacting particle system where the graph of interactions is the complete graph and time and space are properly rescaled. It is of particular interest that the mean-field models corresponding to each of the interacting particle systems analyzed in chapters 3-4 and appendix A have the same long-therm behavior as the replicator equation analyzed above, yet the behavior of these three models are all different when interactions are local. In nature, the assumption of a well-mixed population is not always realistic, so one needs to consider models that include a local interaction structure.

# Chapter 2

# SPATIAL EVOLUTIONARY GAMES

## 2.1 The Framework of Interacting Particle Systems

In order to include space and stochasticity in models of evolutionary games, we now describe the framework of interacting particle systems. An interacting particle system is a continuous-time Markov process $\xi_t$ on a configuration space $X := F^V$, where $V$ is a countable set of particles and $F$ is a compact metric space, which serves as the state space for each individual vertex $v \in V$. Throughout this document, we will take $V = \mathbb{Z}^d$ and $F = \{1, 2\}$. Thus,

$$\xi : (\mathbb{R}_+, \mathbb{Z}^d) \longrightarrow \{1, 2\}$$

with

$$\xi_t(x) := \xi(t, x) = \text{the state of vertex } x \in \mathbb{Z}^d \text{ at time } t \in \mathbb{R}_+.$$

Additionally, the state of any particle at a particular time depends only on its neighbors at an infinitesimally prior time, where the neighbors are determined by a prescribed graph and interaction (or dispersal) range. Interacting particle systems may be thought of as a more general framework in which Markov chains lie by considering the interacting particle system as a collection of Markovian particles whose fates are intertwined (and made non-markovian) by random interactions with one another. Another view of interacting particle systems is as a continuous-time and stochastic version of cellular automata.

The study of interacting particle systems began in the 1960s and early 1970s with the separate works of Harris (Harris 1965), Spitzer (Spitzer 1969; Spitzer 1970), Dobrushin (Dobrushin 1965; Dobrushin 1968), and Holley (Holley 1970), each constructing and studying a particular interacting particle system under various assumptions. Particular existence theorems were proved by each, and in Liggett 1972, Liggett proved existence for a general class of interacting particle systems. Interacting particle systems has been one of the most vibrant areas of probability theory since then. For a great introduction to and survey of early results in interacting particle systems, the reader is directed to Liggett 1985.

Models in the framework of interacting particle systems are not difficult to simulate, and these simulations can be very informative, but they can also be difficult to interpret and lead to erroneous conclusions. It is imperative, then, that analytic techniques be developed for studying these models. This is supported in several places, including Evilsizor and Lanchier 2014, where it is shown that on a $d$-dimensional lattice the dynamics of evolutionary games with the best-response update process contrast significantly with the dynamics suggested by simulation, which can only be conducted on finite graphs. Specifically, for an important parameter region, simulation suggests an end behavior of stationary coexistence, whereas we proved that in this region there is no coexistence and the more selfish strategy will take over at every location.

Some of the standard techniques in studying interacting particle systems include coupling arguments, duality techniques, and comparison to percolation models, but the study of interacting particle systems is rarely routine application of standard techniques.

## 2.2 Evolutionary Games as Interacting Particle Systems

The framework of interacting particle systems as applied to an individual-based evolutionary game model includes: an update rule for determining which individuals in a population are modified at any given time; an interaction structure, which determines which individuals can interact to influence these modifications; and a game description, which describes the payoff or fitness given to an individual by an interaction with others. In nature, interactions between individuals occur in a stochastic and spatially-dependent manner. The framework of interacting particle systems is suitable to explicitly incorporate these important factors. We focus on two-strategy, symmetric games played by individuals on a $d$-dimensional integer lattice.

In terms of interacting particle systems, we describe this process by a continuous-time Markov chain whose state at time $t$ is a spatial configuration

$$\xi_t : \mathbb{Z}^d \longrightarrow \{1, 2\},$$

In words, each site on the integer lattice $\mathbb{Z}^d$ is occupied by exactly one player who is characterized by her strategy. For this reason, we use the terms 'site,' 'vertex,' 'player,' and 'individual' interchangeably to describe $x \in \mathbb{Z}^d$ or the player at site $x$. The strategy at time $t \in [0, \infty)$ of the player occupying site $x \in \mathbb{Z}^d$ is given by $\xi_t(x)$. For a fixed interaction range $M$, we define the neighborhood of a site

$x \in \mathbb{Z}^d$ as

$$\mathcal{N}_x := \left\{ y \in \mathbb{Z}^d \mid 0 < d(x,y) \leq M \right\},$$

where $d(x,y)$ is a prescribed metric on $\mathbb{Z}^d$ (typically either the Manhattan distance or Chebyshev distance). The model's spatial structure is included in the form of local interactions according to $\mathcal{N}_x$ by assuming that each player's payoff only depends on the strategy of individuals in her neighborhood. More precisely, we define the $2 \times 2$ payoff matrix $A = (a_{ij})$ where $a_{ij}$ is interpreted as the payoff of a player holding strategy $i$ interacting with a player holding strategy $j$. Then the payoff to a player $x$ given the spatial configuration $\xi_t$ is defined as

$$\phi(x, \xi_t) := a_{i1}\, N_1(x, \xi_t) + a_{i2}\, N_2(x, \xi_t),$$

where $i = \xi_t(x)$ and $N_j(x, \xi)$ is the number of type $j$ neighbors of vertex $x$, i.e.,

$$N_j(x, \xi) \;:=\; \text{card}\,\{y \in \mathcal{N}_x \mid \xi(y) = j\}.$$

In this way, the payoff matrix $A$ together with the interaction neighborhood induce a payoff landscape for each configuration $\xi_t$. The strategy at each vertex is updated according to a given update rule, of which there are many choices. The update rule describes the rate at which a vertex updates its strategy and the way in which the strategy is updated. These choices depend in a prescribed way on the payoff landscape. In the following chapters, several different update rules will be considered.

### 2.2.1 Previous Work

A few others have studied evolutionary games using the framework of interacting particle systems. In Nowak and May 1992 and Nowak and May 1993, evolutionary games were analyzed as cellular automata. Subsequently Nowak, Bonhoeffer, and May 1994a and Nowak, Bonhoeffer, and May 1994b used simulations to explore the ways in which incorporation of continuous time and stochastic update rules affect the dynamics. Using continuous time and stochastic update rules is essentially a migration from cellular automata to interacting particle systems. In these articles, the authors focused primarily on the prisoner's dilemma and showed by simulation that local interactions can promote the coexistence of strategies in situations where cooperators would be eliminated by defectors if interactions were homogeneous. Langer, Nowak, and Hauert 2008 examines the prisoner's dilemma as interacting particle systems where the underlying graph is a finite square lattice in two dimensions, and the update rule is similar to that of the popular voter model (introduced in Holley and Liggett 1975 and Clifford and Sudbury 1973 as a model for spatial conflict), except that it deterministically excludes transitions to less-fit states and preferentially chooses the state of more successful neighbors. Fu, Nowak, and Hauert 2010 used the same interaction structure and update process as in the work of Langer et al. to compare spatial invasion in the prisoner's dilemma to that in the snowdrift game. In both games, simulations suggest that coexistence occurs when conditions are sufficiently favorable to cooperators, but there is a phase transition after which cooperators go extinct. By comparison with the replicator dynamics, these results suggest that spatial structure may inhibit or eliminate cooperation in the snowdrift game, but promote cooperation in the prisoner's dilemma. In both games, cooperators survive by clustering together, but the form of the clusters are qualitatively different between the two games.

The results given in the references above are mostly observations based on simulations. But interacting particle systems is a rigorous mathematical framework, and we need not rely on simulations alone. A few authors have carried out a rigorous analysis of evolutionary games using interacting particle systems. Some of that work is described next, and continuing that work is the focus of this thesis. Perhaps the first rigorous analysis of interacting particle models in evolutionary game theory are the works of Chen 2013 and Cox, Durrett, and Perkins 2013 verifying the benefit-to-cost ratio rule that was claimed by Ohtsuki et al. 2006 using compelling, though non-rigorous, arguments. The models considered by these authors used so-called weak selection, in which

$$\text{fitness } = (1-w) + w \times \text{ payoff} \quad \text{and} \quad w \to 0,$$

and relied on voter model perturbation techniques. The work of Durrett 2014 continues this analysis in a broader setting, using that in the case of weak selection, as $w \to 0$, the dynamics of the process approach that of reaction-diffusion equations, and the analysis again relies on voter model perturbations. In contrast, the perspective taken throughout this thesis uses strong selection, where $w = 1$ so that fitness = payoff, and the techniques involved vary widely according to the model and game (or parameter region) under consideration.

### 2.2.2 Three Spatial Evolutionary Game Models

In the present work, we consider three different models of evolutionary games. All three are models of stochastic, spatial evolutionary games on a $d$-dimensional integer lattice. The differences between the three models is in their update rule, and they are named accordingly. The three models are as described below:

1. Payoffs affecting birth and death rates: Players either die or give birth depending on the sign of their payoff, and at an exponential rate equal to the magnitude of their payoff. If the player dies, she is replaced by a neighbor chosen uniformly at random, and if the player gives birth, her offspring replaces a neighbor chosen uniformly at random.
2. Best response dynamics: Individuals are assumed to be rational. Each individual has the opportunity to update her strategy at rate one, and she changes her strategy if and only if it increases her payoff to do so.
3. Death-birth process: Individuals die at rate one and are replaced by a neighbor chosen at random with probability proportional to their payoff.

In this thesis, the dynamics of these three models are compared and contrasted with one another and with the replicator equation to illustrate the effect of space and stochasticity on evolutionary dynamics.

## 2.3 Main Results

In order to properly compare the spatial game models with their nonspatial analog (the replicator equation), we assume that each spatial game process starts from a spatially homogeneous distribution, i.e. a product measure in which the density of each of the two strategies is constant across the lattice. In order to state the following results, we must first define what we mean by several terms. For a spatial game $\eta_t(x)$:

- We say that strategy $i$ survives if there exists an $x \in \mathbb{Z}^d$ such that,

$$\liminf_{t\to\infty} P(\eta_t(x) = i) > 0.$$

- We say that two strategies, $i$ and $j$, coexist if

$$\liminf_{t\to\infty} P(\eta_t(x) = i, \eta_t(y) = j) > 0 \quad \text{for all } x, y \in \mathbb{Z}^d,\ x \neq y.$$

- We say that strategy $i$ wins if for each $x \in \mathbb{Z}^d$,

$$\lim_{t\to\infty} P(\eta_t(x) = i) = 1.$$

We also refer to a strategy $i$ as an evolutionarily stable strategy (ESS) if a population consisting only of strategy $i$ players cannot be invaded by an alternative strategy, given that the alternative strategy has a sufficiently small initial density. More precisely, strategy $i$ is an ESS if there exists a $p > 0$ sufficiently small that strategy $i$ wins for the process $\eta_t$ starting from the product measure with $P(\eta_0(x) = i) = 1 - p$ for each $x \in \mathbb{Z}^d$.

In the following description of the results, we always assume that the strategies are named strategy 1 and strategy 2 with the numbers always chosen such that $a_{21} > a_{12}$. This simplifies our description of the results without a loss of generality. We also use the terms selfish and altruistic as defined above. For a meaningful comparison with the replicator dynamics, we always assume that the initial state, $\eta_0$, for the spatial game is a product measure with density $p \in (0, 1)$ and $P(\eta_0(x) = 1) = p$ for each $x \in \mathbb{Z}^d$. Phase portraits giving an overview of the results for each model are given in Figures $3 - 6$. For the Death-Birth process and Payoffs Affecting Birth and Death rates, more details are available for the 1-dimensional, nearest-neighbors case, so a phase portrait in this case is provided for these two models in Figures 7 and 8. Just after the following description of the results, the results for the death-birth process and best-response dynamics are proved. These results and proofs constitute the main work of this thesis. The results shown here for the Best Response Dynamics also appear in the author's publication Evilsizor and Lanchier 2014. The results shown here for the Death-Birth Process also appear in the work Evilsizor and Lanchier 2016. Both works were co-authored with Nicolas Lanchier.

### Dynamics of the spatial games when both strategies are selfish

In this section, we assume that both strategies are selfish: that is, $a_{11} > a_{21}$ and $a_{22} > a_{12}$. We have shown that under replicator dynamics, the system is bistable. Strategy 1 wins if the initial density of strategy 1 players is sufficiently high, and strategy 2 wins otherwise. We compare this result with the dynamics of the three spatial game models mentioned above and show that in the spatial games, this region of bistability is reduced or eliminated. In all three spatial game models, we show that there are regions in the selfish-selfish parameter space for which strategy 2 wins regardless of the initial density $p$, and other regions where strategy 1 wins regardless of the initial density $p$. In particular, for Best Response Dynamics, we show that the bistability region is completely eliminated and the more selfish strategy wins regardless of the initial density. Here, we consider strategy $i$ to be "more selfish" if $a_{ii} - a_{ji} > a_{jj} - a_{ij}$.

In terms of the most popular games, mentioned in section 1.2, we may partially summarize these results by saying that under replicator dynamics, the stag hunt game is always bistable, while including space and stochasticity encourages one or the other strategy to win regardless of the initial density. In particular, for Best Response Dynamics, the whole population becomes stag-hunters for a portion of the parameter region, while the whole population becomes hare-hunters for the remaining portion of the parameter region.

Dynamics of the spatial games when both strategies are altruistic

In this section, we assume that both strategies are altruistic: that is, $a_{11} < a_{21}$ and $a_{22} < a_{12}$. In the case of replicator dynamics, we have shown that the population reaches a stable equilibrium with coexistence of the two strategies. Our results for the spatial games are of two forms. First, we show that this region of coexistence is reduced for two of the three spatial games, but not for the Best Response Dynamics. In particular, for the Death-Birth process and for Payoffs Affecting Birth and Death Rates, we exhibit a parameter region where strategy 2 wins in these two models, but where there was coexistence in the replicator dynamics and in Best Response Dynamics. This region of coexistence can be even further reduced with Payoffs Affecting Birth and Death Rates, and we show that for a particular parameter region, type 1 wins where there is coexistence in the replicator dynamics and the best response dynamics.

The second sort of result we have in the case of two altruistic strategies is to show that while the coexistence region is reduced in the Death-Birth process and Payoffs Affecting Birth and Death Rates, it is not removed entirely. That is, we can still find parameter choices such that these spatial games exhibit coexistence. For the Death-Birth process and payoffs affecting birth and death rates, coexistence requires a large interaction range as well as a particular choice of parameters.

In terms of the most popular games, mentioned in section 1.2, we may partially summarize these results by saying that under replicator dynamics and Best Response Dynamics, there is always coexistence between strategies in the Battle of the Sexes, Leader, and Hawk Dove games. There is also coexistence in the Death-Birth process and for Payoffs Affecting Birth and Death Rates, but only for a reduced parameter space. In these two spatial game models, there are some parameterizations of all three games in which strategy 2 wins. For Payoffs Affecting Birth and Death Rates, it is also possible for strategy 1 (doves) to win in particular parameterizations of the Hawk-Dove game.

### Dynamics of the spatial games when one strategy is selfish and the other is altruistic

Lastly, we describe the dynamics when one strategy is altruistic and the other is selfish. This is the case for the deadlock game, the prisoner's dilemma, and the harmony game. In all four models, strategy 2 (the selfish strategy) wins for the deadlock game. Some disagreements between the models arise in the prisoner's dilemma game. For the replicator equation and Best Response Dynamics, strategy 2 (the defectors) always win in the prisoner's dilemma. The same is true for Payoffs Affecting Birth and Death Rates if we restrict ourselves to the one-dimensional case with nearest neighbor interactions ($M = 1$), and simulations suggest that this is the case for higher dimensions and longer interaction ranges as well. This is not the case for the Death-Birth process. In the Death-Birth process, we can show that in the one-dimensional, nearest neighbor case, the cooperators (strategy 1) will actually win in the prisoner's dilemma for a particular parameter region, and simulations suggest that there will be coexistence for a third portion of the prisoner's dilemma parameter region.

In the harmony game, the dynamics are again simple to describe for the replicator equation and best response dynamics. In both cases, strategy 1 (the strategy of cooperation) wins. This is also true of the Death-Birth process and Payoffs Affecting Birth and Death rates, but only in a particular subset of the harmony game parameter region. Simulations suggest that defectors (strategy 2) can win in the harmony game if $a_{21} - a_{12}$ is sufficiently large in relation to $a_{11}$.

### An important note on the results

It is important to note that the dynamics of all four models differ from one another. The replicator dynamics were most similar to the best response dynamics, while the Death-Birth Process and Payoffs Affecting Birth and Death Rates were similar to one another and very different from the previous two models. This indicates that both the interaction structure and the update rule is a very influential characteristic of evolutionary game models. We observe that the inclusion of local interactions may or may not affect dynamics in each particular game, and it is difficult to describe in broad strokes the way that local interactions will change the dynamics. Namely, we observe that the inclusion of space favored the selfish strategy in some cases and favored the altruistic strategy in others. It had a tendency to reduce coexistence and bistability for the most part, but it expanded the region of coexistence into the prisoner's dilemma region for the death-birth process.

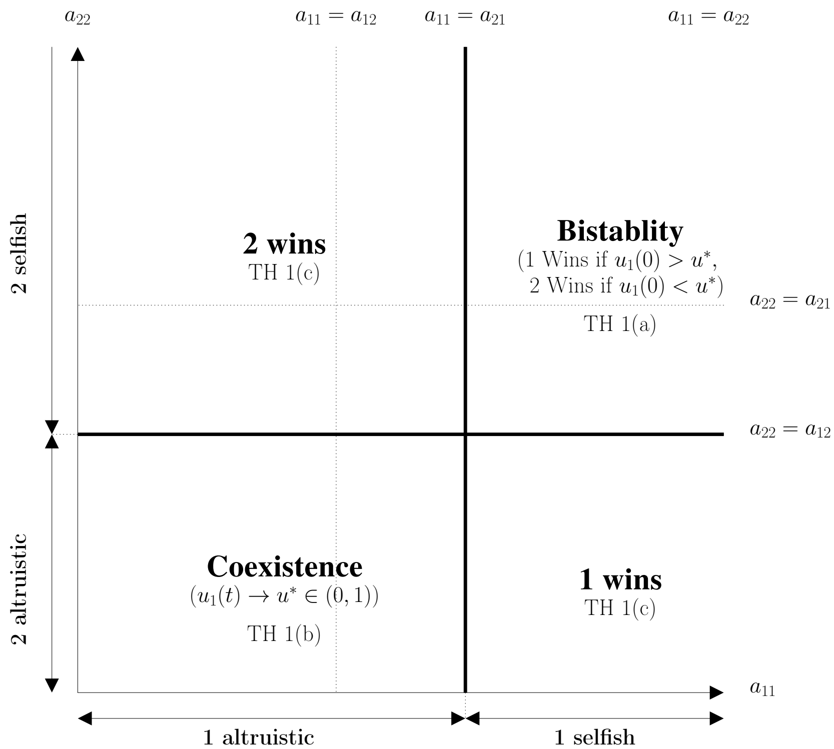



Figure 3: *Phase portrait for the replicator dynamics on* $\mathbb{Z}^d$.

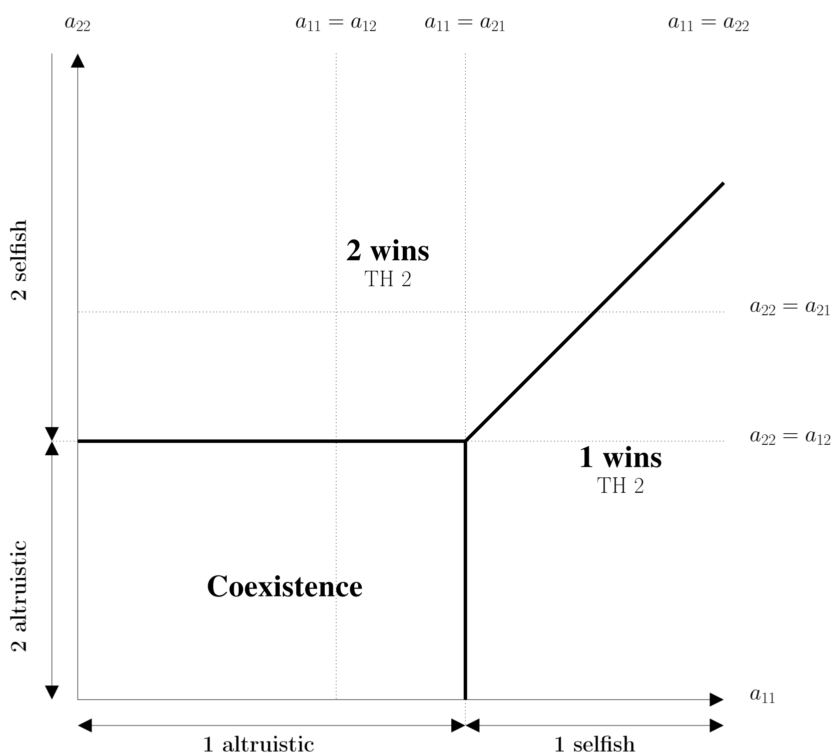



Figure 4: *Phase portrait for the best response dynamics on* $\mathbb{Z}^d$.

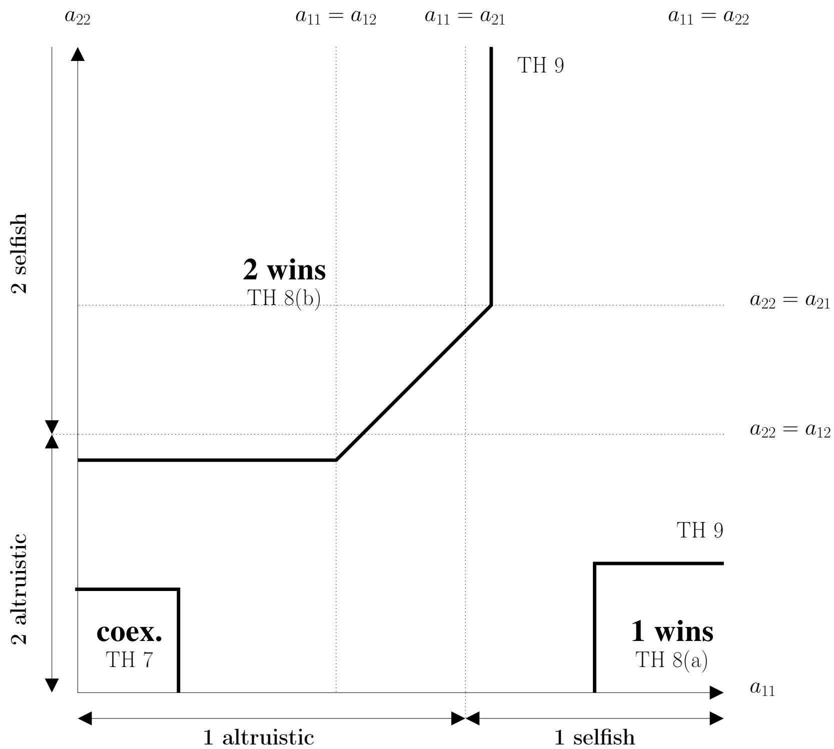



Figure 5: *Phase portrait for the death birth process on* $\mathbb{Z}^d$.

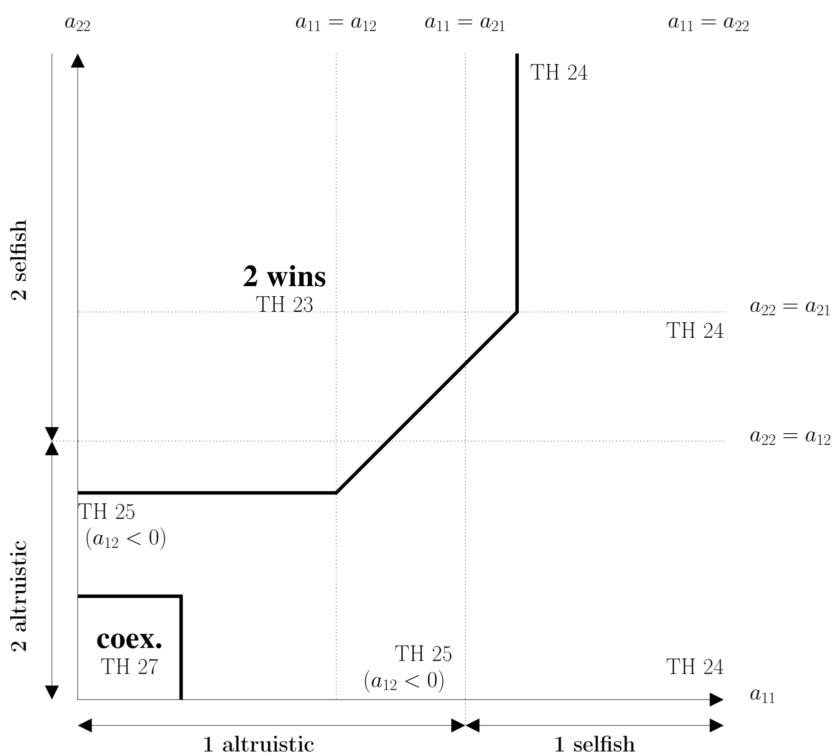



Figure 6: *Phase portrait for the payoffs affecting birth and death rates on* $\mathbb{Z}^d$.

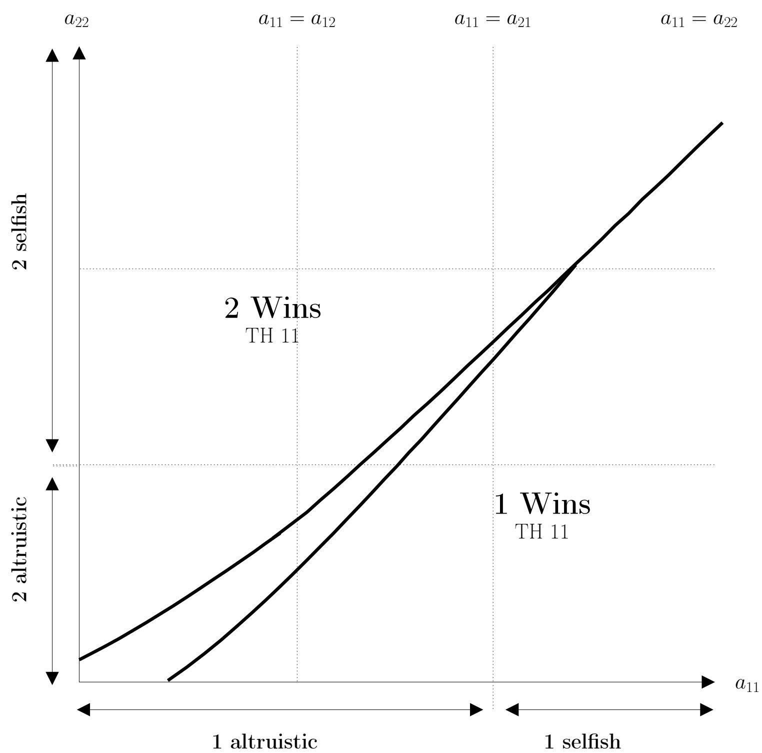



Figure 7: *Phase portrait for nearest-neighbor death birth process on $\mathbb{Z}$.*

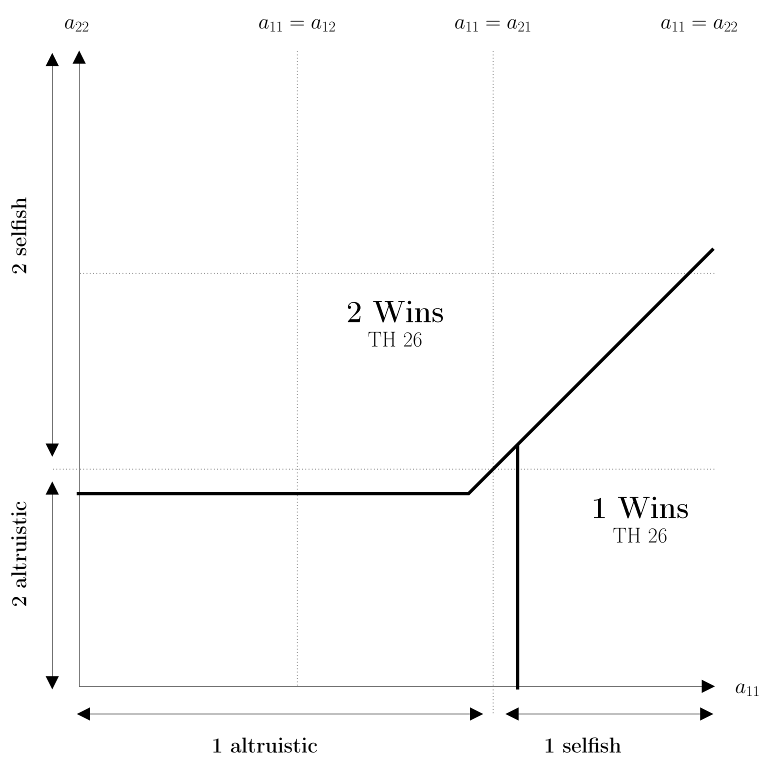



Figure 8: *Phase portrait for nearest-neighbor process of payoffs affecting birth and death rates on $\mathbb{Z}$.*

Chapter 3

# BEST-RESPONSE DYNAMICS

## 3.1 Introduction

The best-response dynamics is an example of an evolutionary game where players update their strategy in order to maximize their payoff. The main objective of this chapter is to study a stochastic spatial version of this game based on the framework of interacting particle systems in which players are located on an infinite square lattice. The contents of this chapter have been previously published as Evilsizor and Lanchier 2014. In the presence of two strategies, and calling a strategy selfish or altruistic depending on a certain ordering of the coefficients of the underlying payoff matrix, a simple analysis of the nonspatial mean-field approximation of the spatial model shows that a strategy is evolutionarily stable if and only if it is selfish, making the system bistable when both strategies are selfish. The spatial and nonspatial models agree when at least one strategy is altruistic. In contrast, we prove that in the presence of two selfish strategies and in any spatial dimension, only the most selfish strategy remains evolutionarily stable. The main ingredients of the proof are monotonicity results and a coupling between the best-response dynamics properly rescaled in space with bootstrap percolation to compare the infinite time limits of both systems.

The framework of evolutionary game theory, which describes the dynamics of populations of individuals identified to players, has been initiated by theoretical biologist Maynard Smith and first appeared in his work with Price (Maynard Smith and Price 1973). Each individual-player is characterized by one of a finite number $n$ of possible strategies and is attributed a payoff that is calculated based on the strategy of the surrounding players and an $n \times n$ payoff matrix. The most popular model of evolutionary game is probably the so-called replicator equation reviewed in Hofbauer and Sigmund 1998, a system of deterministic differential equations for the frequencies of players holding a given strategy. This chapter is a sequel of the work in Lanchier 2015 continuing the analytical study of evolutionary games based on the framework of interacting particle systems which, in contrast with the replicator equation, also includes stochasticity and space in the form of local interactions.

**Model description** – The version of the best-response dynamics we consider in this chapter is a continuous-time Markov chain whose state at time $t$ is a spatial configuration

$$\eta_t : \mathbb{Z}^d \longrightarrow \{1, 2\} := \text{the set of strategies.}$$

In words, each point of the $d$-dimensional square lattice is occupied by exactly one player who is characterized by her strategy. The spatial structure is included in the form of local interactions assuming that each player's payoff only depends on the strategy of her $2d$ neighbors. More precisely, having a two by two payoff matrix $A = (a_{ij})$ where $a_{ij}$ is interpreted as the payoff of a player holding strategy $i$ interacting with a player holding strategy $j$, each configuration is turned into a so-called payoff landscape that attributes a payoff to each vertex as follows:

$$\begin{aligned} \phi(x, \eta_t) \ &:= \ (a_{11}\, N_1(x, \eta_t) + a_{12}\, N_2(x, \eta_t))\ \mathbf{1}\{\eta_t(x) = 1\} \\ &\qquad + \ (a_{21}\, N_1(x, \eta_t) + a_{22}\, N_2(x, \eta_t))\ \mathbf{1}\{\eta_t(x) = 2\} \quad \text{for all} \quad x \in \mathbb{Z}^d \end{aligned}$$

where $N_j(x, \eta_t)$ is the number of type $j$ neighbors of vertex $x$, i.e.,

$$N_j(x, \eta_t) \ := \ \text{card}\, \{y \in \mathbb{Z}^d : y \sim x \text{ and } \eta_t(y) = j\}$$

where the binary relationship $\sim$ indicates that two vertices are neighbors. In the traditional framework of evolutionary game theory, each strategy is often interpreted as a trait and each payoff defined through the payoff landscape as a fitness or reproductive success. In particular, evolutionary game theory makes the implicit assumption that players are not rational decision-makers who can choose their strategy and that the evolution of the system is driven by births and deaths. In contrast, the best-response dynamics assumes that players are rational decision-makers changing their strategy in order to maximize their payoff. Specifically, we assume that each player updates her strategy at an exponential rate one choosing to change her strategy if and only if it increases her payoff. In particular, in case of a tie, i.e., the player would not change her payoff by changing her strategy, nothing happens. More precisely, letting

$$\begin{aligned} \phi_1(x, \eta_t) \ &:= \ a_{11}\, N_1(x, \eta_t) + a_{12}\, N_2(x, \eta_t) \quad \text{for all} \quad x \in \mathbb{Z}^d \\ \phi_2(x, \eta_t) \ &:= \ a_{21}\, N_1(x, \eta_t) + a_{22}\, N_2(x, \eta_t) \quad \text{for all} \quad x \in \mathbb{Z}^d \end{aligned} \tag{3.1}$$

be the payoff that the player at $x$ would receive if she followed strategy 1 and 2, respectively, the best-response dynamics is formally described by the Markov generator

$$\begin{aligned} Lf(\eta_t) \ &= \ \textstyle\sum_x \mathbf{1}\{\phi_1(x, \eta_t) > \phi_2(x, \eta_t)\}\ [f(\eta_t^{x,1}) - f(\eta_t)] \\ &\qquad + \ \textstyle\sum_x \mathbf{1}\{\phi_1(x, \eta_t) < \phi_2(x, \eta_t)\}\ [f(\eta_t^{x,2}) - f(\eta)] \end{aligned} \tag{3.2}$$

where the configuration $\eta_t^{x,i}$ is obtained from $\eta_t$ by setting to $i$ the strategy at $x$ and leaving the strategy at the other vertices unchanged. Note that, for any given vertex $x$, the difference between the two alternative payoffs in (3.1) can be written as

$$\begin{aligned} \phi_1(x, \eta_t) - \phi_2(x, \eta_t) &= (a_{11}\, N_1(x, \eta_t) + a_{12}\, N_2(x, \eta_t)) - (a_{21}\, N_1(x, \eta_t) + a_{22}\, N_2(x, \eta_t)) \\ &= (a_{11} - a_{21})\, N_1(x, \eta_t) - (a_{22} - a_{12})\, N_2(x, \eta_t). \end{aligned}$$

In particular, the dynamics only depends on $a_1 := a_{11} - a_{21}$ and $a_2 := a_{22} - a_{12}$ rather than all four coefficients of the payoff matrix so the Markov generator (3.2) can be written as

$$\begin{aligned} Lf(\eta_t) \ = \ \textstyle\sum_x \mathbf{1}\{a_1\, N_1(x, \eta_t) > a_2\, N_2(x, \eta_t)\}\ [f(\eta_t^{x,1}) - f(\eta_t)] \\ + \textstyle\sum_x \mathbf{1}\{a_1\, N_1(x, \eta_t) < a_2\, N_2(x, \eta_t)\}\ [f(\eta_t^{x,2}) - f(\eta_t)]. \end{aligned} \tag{3.3}$$

Since the behavior of the system strongly depends on the sign of $a_1$ and $a_2$, it is convenient to use the terminology introduced in Lanchier 2013, 2015 by declaring strategy $i$ to be

- **altruistic** when $a_i < 0$, meaning that a player with strategy $i$ confers a lower payoff to a player following the same strategy than to a player following the other strategy,
- **selfish** when $a_i > 0$, meaning that a player with strategy $i$ confers a higher payoff to a player following the same strategy than to a player following the other strategy.

**Mean-field approximation** – To understand the role of space in the long-term behavior of the best-response dynamics, the first step is to look at the deterministic nonspatial version, or mean-field approximation, of the process (3.3). This mean-field model is obtained under the assumption that the population is well-mixed, and more precisely by looking at the process on the complete graph in which any two players are neighbors and then taking the limit as the number of vertices tends to infinity. This results in a system of differential equations for the frequency of players holding strategy $i$ that we denote by $u_i$. In the absence of a spatial structure, the payoff that a player would receive if she followed strategy 1 and 2, respectively, is

$$\phi_1(u_1, u_2) \ = \ a_{11}\, u_1 + a_{12}\, u_2 \qquad \text{and} \qquad \phi_2(u_1, u_2) \ = \ a_{21}\, u_1 + a_{22}\, u_2$$

which can be viewed as the nonspatial analog of (3.1). Also, under the evolution rules of the best-response dynamics, either each type 1 player or each type 2 player changes her strategy at an exponential rate one depending on whether $\phi_1 - \phi_2$ is negative or positive, respectively. Then, rescaling time by the number of vertices and taking the limit as the number of vertices tends to infinity gives the following differential equation for the frequency of type 1 players:

$$\begin{aligned} u_1'(t) &= u_2\ \mathbf{1}\{\phi_1(u_1,u_2) > \phi_2(u_1,u_2)\}\ -\ u_1\ \mathbf{1}\{\phi_1(u_1,u_2) < \phi_2(u_1,u_2)\} \\ &= u_2\ \mathbf{1}\{a_1\,u_1 > a_2\,u_2\}\ -\ u_1\ \mathbf{1}\{a_1\,u_1 < a_2\,u_2\} \\ &= u_2\ \mathbf{1}\{(a_1+a_2)\,u_1 > a_2\}\ -\ u_1\ \mathbf{1}\{(a_1+a_2)\,u_1 < a_2\} \end{aligned} \tag{3.4}$$

where we used that $u_1 + u_2 = 1$. Letting $u_* := a_2\,(a_1+a_2)^{-1}$, we have

$$\begin{aligned} u_1'(t) &= +u_2 \quad \text{when} \quad (u_1 > u_* \text{ and } a_1+a_2 > 0) \quad \text{or} \quad (u_1 < u_* \text{ and } a_1+a_2 < 0) \\ u_1'(t) &= -u_1 \quad \text{when} \quad (u_1 > u_* \text{ and } a_1+a_2 < 0) \quad \text{or} \quad (u_1 < u_* \text{ and } a_1+a_2 > 0) \end{aligned}$$

which shows the following four possible regimes:

- when strategy 1 is selfish and strategy 2 altruistic, strategy 1 wins in the sense that starting from any initial condition $u_1(t) \to 1$ as $t \to \infty$.
- when strategy 1 is altruistic and strategy 2 selfish, strategy 2 wins in the sense that starting from any initial condition $u_1(t) \to 0$ as $t \to \infty$.
- when both strategies are altruistic, coexistence occurs in the sense that starting from any initial condition $u_1(t) \to u_* \in (0,1)$ as $t \to \infty$.
- when both strategies are selfish, the system is bistable:

$$\begin{aligned} u_1(t) \to 0 \ \text{ as } \ t \to \infty \quad \text{when} \quad u_1(0) < u_* \in (0,1) \\ u_1(t) \to 1 \ \text{ as } \ t \to \infty \quad \text{when} \quad u_1(0) > u_* \in (0,1). \end{aligned}$$

In terms of evolutionarily stable strategy, this indicates that, for well-mixed populations, a strategy is evolutionarily stable if it is selfish but not if it is altruistic. Recall that an evolutionarily stable strategy is defined as a strategy which, if adopted by a population, cannot be invaded by any alternative strategy starting at an infinitesimally small frequency. We point out that the mean-field model derived here is slightly different than the replicator equation described by section 1.3, but the end behavior is the same and is summarized by Theorem 1.

**Spatial stochastic model** – We now return to the spatial model (3.3) looking at the four parameter regions corresponding to the four possible regimes of the mean-field approximation. Assuming first that strategy 1 is selfish and strategy 2 altruistic, we get

$$a_1\,N_1(x,\eta_t) - a_2\,N_2(x,\eta_t) \ = \ a_1\,N_1(x,\eta_t) + (-a_2)(2d - N_1(x,\eta_t)) \ > \ 0$$

for all $x \in \mathbb{Z}^d$ and all configuration $\eta_t$. This shows that each type 2 player changes her strategy at an exponential rate one whereas each type 1 player sticks to her strategy, therefore strategy 1 wins, just as in the mean-field model, in the sense that for any initial configuration

$$\lim_{t \to \infty} P(\eta_t(x) = 1) \ = \ 1 \quad \text{for all} \quad x \in \mathbb{Z}^d.$$

By symmetry, strategy 2 wins whenever strategy 1 is altruistic and strategy 2 selfish. Note in particular that the "all 1" and "all 2" configurations are not necessarily absorbing states for the process. This is due to the fact that, though the new strategy is chosen based on the strategy of the neighbors, it is not chosen from the neighborhood.

Looking now at altruistic-altruistic interactions, whenever the player at $x$ and all her neighbors follow the same strategy,

$$\begin{array}{rcl} a_1 \, N_1(x, \eta_t) - a_2 \, N_2(x, \eta_t) & = & + \, 2d \, a_1 \ < \ 0 \quad \text{when} \quad \eta_t(x) = 1 \\ a_1 \, N_1(x, \eta_t) - a_2 \, N_2(x, \eta_t) & = & - \, 2d \, a_2 \ > \ 0 \quad \text{when} \quad \eta_t(x) = 2. \end{array}$$

In either case, the player at $x$ changes her strategy at an exponential rate one, indicating that, as in the mean-field model, two altruistic strategies coexist in the sense that

$$\lim_{t \to \infty} P(\eta_t(x) = \eta_t(y)) \ < \ 1 \quad \text{for all} \quad x, y \in \mathbb{Z}^d, \ x \neq y.$$

We now study the process when both strategies are selfish, a case more challenging mathematically and also more interesting as it shows some important disagreements between the spatial and nonspatial models. To confront our results for the spatial model with the bistability displayed by its nonspatial counterpart, we consider the process starting from the product measure with

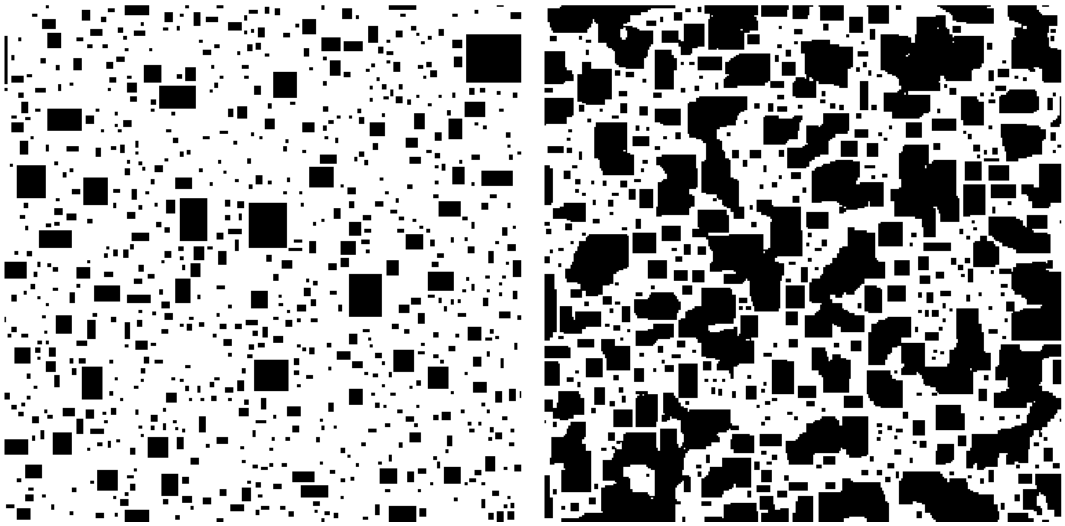
$$P(\eta_0(x) = 1) \ =: \ p \quad \text{for all} \quad x \in \mathbb{Z}^d$$

and compare the models when $p = u_1(0)$. The fact that the inclusion of space in the form of local interactions strongly affects the long-term behavior of the system can be seen in a specific parameter region using a standard coupling with the Richardson model (Richardson 1973). Indeed, let

$$c(x, \eta_t) \ := \ \lim_{h \to 0} P(\eta_{t+h}(x) \neq \eta_t(x) \,|\, \eta_t). \tag{3.5}$$

Then, when $a_1 > (2d - 1) \, a_2 > 0$ and $N_1(x, \eta_t) \geq 1$, we have

$$\begin{array}{rcl} c(x, \eta_t \,|\, \eta_t(x) = 1) & = & \mathbf{1}\{a_1 \, N_1(x, \eta_t) < a_2 \, N_2(x, \eta_t)\} \ \leq \ \mathbf{1}\{a_1 < (2d - 1) \, a_2\} \ = \ 0 \\ c(x, \eta_t \,|\, \eta_t(x) = 2) & = & \mathbf{1}\{a_1 \, N_1(x, \eta_t) > a_2 \, N_2(x, \eta_t)\} \ \geq \ \mathbf{1}\{a_1 > (2d - 1) \, a_2\} \ = \ 1 \end{array}$$



(a) $a_1 = 1.01 > a_2 = 1$ *and* $p = 0.15$ (b) $a_1 = 1.01 > a_2 = 1$ *and* $p = 0.20$

Figure 9: *Best-response dynamics on a* $300 \times 300$ *lattice with periodic boundary conditions starting from a product measure with density* p *of type 1 players in black. In (a), the process hits an absorbing state in which both types are present, whereas in (b), which shows a snapshot of the process at time 25, the system is converging to the all black configuration: strategy 1 wins.*

almost surely. These two inequalities imply that the set of type 1 players dominates stochastically the set of infected sites in the Richardson model $\pi_t$ with initial configuration

$$\pi_0(x) \ = \ \mathbf{1}\,\{\eta_0(x) = 1 \text{ and } \eta_0(y) = 1 \text{ for some } y \sim x\}$$

which, in turn, implies that strategy 1 wins whenever $p > 0$. This shows in particular the existence of parameter regions in which, in contrast with the nonspatial model, only the most selfish strategy is evolutionarily stable for the spatial model. Returning to general selfish-selfish interactions, the numerical simulations of the two-dimensional process displayed in Figure 9 suggest that, when $a_1$ is slightly larger than $a_2$ and the initial density $p > 0$ is small, the system fixates to a configuration in which the set of type 1 players consists of a union of disjoint rectangles, indicating that strategy 1 is unable to invade strategy 2. These simulations, however, are misleading due to the finiteness of the graph, and it can be proved that, in any dimension, the most selfish strategy always wins even when starting at a low density. More precisely, we have the following theorem.

**Theorem 2** – Assume that $a_1 > a_2 > 0$ and $p > 0$. Then,

$$\lim_{t \to \infty} P\,(\eta_t(x) = 1) \ = \ 1 \quad \text{for all} \quad x \in \mathbb{Z}^d.$$

In particular, while any selfish strategy is evolutionarily stable in the nonspatial model, only the most selfish strategy is evolutionarily stable in the spatial model. The result in one dimension directly follows from our coupling with the Richardson model since

$$(2d-1)\,a_2 \;=\; a_2 \quad \text{when} \quad d=1$$

while the general result relies on a combination of monotonicity results and coupling arguments to compare the best-response dynamics with bootstrap percolation. More precisely, we first prove that, in the presence of selfish-selfish interactions, the best-response dynamics is attractive, which allows us to focus on the process starting from a certain reduced configuration that consists of a union of hyperrectangles. The second ingredient is to show that, for the process starting from this reduced configuration, the set of type 1 players is a pure growth process, just like the Richardson model. This strong monotonicity result is then applied repeatedly to show that the best-response dynamics properly rescaled in space dominates stochastically bootstrap percolation with parameter $d$. From this domination and a result due to Schonmann Schonmann 1992, Theorem 3.1, we finally deduce that, unlike what Figure 9 suggests, the most selfish strategy indeed invades the entire lattice.

### 3.2 Some Monotonicity Results

To avoid cumbersome notations, it is convenient to sometimes think of the state of the process as a subset rather than a function by using the identification:

$$\eta_t \;\equiv\; \{x \in \mathbb{Z}^d : \eta_t(x) = 1\} \;\subset\; \mathbb{Z}^d.$$

One key ingredient is to think of the process as being constructed from a so-called Harris' graphical representation (Harris 1972) which, in the case of the best-response dynamics, reduces to a collection of independent Poisson processes. More precisely,

- for each $x \in \mathbb{Z}^d$, we let $(N_t(x) : t \geq 0)$ be a rate one Poisson process and
- we denote by $T_n(x) := \inf\{t : N_t(x) = n\}$ its $n$th arrival time.

The configuration at time $t := T_n(x)$ is obtained from $\eta_{t-} := \lim_{s \uparrow t} \eta_s$ by

$$\begin{array}{rcl} \text{adding} \;\; x & \text{when} & a_1\, N_1(x, \eta_{t-}) > a_2\, N_2(x, \eta_{t-}) \\ \text{removing} \;\; x & \text{when} & a_1\, N_1(x, \eta_{t-}) < a_2\, N_2(x, \eta_{t-}). \end{array}$$

An argument due to Harris (Harris 1972) implies that the best-response dynamics starting from any initial configuration can indeed be constructed using this rule. The next lemma shows that, in the presence of selfish-selfish interactions, the best-response dynamics is attractive.

**Lemma 3** – The process with $a_1 > 0$ and $a_2 > 0$ is attractive:

$$P(x \in \bar\eta_t) \ \leq \ P(x \in \eta_t) \quad \text{for all} \quad (x,t) \in \mathbb{Z}^d \times \mathbb{R}_+ \quad \text{whenever} \quad \bar\eta_0 \subset \eta_0.$$

*Proof.* Let $\bar\eta_t \subset \eta_t$. Since $a_1 > 0$ and $a_2 > 0$,

$$a_1 N_1(x, \bar\eta_t) \ \leq \ a_1 N_1(x, \eta_t) \quad \text{and} \quad a_2 N_2(x, \bar\eta_t) \ \geq \ a_2 N_2(x, \eta_t). \tag{3.6}$$

Let $c(x, \eta_t)$ be defined as in (3.5). Using (3.6), we obtain that, for all $x \in \bar\eta_t$,

$$\begin{aligned} c(x, \bar\eta_t) \ &= \ \mathbf{1}\{a_1 N_1(x, \bar\eta_t) < a_2 N_2(x, \bar\eta_t)\} \\ &\geq \ \mathbf{1}\{a_1 N_1(x, \eta_t) < a_2 N_2(x, \eta_t)\} \ = \ c(x, \eta_t). \end{aligned} \tag{3.7}$$

Similarly, for all $x \notin \eta_t$, we have

$$\begin{aligned} c(x, \bar\eta_t) \ &= \ \mathbf{1}\{a_1 N_1(x, \bar\eta_t) > a_2 N_2(x, \bar\eta_t)\} \\ &\leq \ \mathbf{1}\{a_1 N_1(x, \eta_t) > a_2 N_2(x, \eta_t)\} \ = \ c(x, \eta_t). \end{aligned} \tag{3.8}$$

The inequalities (3.7)–(3.8) show that condition (B14) in Liggett Liggett 1999 are satisfied, which proves that, in the presence of selfish-selfish interactions, the process is attractive. □

In addition to attractiveness, a key ingredient to prove our theorem is to replace the initial configuration $\eta_0$ with a specific reduced initial configuration $\bar\eta_0$. To define this new initial configuration, we introduce the following collection of hypercubes:

$$H_z \ := \ 2z + \{0,1\}^d \quad \text{for all} \quad z \in \mathbb{Z}^d.$$

Then, given $\eta_0$, we say that $H_z$ is a type 1 hypercube whenever $H_z \subset \eta_0$ and define

$$\begin{aligned} \bar\eta_0 \ &:= \ \{x \in \mathbb{Z}^d : x \in H_z \text{ and } H_z \subset \eta_0 \text{ for some } z \in \mathbb{Z}^d\} \\ &= \ \text{the union of all type 1 hypercubes.} \end{aligned} \tag{3.9}$$

Note that $\bar\eta_0 \subset \eta_0$ therefore, according to Lemma 3,

$$P(x \in \bar\eta_t) \ \leq \ P(x \in \eta_t) \quad \text{for all} \quad (x,t) \in \mathbb{Z}^d \times \mathbb{R}_+.$$

In particular, it suffices to prove the theorem for the modified process $\bar\eta_t$ that we call from now on the **sparse** best-response dynamics. The main reason for working with this process appears in the next lemma which states that, starting from any configuration that consists of a union of hypercubes, the process can only increase. This somewhat strong result is due in part to the fact that, while the time of the updates are random, the outcome at each update is deterministic.

**Lemma 4** – Assume that $a_1 > a_2 > 0$. Then, $P\,(\bar\eta_s \subset \bar\eta_t \text{ for all } s < t) = 1$.

*Proof.* Let $\Phi$ be the function defined on the set of configurations by

$$\begin{aligned} \Phi(\eta_t) \ &:= \ \{x \in \mathbb{Z}^d : a_1\,N_1(x,\eta_t) > a_2\,N_2(x,\eta_t) \\ &\qquad \text{or } (x \in \eta_t \text{ and } a_1\,N_1(x,\eta_t) = a_2\,N_2(x,\eta_t))\}. \end{aligned} \tag{3.10}$$

In words, while $\eta_t$ represents the set of vertices following strategy 1, configuration $\Phi(\eta_t)$ can be seen as the set of vertices that will become or stay of type 1 at the next update provided the configuration in their neighborhood does not change by the time of the update. Note that, due to the presence of selfish-selfish interactions: $a_1 > 0$ and $a_2 > 0$, we have

$$\begin{aligned} \eta_t \subset \eta_t' \quad &\text{implies that} \quad N_1(x,\eta_t) \leq N_1(x,\eta_t') \text{ and } N_2(x,\eta_t) \geq N_2(x,\eta_t') \\ &\text{implies that} \quad a_1\,N_1(x,\eta_t) - a_2\,N_2(x,\eta_t) \leq a_1\,N_1(x,\eta_t') - a_2\,N_2(x,\eta_t') \\ &\text{implies that} \quad \Phi(\eta_t) \subset \Phi(\eta_t') \end{aligned} \tag{3.11}$$

indicating that the function $\Phi$ is nondecreasing. In addition, for any configuration $\bar\eta_0$ obtained by reduction of an arbitrary initial configuration using the partition into hypercubes, since each type 1 player has at least $d$ type 1 neighbors and $a_1 > a_2 > 0$, we also have

$$\begin{aligned} x \in \bar\eta_0 \quad &\text{implies that} \quad N_1(x,\bar\eta_0) \geq d \text{ and } N_2(x,\bar\eta_0) \leq d \\ &\text{implies that} \quad a_1\,N_1(x,\bar\eta_0) > a_2\,N_2(x,\bar\eta_0) \\ &\text{implies that} \quad x \in \Phi(\bar\eta_0) \end{aligned} \tag{3.12}$$

indicating that $\bar\eta_0 \subset \Phi(\bar\eta_0)$.

Monotonicity (3.11) and the generalization of (3.12) to all times are the main two ingredients to establish the lemma that we prove by induction. Since the lattice is infinite, the time of the first update does not exist. Also, in order to prove the result inductively, the next step is to use an idea of Harris Harris 1972 to break down the lattice into finite islands that do not interact with each other for a short time. More precisely, we do the following construction:

- we let $\epsilon > 0$ be small and, for each vertex $x$ such that $T_1(x) < \epsilon$, draw a line segment between $x$ and each of its $2d$ nearest neighbors.

This construction naturally induces a partition of the lattice into clusters, where two vertices belong to the same cluster if there is a sequence of line segments connecting them. In addition, since the probability of two neighbors $x \sim y$ being connected by a line segment

$$P\,(\text{there is a line segment between } x \text{ and } y)$$
$$= \; P\,(\min(T_1(x), T_1(y)) < \epsilon) \; = \; 1 - e^{-2\epsilon}$$

can be made arbitrarily small by choosing time $\epsilon > 0$ small, Theorem 1.33 in Grimmett 1989 implies that there exists $\epsilon > 0$ small, fixed from now on, such that each cluster is almost surely finite. Letting $A$ be an arbitrary, necessarily finite, cluster, we have the following two properties:

(a) the configuration in $A$ at time $\epsilon$ only depends on the initial configuration of the process and its graphical representation restricted to the cluster $A$.

(b) whenever ($x \in A$ and $N_x \not\subset A$) or ($x \in A^c$ and $N_x \not\subset A^c$) where $N_x$ refers to the interaction neighborhood of vertex $x$, the strategy at $x$ is not updated before time $\epsilon$.

Now, since $A$ is finite, the number of updates in $A$ up to time $\epsilon$ is almost surely finite and therefore can be ordered. Let the times of these updates and their corresponding locations be

$$s_0 := 0 < s_1 < s_2 < \cdots < s_m < \epsilon \quad \text{and} \quad x_1, x_2, \ldots, x_m \in A.$$

By (a) and the definition of the function $\Phi$, we have

$$x_1 \in \bar\eta_{s_1} \quad \text{if and only if} \quad x_1 \in \Phi(\bar\eta_0).$$

But according to (3.12), we also have $\bar\eta_0 \subset \Phi(\bar\eta_0)$ therefore

$$(x_1 \in \bar\eta_{s_0} \;\text{ implies }\; x_1 \in \bar\eta_{s_1}) \quad \text{so} \quad (\bar\eta_{s_0} \cap A) \subset (\bar\eta_{s_1} \cap A) \subset (\Phi(\bar\eta_{s_0}) \cap A). \tag{3.13}$$

This, together with (b) and the monotonicity of $\Phi$ in (3.11), implies

$$(\Phi(\bar\eta_{s_0}) \cap A) \subset (\Phi(\bar\eta_{s_1}) \cap A) \quad \text{and} \quad (\bar\eta_{s_1} \cap A) \subset (\Phi(\bar\eta_{s_1}) \cap A). \tag{3.14}$$

The last inclusion in (3.14) allows us to repeat the same reasoning to get (3.13)–(3.14) at the next update time, and so on up to time $s_m$. Using in addition the obvious fact that the configuration in the cluster $A$ does not change between two consecutive updates implies that the property to be proved holds at all times smaller than $\epsilon$ so we have

$$(\bar\eta_s \cap A) \subset (\bar\eta_t \cap A) \quad \text{and} \quad (\bar\eta_t \cap A) \subset (\Phi(\bar\eta_t) \cap A) \quad \text{for all} \quad s < t \leq \epsilon. \tag{3.15}$$

This only proves the result for the process restricted to $A$ and up to time $\epsilon$. To extend the result across the lattice and for all times, we first use that the set of all the clusters forms a cover of the lattice and union (3.15) over all the possible clusters:

$$\begin{aligned} \bar\eta_s \ &= \ \bigcup_A (\bar\eta_s \cap A) \ \subset \ \bigcup_A (\bar\eta_t \cap A) \ = \ \bar\eta_t \quad \text{for all} \quad s < t \leq \epsilon \\ \bar\eta_\epsilon \ &= \ \bigcup_A (\bar\eta_\epsilon \cap A) \ \subset \ \bigcup_A (\Phi(\bar\eta_\epsilon) \cap A) \ = \ \Phi(\bar\eta_\epsilon). \end{aligned} \tag{3.16}$$

This first inclusion proves the lemma up to time $\epsilon$ while the second inclusion can be used, together with the fact that the process is Markov, to restart the argument and extend the result inductively up to time $2\epsilon$, then $3\epsilon$, and so on. This proves the result at all times. □

## 3.3 Coupling with Bootstrap Percolation

This section is devoted to the proof of the theorem, which relies on a coupling between bootstrap percolation and the best-response dynamics. Bootstrap percolation with parameter $m$ is the discrete-time process whose state at time $t$ is a spatial configuration

$$\psi_t : \mathbb{Z}^d \longrightarrow \{0, 1\} \quad \text{where} \quad 0 = \text{empty} \quad \text{and} \quad 1 = \text{occupied}$$

that evolves deterministically as follows: for all $z \in \mathbb{Z}^d$ and $t \in \mathbb{N}$,

$$\begin{aligned} &\psi_t(z) = 1 \quad \text{implies that} \quad \psi_{t+1}(z) = 1 \\ &\psi_t(z) = 0 \quad \text{implies that} \quad \psi_{t+1}(z) = 1 \ \text{ if and only if } \ \text{card}\,\{w \sim z : \psi_t(w) = 1\} \ \geq \ m. \end{aligned}$$

We will couple the best response dynamics with a modified bootstrap percolation model, $\xi_t$, which we call the modified basic bootstrap percolation model as named in Schonmann 1992. $\xi_t$ evolves identically to $\psi_t$ except that $m$ is taken to be always equal to $d$, the dimension of the lattice, and

$$\xi_t(z) = 0 \quad \text{implies that} \quad \xi_{t+1}(z) = 1 \ \text{ if and only if } \ 1 \in \{\xi_t(z + e_i), \xi_t(z - e_i)\} \text{ for each } i = 1, \ldots, d.$$

In words, that is, a 0 becomes a 1 if in each one of the $d$ coordinate directions, it has at least one neighbor which is a 1.

In view of Lemma 4 for the sparse best-response dynamics and the evolution rules of bootstrap percolation, both processes are almost surely monotone, therefore the limits

$$\bar\eta_\infty \ := \ \lim_{t \to \infty} \bar\eta_t \qquad \text{and} \qquad \xi_\infty \ := \ \lim_{t \to \infty} \xi_t \qquad \text{exist.}$$

Here, we again identify configurations with the set of vertices in state 1. From now on, we call the two limit sets above, the **infinite time limits** of the sparse best-response dynamics and bootstrap percolation, respectively. To prove the theorem, we first rely on the monotonicity results of the previous section to show that the infinite time limit of the sparse best-response dynamics properly rescaled in space dominates its counterpart for bootstrap percolation. The main ingredient is to couple both systems using the key function introduced in (3.10). Based on this coupling, we can directly deduce the theorem from its analog for modified basic bootstrap percolation on the infinite lattice starting from a product measure, a result due to Schonmann (Schonmann 1992, Theorem 3.1).

**Lemma 5** – Assume that $a_1 > a_2 > 0$. Then,

$$\Phi^n(\bar{\eta}_s) := (\Phi \circ \Phi \circ \cdots \circ \Phi)(\bar{\eta}_s) \subset \bar{\eta}_\infty \quad \text{almost surely for all} \quad s > 0 \quad \text{and} \quad n \geq 0.$$

*Proof.* We prove the result by induction with respect to $n$.

**Base case** – This follows from Lemma 4 which gives

$$P\,(\Phi^0(\bar{\eta}_s) \subset \bar{\eta}_\infty) \ = \ P\,(\bar{\eta}_s \subset \bar{\eta}_\infty) \ \geq \ P\,(\bar{\eta}_s \subset \bar{\eta}_t \text{ for all } s < t) \ = \ 1.$$

**Inductive step** – Assume $\Phi^n(\bar{\eta}_s) \subset \bar{\eta}_\infty$ and $x \in \Phi^{n+1}(\bar{\eta}_s) \setminus \Phi^n(\bar{\eta}_s)$. Then,

$$\begin{aligned} T_y \ &:= \ \inf\{T > 0 : y \in \bar{\eta}_T\} < \infty \ \text{ a.s. for all } \ y \in \Phi^n(\bar{\eta}_s) \ \text{ and} \\ \tau_x \ &:= \ \max\{T_y : y \sim x \text{ and } y \in \Phi^n(\bar{\eta}_s)\} < \infty \ \text{ a.s.} \end{aligned} \tag{3.17}$$

In addition, the choice of $x$ implies that

$$a_1\, N_1(x, \Phi^n(\bar{\eta}_s)) \ > \ a_2\, N_2(x, \Phi^n(\bar{\eta}_s)) \quad \text{since} \quad x \in \Phi\,(\Phi^n(\bar{\eta}_s)) \setminus \Phi^n(\bar{\eta}_s) \tag{3.18}$$

while a new application of Lemma 4 gives

$$N_1(x, \bar{\eta}_t) \ \geq \ N_1(x, \Phi^n(\bar{\eta}_s)) \quad \text{and} \quad N_2(x, \bar{\eta}_t) \ \leq \ N_2(x, \Phi^n(\bar{\eta}_s)) \tag{3.19}$$

for all $t > \tau_x$. Combining (3.18)–(3.19) and using that $a_1 > 0$ and $a_2 > 0$, we get

$$\begin{aligned} a_1\, N_1(x, \bar{\eta}_t) \ &\geq \ a_1\, N_1(x, \Phi^n(\bar{\eta}_s)) \\ &> \ a_2\, N_2(x, \Phi^n(\bar{\eta}_s)) \ \geq \ a_2\, N_2(x, \bar{\eta}_t) \quad \text{for all} \quad t > \tau_x. \end{aligned}$$

It follows that, given that the player at vertex $x$ follows strategy 2 after time $\tau_x$, she switches to strategy 1 at rate one. This together with (3.17) implies that

$$T_x \ = \ \inf\{t > 0 : x \in \bar{\eta}_t\} < \infty \ \text{ a.s.} \quad \text{therefore} \quad x \in \bar{\eta}_\infty. \tag{3.20}$$

Finally, using consecutively (3.11) and (3.16) and then (3.20), we deduce that

$$\begin{aligned} \Phi^n(\bar\eta_s) \ \subset \ \Phi\,(\Phi^n(\bar\eta_s)) \ = \ \Phi^{n+1}(\bar\eta_s) \\ \text{and} \quad \Phi^{n+1}(\bar\eta_s) \ = \ (\Phi^{n+1}(\bar\eta_s) \setminus \Phi^n(\bar\eta_s)) \cup \Phi^n(\bar\eta_s) \ \subset \ \bar\eta_\infty \end{aligned}$$

which shows the result at step $n+1$ and completes the proof. □

We are now ready to prove that the infinite time limit of the best-response dynamics properly rescaled in space dominates the infinite time limit of bootstrap percolation. More precisely, we look at the best-response dynamics viewed at the hypercube level by introducing

$$\zeta_t : \mathbb{Z}^d \longrightarrow \{0,1\} \quad \text{where} \quad \zeta_t(z) \ := \ \mathbf{1}\{H_z \subset \bar\eta_t\} \quad \text{for all} \quad z \in \mathbb{Z}^d. \tag{3.21}$$

From now on, we call this process the **hypercubic** best-response dynamics. Identifying once more configurations with the set of vertices in state 1 and using again the monotonicity of the sparse best-response dynamics given by Lemma 4, we note that

$$\begin{aligned} \zeta_\infty \ := \ \lim_{t\to\infty} \zeta_t \ &= \ \lim_{t\to\infty} \{z : H_z \subset \bar\eta_t\} \\ &= \ \{z : H_z \subset \lim_{t\to\infty} \bar\eta_t\} \ = \ \{z : H_z \subset \bar\eta_\infty\} \end{aligned} \tag{3.22}$$

therefore the infinite time limit $\zeta_\infty$ is well-defined.

**Lemma 6** – Assume that $a_1 > a_2 > 0$. Then,

$$\xi_\infty \subset \zeta_\infty \quad \text{almost surely whenever} \quad \xi_0 = \zeta_0.$$

*Proof.* Let $z \in \mathbb{Z}^d$ and $s > 0$, and assume that

$$\zeta_s(z) = 0 \quad \text{and} \quad 1 \in \{\zeta_s(z+e_i), \zeta_s(z-e_i)\} \text{ for each } i = 1, \ldots, d. \tag{3.23}$$

Recalling (3.21), this indicates that there are at least $d$ hypercubes adjacent to $H_z$ that are completely occupied by players of type 1. Invoking the invariance by symmetry of the best-response dynamics, we may assume without loss of generality that

$$H_{z-e_j} \subset \bar\eta_s \quad \text{for} \quad j = 1, 2, \ldots, d \quad \text{where} \quad e_j := j\text{th unit vector}. \tag{3.24}$$

Since $a_1 > a_2 > 0$, we also have

$$\Phi(\bar\eta_s) \ \supset \ \{x \in \mathbb{Z}^d : N_1(x, \bar\eta_s) \geq N_2(x, \bar\eta_s)\} \ = \ \{x \in \mathbb{Z}^d : N_1(x, \bar\eta_s) \geq d\}. \tag{3.25}$$

Combining (3.24)–(3.25) together with Lemma 4 and some basic geometry, we get

$$2z + \{x \in \{0,1\}^d : \textstyle\sum_{j=1,2,\ldots,d} x_j < n\} \ \subset \ \Phi^n(\bar\eta_s) \quad \text{for} \quad n = 1, 2, \ldots, d+1. \tag{3.26}$$

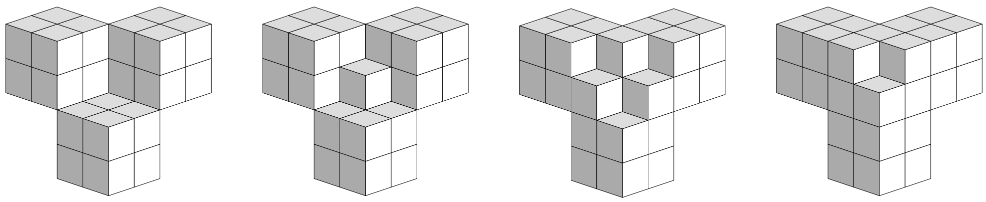


Figure 10: *Picture of the progression* $\bar{\eta}_s \to \Phi(\bar{\eta}_s) \to \Phi^2(\bar{\eta}_s) \to \Phi^3(\bar{\eta}_s)$ *starting from the union of three hypercubes adjacent to the same hypercube. The figure gives an illustration of the inclusions in* (3.26).

For an illustration in three dimensions, we refer to Figure 10 where configuration $\bar{\eta}_s$ consists of the union of three hypercubes. In particular, taking $n = d + 1$ gives

$$H_z \;=\; 2z + \{x \in \{0,1\}^d : \textstyle\sum_{j=1,2,\ldots,d} x_j \leq d\} \;\subset\; \Phi^{d+1}(\bar{\eta}_s).$$

Applying Lemma 5, we then obtain

$$H_z \;\subset\; \Phi^{d+1}(\bar{\eta}_s) \;\subset\; \bar{\eta}_\infty \quad \text{therefore} \quad \zeta_t(z) = 1 \;\text{ for some time } t < \infty \;\text{ a.s.} \tag{3.27}$$

in view of (3.22). In addition, since the hypercubic process clearly inherits the monotonicity property of the sparse best-response dynamics given by Lemma 4,

$$\zeta_s(z) = 1 \quad \text{implies that} \quad P\,(\zeta_t(z) = 1 \text{ for all } t > s) = 1. \tag{3.28}$$

In summary, (3.28) and the fact that (3.23) implies (3.27) indicate that: for the hypercubic process, once a vertex is occupied it remains occupied forever, and if an empty vertex has at least one occupied neighbor in each of the $d$ coordinate directions then it becomes occupied after an almost surely finite time. Recalling the evolution rules of modified basic bootstrap percolation, the result follows. $\square$

Combining the previous lemma with a result of Schonmann 1992, Theorem 3.1 on bootstrap percolation on the infinite lattice, we now deduce Theorem 2:

$$\lim_{t\to\infty} P\,(\eta_t(x) = 1) = 1 \quad \text{for all} \quad x \in \mathbb{Z}^d \quad \text{when} \quad a_1 > a_2 > 0 \;\text{ and }\; p > 0.$$

*Proof.* To begin with, we consider the modified basic bootstrap percolation model starting from the product measure with density $q$. That is, the initial configuration satisfies

$$P\,(\xi_0(z_1) = \xi_0(z_2) = \cdots = \xi_0(z_n) = 1) \;=\; q^n \quad \text{for} \quad z_1, z_2, \ldots, z_n \in \mathbb{Z}^d \quad \text{all distinct.}$$

Whether the set of occupied vertices ultimately covers the entire lattice depends on the initial density, and the fact that bootstrap percolation is clearly attractive motivates the introduction of the following critical value for the initial density:

$$q_c \ := \ \inf \{q \in [0,1] : P(\xi_\infty = \mathbb{Z}^d) = 1\}.$$

Schonmann 1992, Theorem 3.1 proved that $q_c = 0$, therefore

$$P(\xi_\infty = \mathbb{Z}^d) = 1 \quad \text{whenever} \quad m = d \quad \text{and} \quad q > 0.$$

In particular, taking $q := p^{2^d}$ so that

$$P(\zeta_0(z) = 1) \ = \ P(\bar\eta_0(x) = 1 \text{ for all } x \in H_z) \ = \ p^{2^d} \ = \ q \ = \ P(\xi_0(z) = 1),$$

assuming that $p > 0$ and applying Lemmas 3 and 6, we get

$$\begin{aligned} \lim\nolimits_{t \to \infty} P(\eta_t(x) = 1) \ &\geq \ \lim\nolimits_{t \to \infty} P(\bar\eta_t(x) = 1) \ = \ P(x \in \bar\eta_\infty) \\ &\geq \ P(\bar\eta_\infty = \mathbb{Z}^d) \ = \ P(\zeta_\infty = \mathbb{Z}^d) \ \geq \ P(\xi_\infty = \mathbb{Z}^d) \ = \ 1 \end{aligned}$$

which completes the proof of Theorem 2. $\square$

Chapter 4

# DEATH-BIRTH UPDATING PROCESS

## 4.1 Introduction

This chapter is concerned with the death-birth updating process, and includes the work that has been previously published as Evilsizor and Lanchier 2016. This model is an example of a spatial game in which players located on the $d$-dimensional integer lattice are characterized by one of two possible strategies and update their strategy at rate one by mimicking one of their neighbors chosen at random with a probability proportional to the neighbor's payoff. The model studied here is closely related to a version of the death-birth updating process in evolutionary game theory introduced in Ohtsuki and Nowak 2006. The model in Ohtsuki and Nowak 2006 assumes that the updates are neutral with high probability and based on the payoff of the neighbors with small probability, which we refer respectively as voter and game steps. In contrast, the model considered in this chapter only accounts for game steps, so the duality techniques Cox, Durrett, and Perkins 2013 developed for voter model perturbations are no longer available tools to study the process. Instead, our analysis is based on comparisons with oriented site percolation, coupling arguments, and martingale techniques. The main objective is to study the limiting behavior of the spatial stochastic process and confront our results with the limiting behavior of the replicator equation in order to understand the effects of the inclusion of space.

**Model description** – The process studied in this chapter, which we again refer to as the death-birth updating process following the terminology in Ohtsuki and Nowak 2006, is a spin system on the $d$-dimensional integer lattice where each vertex is occupied by a player characterized by one of two possible strategies, say strategy 1 and strategy 2. The state at time $t$ is a function

$$\xi_t : \mathbb{Z}^d \to \{1, 2\} \quad \text{where} \quad \xi_t(x) = \text{strategy at vertex } x \text{ at time } t.$$

The dynamics of this process or any other spatial game is defined in a couple of steps: we first fix a payoff matrix, which allows us to turn every spatial configuration of strategies into a so-called payoff landscape, which can then be used to define the transition rates at each vertex. Since we focus on games with two strategies, the payoff matrix is a $2 \times 2$ matrix $A = (a_{ij})$ whose coefficients are positive real numbers interpreted as

$$a_{ij} = \text{payoff of a type } i \text{ player interacting with a type } j \text{ player.}$$

In nonspatial evolutionary games, players interact equally with any other player in the population, making their payoff a function of the global frequency of representatives of each strategy. In contrast, spatial games assume that the payoff of a player depends exclusively on the strategy of a finite set of neighbors, which is the key to designing more realistic models with local interactions. Throughout this chapter, the interaction neighborhood of vertex $x$ is the set

$$N_x := \{y \in \mathbb{Z}^d : y \neq x \text{ and } \max_{j=1,2,\ldots,d} |x_j - y_j| \leq M\}$$

where the constant $M$ is called the range of the interactions. Letting $N_j(x, \xi)$ be the number of neighbors of the player at vertex $x$ following strategy $j$, every spatial configuration $\xi$ is then turned into a payoff landscape by attributing the payoff

$$\phi(x, \xi) := \textstyle\sum_j a_{ij} N_j(x, \xi)\} \quad \text{where} \quad i = \xi(x), \tag{4.1}$$

to the player at vertex $x$. In words, each type $i$ player receives $a_{ij}$ from each of her neighbors following strategy $j$. The last step to define the dynamics of the process is to follow Maynard Smith and Price 1973 and interpret the payoff as fitness. The basic idea here is to write the rate at which a player changes her strategy as a function of her payoff and the payoff of her neighbors in such a way that players with a larger payoff are more likely to spread their strategy. There are multiple options. For instance, the updating rules considered in Evilsizor and Lanchier 2014; Lanchier 2015 are as follows.

- Best-response dynamics Evilsizor and Lanchier 2014. Players update their strategy at rate one in order to maximize their payoff, which depends on the strategy of their neighbors.
- Payoff affecting birth and death rates Lanchier 2015. In this process, when a player has a positive payoff, at rate this payoff, one of her neighbors chosen at random adopts her strategy, whereas when her payoff is negative, at rate minus this payoff, she adopts the strategy of one of her neighbors chosen at random. This updating rule is inspired from Brown and Hansell 1987.

The dynamics of the death-birth updating process is built using a similar approach: we assume that players update their strategy at rate one by mimicking a random neighbor, with each neighbor being chosen with a probability proportional to her payoff. More precisely, letting $\xi$ be the configuration of the system, the player at $x$ switches her strategy $i \to j$ at rate

$$p_{i\to j}(x,\xi) := \frac{\sum_{y\in N_x} \phi(y,\xi)\ \mathbf{1}\{\xi(y)=j\}}{\sum_{y\in N_x} \phi(y,\xi)} \quad \text{for} \quad \{i,j\}=\{1,2\}. \tag{4.2}$$

Note that, when all four payoff coefficients are equal, the expression above is equal to the fraction of type $j$ neighbors, thus showing that, in this particular case, the death-birth updating process reduces to the voter model introduced independently in Clifford and Sudbury 1973 and Holley and Liggett 1975.

The model described by the two transition rates in 4.2, or to be more specific, a closely related version of this model, has been introduced and studied heuristically in Ohtsuki and Nowak 2006 while Chen 2013; Cox, Durrett, and Perkins 2013 give rigorous results. The process considered in these works can be seen as the weak selection approximation of the model described by (4.2). Players again update their strategy at rate one but, at the time of the update,

- with probability $1-\epsilon$, the player mimics the strategy of a neighbor chosen uniformly at random, just like in the voter model Clifford and Sudbury 1973; Holley and Liggett 1975, while
- with probability $\epsilon$, the player mimics a neighbor chosen at random according to probabilities that are proportional to the neighbors' payoff, as described by (4.2).

This model is studied in Chen 2013; Cox, Durrett, and Perkins 2013; Ohtsuki and Nowak 2006 when $\epsilon$ is small, in which case duality techniques for voter model perturbations are available. For a general definition of duality for interacting particle systems, we refer the reader to Liggett 1985, section II.3. In contrast, we study the process when $\epsilon = 1$, in which case duality cannot be used, which leads to more qualitative and less quantitative results.

**The replicator equation** – Before studying the spatial game, it is worth taking a quick look at its nonspatial deterministic analog to later identify disagreements between both models and thus understand the effect of the inclusion of space in the form of local interactions. The nonspatial model is obtained by assuming that the population of players is well-mixed, which results in a system of ordinary differential equations for the frequency of each strategy. In the case of the death-birth process, this is a time-change of the replicator equation:

$$u_1' = u_1\, u_2\, (\phi_1(u_1,u_2) - \phi_2(u_1,u_2)) \tag{4.3}$$

where $u_j$ is the frequency of players following strategy $j$ and

$$\phi_1(u_1, u_2) = a_{11} u_1 + a_{12} u_2 \quad \text{and} \quad \phi_2(u_1, u_2) = a_{21} u_1 + a_{22} u_2$$

are the common payoffs of all type 1 and all type 2 players, respectively. This can be viewed as the nonspatial analog of the payoff landscape (4.1). Since each player follows either strategy 1 or strategy 2, we always have $u_1 + u_2 = 1$, which implies that, in the presence of two strategies, the replicator equation is only one-dimensional and easy to analyze. As pointed out in Lanchier 2013, the limiting behavior can be conveniently described by introducing the parameters

$$a_1 := a_{11} - a_{21} \quad \text{and} \quad a_2 := a_{22} - a_{12}$$

and calling strategy $i$ selfish whenever $a_i > 0$ and altruistic whenever $a_i < 0$. Then, following the usual terminology by calling a strategy an evolutionarily stable strategy if it cannot be invaded by any alternative strategy starting at an infinitesimally small frequency, some basic algebra shows that the behavior of the replicator equation (4.3) is as follows:

- when $a_1 \, a_2 < 0$, the selfish strategy always outcompetes the altruistic strategy, showing that the selfish strategy is the only evolutionarily stable strategy,
- when $a_1, a_2 > 0$, there is an unstable interior fixed point so the system is bistable, showing that the two (selfish) strategies are evolutionarily stable,
- when $a_1, a_2 < 0$, there is a globally stable interior fixed point so both strategies coexist and none of the two (altruistic) strategies is evolutionarily stable.

In summary, the analysis of the replicator equation shows that, when the population is well-mixed, a strategy is an evolutionarily stable strategy if it is selfish but not if it is altruistic.

**Main results for the spatial game** – In order to compare the spatial game with its nonspatial analog, we assume that the process starts from a translation invariant product measure in which the density of each of the two strategies is constant across the lattice. Since the two configurations in which all players follow the same strategy are absorbing states, we also assume, to avoid trivialities, that the density of each strategy is positive. From now on, we assume without loss of generality that $a_{21} > a_{12} > 0$ and study the limiting behavior of the process as the other two payoffs vary.

To begin with, we look at the parameter region where both strategies are altruistic. In this case, coexistence occurs when the population is well-mixed, i.e., when the dynamics is described by the replicator equation 4.3. Numerical simulations suggest that, except in the one-dimensional nearest neighbor case, coexistence is again possible for the spatial game though the coexistence region is reduced. The smaller the spatial dimension and/or the range of the interactions, the smaller the coexistence region. Our first two theorems show that coexistence is indeed possible and that the coexistence region for the spatial game is indeed reduced. More precisely, Theorem 7 shows that, regardless of the spatial dimension, both strategies coexist when they are sufficiently altruistic and the range of the interactions is sufficiently large.

**Theorem 7** – Let $a_{21} > a_{12} > 0$. Then, there exist $a > 0$ and $M_0 < \infty$, which depends on $a$, such that the death-birth updating process coexists when

$$\max(a_{11}, a_{22}) \leq a \quad \text{and} \quad M > M_0.$$

To prove that the coexistence region is reduced, and more generally identify parameter regions in which one strategy wins, we first observe that, when $a_{11} = a_{12}$ and $a_{22} = a_{21}$, the process is significantly simplified because the payoff of the players only depends on their strategy but not on the strategy of their neighbors. Indeed, in this case (4.1) reduces to

$$\phi(x,\xi) = \textstyle\sum_j a_{ij}\, N_j(x,\xi)\, \mathbf{1}\{\xi(x) = i\} = a_{ii}\, ((2M+1)^d - 1)\, \mathbf{1}\{\xi(x) = i\}$$

for $i = 1, 2$, therefore the transition rates become

$$p_{i \to j}(x,\xi) = \frac{\sum_{y \in N_x} \phi(y,\xi)\ \mathbf{1}\{\xi(y) = j\}}{\sum_{y \in N_x} \phi(y,\xi)} = \frac{a_{jj}\, N_j(x,\xi)}{a_{ii}\, N_i(x,\xi) + a_{jj}\, N_j(x,\xi)}$$

for $\{i, j\} = \{1, 2\}$. It follows that, under our general assumption $a_{12} < a_{21}$, the set of type 2 players dominates stochastically a certain biased voter model Bramson and Griffeath 1980, 1981, thus showing that, in this very special case, strategy 2 wins. Elaborating on this idea but using coupling arguments to compare the death-birth updating process with spin systems which are more complicated than the biased voter model, we can prove much more, as shown in the next theorem.

**Theorem 8** – Let $a_{21} > a_{12}$ and let $N := \operatorname{card} N_x = (2M+1)^d - 1$. Then,

(a) strategy 1 wins when $\min(a_{12} - a_{22}, a_{11} - a_{21}) > (N-1)(a_{21} - a_{12})$,

(b) strategy 2 wins when $(M, d) \neq (1, 1)$ and

$$(N^2 - N - 1)\, \max(a_{11} - a_{21}, a_{12} - a_{22}, a_{11} - a_{22}) < a_{21} - a_{12}.$$

Figure 11 shows the parameter regions in both parts of the theorem. Note that the parameter region in the first part of the theorem is nonempty if and only if

$$a_{12} - (N-1)(a_{21} - a_{12}) > 0 \quad \text{if and only if} \quad a_{12} > (1 - 1/N)\, a_{21}.$$

The figure shows this region for $N = 2$, i.e., in the presence of one-dimensional nearest neighbor interactions. In contrast, the parameter region in the second part of the theorem is always nonempty and, more interestingly, it always overlaps the region where both strategies are altruistic as well as the region where both strategies are selfish. This shows that the inclusion of a spatial structure in the form of local interactions indeed reduces the coexistence region, as mentioned above. This also shows that, in a subset of the parameter region where the replicator equation is bistable, there is instead a strong type for the spatial game that wins even starting at low density. The next theorem goes a little bit further in this direction by showing that, no matter how selfish a strategy is, the other strategy always wins if it is selfish enough.

**Theorem 9** – For all $a > 0$ there is $A < \infty$ such that strategy 1 wins when

$$\max(a_{21}, a_{22}) \leq a \quad \text{and} \quad a_{11} > A.$$

This theorem extends the parameter region where strategy 1 wins found in Theorem 8. In addition, contrary to Theorem 8, this theorem does not assume that $a_{12} < a_{21}$. Therefore, its dual statement obtained by exchanging the role of both strategies holds as well, showing that Theorem 9 also extends the parameter region where strategy 2 wins found in Theorem 8. We state this explicitly in the following remark:

**Remark 10** – For all $a > 0$ there is $A < \infty$ such that strategy 2 wins when

$$\max(a_{12}, a_{11}) \leq a \quad \text{and} \quad a_{22} > A.$$

The results collected so far indicate interesting discrepancies between the death-birth updating process and the replicator equation, showing the importance of local interactions. The most interesting aspect suggested by spatial simulations is the existence of a subset of the parameter region corresponding to the prisoner's dilemma game in which cooperators win on the lattice whereas they always lose when the population is well-mixed. As stated in Section 1.2 the prisoner's dilemma game

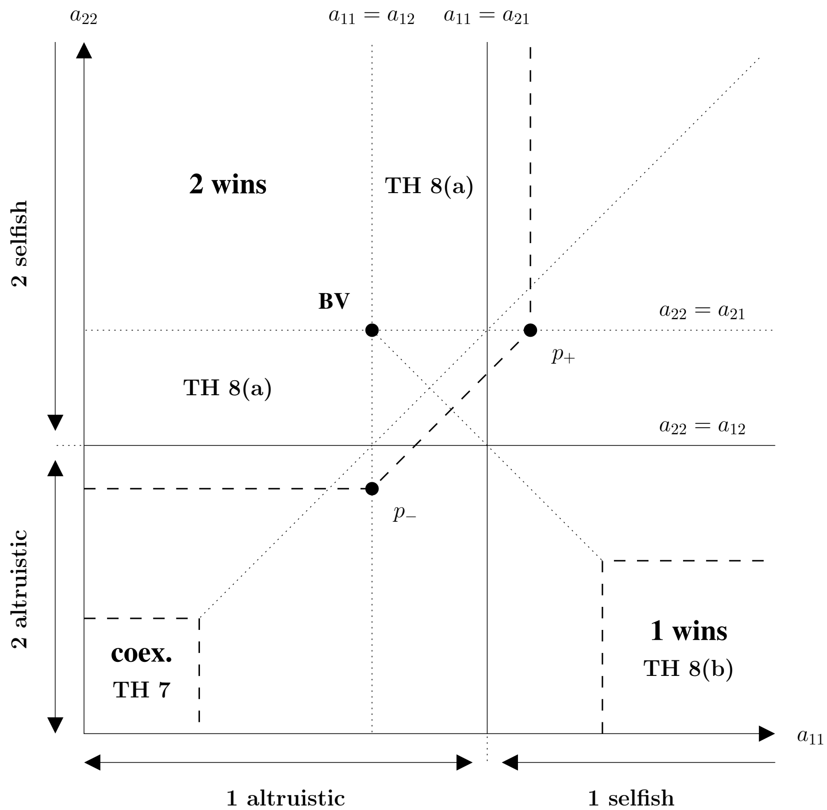



Figure 11: *Phase diagram of the spatial game along with a summary of the theorems in the* $a_{11} - a_{22}$ *plane. In the picture, the points* $p_-$ *and* $p_+$ *are the two points introduced in the proof of Lemma 18.*

is characterized by the following ordering and terminology of the four payoffs:

$$a_{12} = \text{sucker's payoff} \ < \ a_{22} = \text{punishment} \ < \ a_{11} = \text{reward} \ < \ a_{21} = \text{temptation}.$$

Figure 12 shows the corresponding triangular region in solid lines. Players with strategy 1 are called cooperators while players with strategy 2 are called defectors. Because the reward is not as good as the temptation, and the punishment is not as bad as the sucker's payoff, cooperators are altruistic and defectors selfish, therefore defectors indeed win when the population is well-mixed. In contrast, the heuristic arguments in Ohtsuki and Nowak 2006 suggest that there is a subset of the prisoner's dilemma triangle in which cooperators are favored over defectors on regular graphs. This has been proved in Chen 2013 for finite, connected, simple graphs, and in Cox, Durrett, and Perkins 2013 for integer lattices with $d > 2$. Their results, however, hold in the weak selection case but not for the process (4.2). We now study the interactions among cooperators and defectors in one dimension, the main difficulty being the lack of attractiveness of the process.

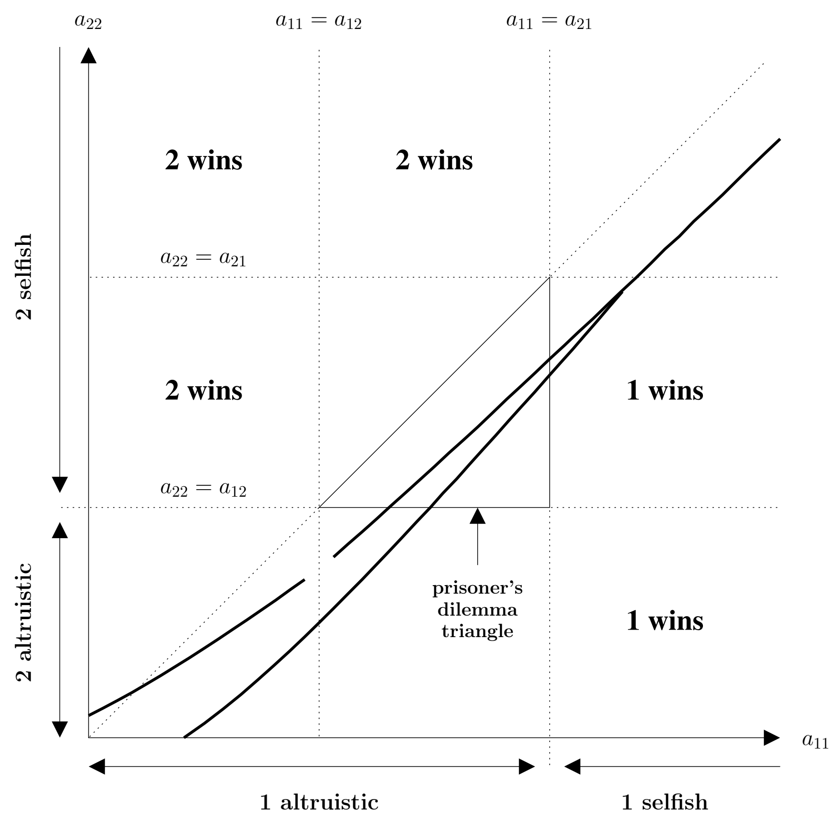



Figure 12: *Phase diagram of the one-dimensional nearest neighbor spatial game when* $a_{21}/a_{12} = 2$. *The lower thick curve is obtained from the conditions in Theorem 11 and the upper thick curve by exchanging the role of strategies 1 and 2 in Theorem 11. In particular, strategy 1 wins below the lower curve while strategy 2 wins above the upper curve. The triangle in solid lines represents the parameter region corresponding to the prisoner's dilemma game.*

To state our last result, we introduce the following quantities that will be interpreted later as drift of a certain interface:

$$D_3 := \frac{a_{11} + a_{12}}{a_{11} + a_{12} + a_{21} + a_{22}} - \frac{a_{21} + a_{22}}{2a_{11} + a_{21} + a_{22}}$$

$$D_4 := \frac{a_{11} + a_{12}}{a_{11} + a_{12} + 2a_{22}} - \frac{a_{21} + a_{22}}{2a_{11} + a_{21} + a_{22}}.$$

Then, we have the following theorem.

**Theorem 11** – Assume $M = d = 1$. Then, strategy 1 wins when

$$(a_{22} < a_{21} \text{ and } D_3 + D_4 > 0) \quad \text{or} \quad (a_{22} > a_{21} \text{ and } D_4 > 0).$$

Note that the parameter region given by the theorem overlaps but is not restricted to the prisoner's dilemma triangle. To see that the theorem implies the existence of a subset of the triangle in which cooperators win, observe that, when $a_{11} = a_{21} > a_{22} = a_{12}$,

$$D_3 + D_4 = \frac{1}{2} + \frac{a_{12} + a_{21}}{3a_{12} + a_{21}} - 2 \times \frac{a_{12} + a_{21}}{a_{12} + 3a_{21}} > \frac{1}{2} + \frac{1}{2} - 2 \times \frac{1}{2} = 0.$$

In particular, the first parameter region given by the theorem in which cooperators win indeed overlap the prisoner's dilemma triangle. The theorem also implies that strategy 2 wins in the parameter regions obtained by exchanging the role of the two strategies, i.e. letting

$$\begin{aligned}\bar{D}_3 &:= \frac{a_{22} + a_{21}}{a_{22} + a_{21} + a_{12} + a_{11}} - \frac{a_{12} + a_{11}}{2a_{22} + a_{12} + a_{11}} \\ \bar{D}_4 &:= \frac{a_{22} + a_{21}}{a_{22} + a_{21} + 2a_{11}} - \frac{a_{12} + a_{11}}{2a_{22} + a_{12} + a_{11}}.\end{aligned}$$

and again $M = d = 1$, strategy 2 wins when

$$(a_{11} < a_{12} \text{ and } \bar{D}_3 + \bar{D}_4 > 0) \quad \text{or} \quad (a_{11} > a_{12} \text{ and } \bar{D}_4 > 0). \tag{4.4}$$

Finally, observing that $D_4 + \bar{D}_4 = 0$, we deduce that

$$\begin{aligned}&\text{strategy 1 wins when} \quad a_{22} > a_{21} \quad \text{and} \quad D_4 > 0 \\ &\text{strategy 2 wins when} \quad a_{11} > a_{12} \quad \text{and} \quad D_4 < 0\end{aligned}$$

showing that, when $\min(a_{11}, a_{22}) > \max(a_{12}, a_{21})$, the condition is sharp. Figure 12 gives a picture of the curves derived from the theorem when $a_{21}/a_{12} = 2$.

## 4.2 Coexistence of Altruistic Strategies

This section is devoted to the proof of our coexistence result Theorem 7. To prove this result, we think of the process as being generated from a graphical representation which, for the death-birth updating process, reduces to a countable collection of independent Poisson processes and uniform random variables. More precisely, for each $(x, n) \in \mathbb{Z}^d \times \mathbb{N}^*$, let

$$\begin{aligned}T_n(x) &:= \text{the } n\text{th arrival time of a Poisson process with intensity one} \\ U_n(x) &:= \text{uniform random variable on the interval } (0, 1).\end{aligned}$$

Then, at time $s := T_n(x)$, the strategy at $x$ is set equal to

$$\begin{aligned}&1 \quad \text{when} \quad U_n(x) < p_{2\to1}(x, \xi_{s-}) \\ &2 \quad \text{when} \quad U_n(x) > p_{2\to1}(x, \xi_{s-})\end{aligned}$$

where $p_{i\to j}(x,\xi)$ has been defined in (4.2). An argument due to Harris 1972 implies that the process starting from any initial configuration can indeed be constructed following these rules.

In the proof of the coexistence result, we focus for simplicity on the two-dimensional case but our approach easily extends to any spatial dimension. Specifically, we will prove that both strategies coexist when $M$ is large and

$$\max(a_{11},a_{22}) \leq 5^{-2}\,2^{-21}\,(c_-)^5\,\min(a_{12},a_{21}) = 2^{-14}\,(c_+)^{-1}\,\min(a_{12},a_{21}) \tag{4.5}$$

where the two key constants $c_-$ and $c_+$ are defined as

$$c_- := 2^{-17}\,\min(a_{12}/a_{21},a_{21}/a_{12}) \quad \text{and} \quad c_+ := 5^2\,2^7(c_-)^{-5}. \tag{4.6}$$

Let $s := \ln(2)$ and $K_r := [-rM,rM)^2$ for all $r>0$, and fix

$$A,B \subset K_{1/2} \quad \text{with} \quad \text{card}(A) = \text{card}(B) = 2^{-2}\,M^2.$$

The proofs of Lemmas 13–14 below hold for such general sets though they will be applied ultimately to more specific space-time boxes. One key to the proof is to observe that

$$\max_{j=1,2}|x_j - y_j| \leq M \quad \text{for all} \quad (x,y) \in A \times B.$$

For all $D \subset \mathbb{Z}^2$ finite and $i=1,2$, we let

$$\zeta_t^i(D) := \text{card}\,\{x \in D : \xi_t(x) = i\}$$

denote the number of type $i$ players in the set $D$ at time $t$. Throughout the proof, we will use repeatedly the large deviation estimate

$$P\,(\text{Binomial}\,(K,p) \leq K\,(p-z)) \leq \exp(-Kz^2/2p) \quad \text{for all} \quad z \in (0,p) \tag{4.7}$$

which follows from Arratia and Gordon 1989, Theorem 1 and the fact that, setting $q = 1-p$, we have

$$\begin{aligned} P\,(\text{Binomial}\,(K,p) \leq K\,(p-z)) &= P\,(\text{Binomial}\,(K,1-p) \geq K - K\,(p-z)) \\ &= P\,(\text{Binomial}\,(K,q) \geq K\,(q+z)). \end{aligned}$$

**Keeping the players in a box** – To begin with, we prove in the next lemma that, the number of players of either type in a given spatial region does not decrease too fast. The idea is to simply find a bound for the number of updates using standard large deviation estimates for the binomial random variable. This lemma will be used repeatedly later.

**Lemma 12** – Let $D \subset \mathbb{Z}^2$ be finite and $n \in \mathbb{N}$. Then, for $i = 1, 2$,

$$P\,(\zeta_t^i(D) \leq 2^{-(n+1)}\,K \text{ for some } t \in (0, ns) \mid \zeta_0^i(D) \geq K) \leq \exp(-2^{-(n+3)}\,K).$$

*Proof.* To begin with, we let

$$u_i(D) := \text{card}\,\{x \in D : \xi_0(x) = i \text{ and } T_1(x) < ns\}$$

denote the total number of players in the set $D$ who are initially of type $i$ and update their strategy at least once by time $ns$. Here and after, updates refer to the death-birth events that occur at the times $T_n(x)$. In particular, an update does not necessarily induce a change of strategy. Now, recalling that the random variables $T_1(x)$ are the first arrival times of the independent rate one Poisson processes used to construct the process and therefore are independent and exponentially distributed with rate one, our choice of $s$ implies that

$$u_i(D) = \text{Binomial}\,(\zeta_0^i(D), 1 - e^{-ns}) = \text{Binomial}\,(\zeta_0^i(D), 1 - 2^{-n}). \tag{4.8}$$

Note also that, since the initial number of type $i$ players in $D$ minus the number of those players that have updated their strategy by time $ns$ must exceed the number of type $i$ players in $D$ at all times before time $ns$, we have for all $t \in (0, ns)$

$$\{\zeta_t^i(D) \leq 2^{-(n+1)}\,K\} \cap \{\zeta_0^i(D) \geq K\} \subset \{u_i(D) \geq (1 - 2^{-(n+1)})\,K\}. \tag{4.9}$$

Using (4.8)–(4.9) and (4.7) with $p = 2^{-n}$ and $z = 2^{-(n+1)}$, we get

$$\begin{aligned} & P\,(\zeta_t^i(D) \leq 2^{-(n+1)}\,K \text{ for some time } t \in (0, ns) \mid \zeta_0^i(D) \geq K) \\ & \quad \leq \; P\,(u_i(D) \geq (1 - 2^{-(n+1)})\,K \mid \zeta_0^i(D) = K) \\ & \quad = \; P\,(\text{Binomial}\,(K, 1 - 2^{-n}) \geq (1 - 2^{-(n+1)})\,K) \\ & \quad = \; P\,(\text{Binomial}\,(K, 2^{-n}) \leq 2^{-(n+1)}\,K) \leq \exp(-2^{-(n+3)}\,K). \end{aligned}$$

This completes the proof of the lemma. □

**Moving the players around** – We now prove that if the region $A$ has a large number of type 1 players then, regardless of the configuration around this region, we can "move" a positive fraction of these players to the nearby region $B$ in $s$ units of time. The constant $c_-$ defined in (4.6) will play the role of the fraction of players we can move.

**Lemma 13** – Assume (4.5) and let $c_-$ as in (4.6) and $a > 0$. Then,

$$P\,(\zeta_s^1(B) \leq c_-(aM) \mid \zeta_0^1(A) \geq aM) \leq \exp(-(aM)^{1/2}) \quad \text{for all } M \text{ large.}$$

*Proof.* The proof of Lemma 12 with $D = B$, $n = 1$, and $K = aM$ gives

$$\begin{aligned} & P\,(\zeta_s^1(B) \leq c_-(aM) \mid \zeta_0^1(A) \geq aM \text{ and } \zeta_0^1(B) \geq aM) \\ & \quad \leq \ P\,(\zeta_t^1(B) \leq 2^{-2}(aM) \text{ for some } t \in (0,s) \mid \zeta_0^1(A) \geq aM \text{ and } \zeta_0^1(B) \geq aM) \\ & \quad \leq \ \exp(-2^{-4}\,aM) \leq \exp(-(aM)^{1/2}) \end{aligned}$$

for all $M$ large. The first inequality follows from the fact that $c_- \leq 2^{-17} < 2^{-2}$ while the second inequality indeed follows from the proof of Lemma 12 since the estimates in this proof only depend on the initial number of type 1 players in $B$ and the number of sites in this set which are updated before time $s$. To complete the proof, it remains to show that

$$P\,(\zeta_s^1(B) \leq c_-(aM) \mid \zeta_0^1(B) \leq aM \leq \zeta_0^1(A)) \leq \exp(-(aM)^{1/2}) \tag{4.10}$$

for all $M$ large. To lighten the notation, we let

$$P^*(\mathbf{E}) := P\,(\mathbf{E} \mid \zeta_0^1(B) \leq aM \leq \zeta_0^1(A)) \quad \text{for any event } \mathbf{E}$$

and introduce the two events

$$\begin{aligned} \mathbf{A} \ &:= \ \{\zeta_t^1(A) \leq 2^{-2}\,aM \text{ for some time } t \in (0,s)\} \\ \mathbf{B} \ &:= \ \{\zeta_t^2(B) \leq 2^{-5}\,M^2 \text{ for some time } t \in (0,s)\}. \end{aligned} \tag{4.11}$$

The proof of Lemma 12 with $K = aM$ then $K = 2^{-3}\,M^2$, gives

$$P^*(\mathbf{A}) \leq \exp(-2^{-4}\,aM) \quad \text{and} \quad P^*(\mathbf{B}) \leq \exp(-2^{-7}\,M^2). \tag{4.12}$$

As previously, this follows from the fact that the estimates in the proof of Lemma 12 only depend on the initial number of players of a given type and the number of sites which are updated in the set under consideration. Now, observe that, on the intersection of the event that $x \in A$ follows strategy 1 and the event $\mathbf{B}^c$, the payoff of the player at $x$ satisfies

$$\phi(x, \xi_t) \geq a_{12}\ 2^{-5}\,M^2 \quad \text{for all} \quad t \in (0,s).$$

It follows that, on the event $(\mathbf{A} \cup \mathbf{B})^c$, each time a player in the set $B$ updates her strategy, she remains/becomes of type 1 with probability at least

$$\begin{aligned} p_1 \ &\geq \ (a_{12}\ 2^{-5}\,M^2)(2^{-2}\,aM)((a_{12}\ 2^{-5}\,M^2)(2^{-2}\,aM) + (a_{21} + a_{22})(2M+1)^4)^{-1} \\ &\geq \ (a_{12}\ 2^{-5}\,M^2)(2^{-2}\,aM)(a_{21}\,2^5\,M^4)^{-1} \geq 2^{-12}\,(a_{12}/a_{21})\,aM^{-1} \\ &\geq \ 2^5\,c_-(aM^{-1}) \end{aligned} \tag{4.13}$$

for all $M$ sufficiently large. Note that the the second inequality in (4.13) follows from (4.5) while the last inequality follows from the choice of $c_-$ in (4.6). In particular, letting

$$\begin{aligned} X_u &:= \text{card}\,\{x \in B : T_1(x) < s\} \\ X_1 &:= \text{card}\,\{x \in B : T_1(x) < s \text{ and } \xi_s(x) = 1\} \end{aligned} \tag{4.14}$$

observing from (4.13) that $2^{-5}\,M^2\,p_1 \geq c_-(aM)$, and using (4.8), we deduce that

$$\begin{aligned} & P\,(X_1 \leq c_-(aM) \mid (\mathbf{A} \cup \mathbf{B})^c) \\ & \quad \leq \; P\,(X_u \leq 2^{-4}\,M^2) + P\,(X_1 \leq c_-(aM) \mid (\mathbf{A} \cup \mathbf{B})^c \text{ and } X_u > 2^{-4}\,M^2) \\ & \quad \leq \; P\,(\text{Binomial}\,(M^2/4, 1/2) \leq 2^{-4}\,M^2) + P\,(\text{Binomial}\,(2^{-4}\,M^2, p_1) \leq c_-(aM)) \\ & \quad \leq \; P\,(\text{Binomial}\,(M^2/4, 1/2) \leq 2^{-4}\,M^2) + P\,(\text{Binomial}\,(2^{-4}\,M^2, p_1) \leq 2^{-5}\,M^2\,p_1) \\ & \quad \leq \; \exp(-2^{-6}\,M^2) + \exp(-2^{-7}\,M^2\,p_1) \leq (1/2)\exp(-(aM)^{1/2}). \end{aligned} \tag{4.15}$$

Using that $X_1 \leq \zeta_s^1(B)$ and combining (4.12) and (4.15), we conclude that

$$\begin{aligned} & P^*(\zeta_s^1(B) \leq c_-(aM)) \leq P^*(\mathbf{A} \cup \mathbf{B}) + P\,(\zeta_s^1(B) \leq c_-(aM) \mid (\mathbf{A} \cup \mathbf{B})^c) \\ & \quad \leq \; P^*(\mathbf{A}) + P^*(\mathbf{B}) + P\,(X_1 \leq c_-(aM) \mid (\mathbf{A} \cup \mathbf{B})^c) \\ & \quad \leq \; \exp(-2^{-4}\,aM) + \exp(-2^{-7}\,M^2) + (1/2)\exp(-(aM)^{1/2}) \leq \exp(-(aM)^{1/2}) \end{aligned}$$

for all $M$ large, which gives (4.10). Note that, for the first inequality above, we also use the fact that the death-birth updating process $\xi_t$ is Markov. This completes the proof. □

**Creating a pile of players** – The next lemma improves the previous one by showing that if the region $A$ has a large number of type 1 players then the same amount of type 1 players can be created in the nearby region $B$. The idea is to first prove that, as long as the number of type 1 players nearby is small, the number of such players can be increased by a factor $c_+$. Once this threshold is reached, one can find a small box with a large number of type 1 players and apply the previous lemma repeatedly to move a fraction of these players to the target set $B$.

**Lemma 14** – Assume that (4.5) holds. Then,

$$P\,(\zeta_{6s}^1(B) \leq M \mid \zeta_0^1(A) \geq M) \leq \exp(-M^{1/2}) \quad \text{for all } M \text{ large.}$$

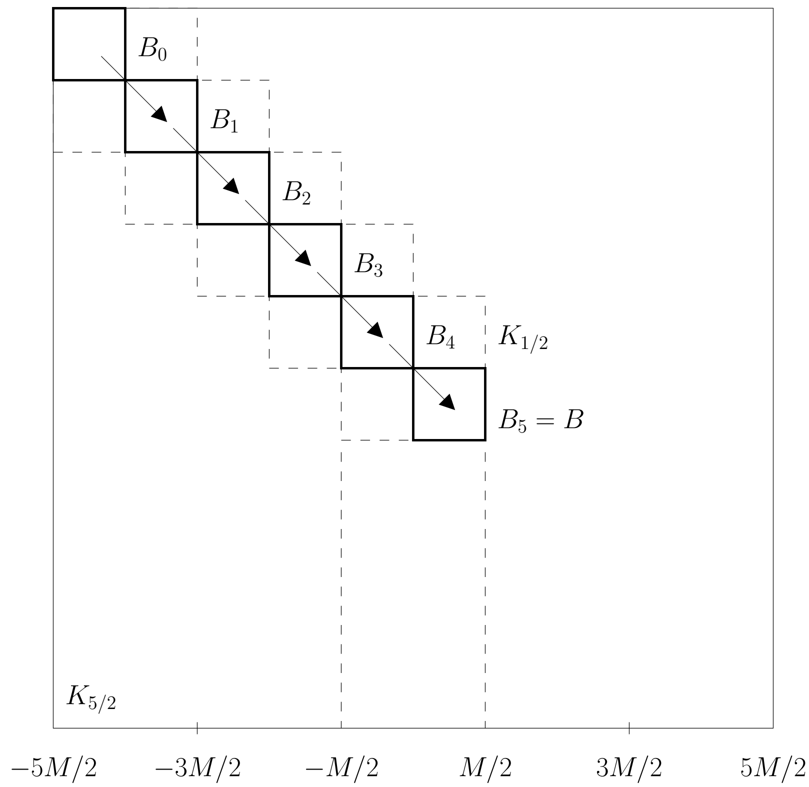



Figure 13: *Picture related to the proof of Lemma 14 illustrating the movement of players to nearby target regions*

*Proof.* To keep track of the amount of type 1 and type 2 players in $K_{5/2}$, which is the key to controlling the payoff of the type 1 players in the set $A$, we set

$$\mathbf{K} := \{\zeta_t^1(K_{5/2}) \geq c_+ M \text{ for some time } t \in (0, s)\}$$

where $c_+$ has been defined in (4.6). The proof is divided into two steps.

**Step 1** – First, we prove that, given $\zeta_0^1(A) \geq M$, the event $\mathbf{K}$ occurs with high probability. The idea is to show that the complement of this event confers a large payoff to the type 1 players in the set $A$, which results in a large production of such players with high probability, thus leading to a contradiction. To make this argument precise, we observe that, with probability one,

$$\zeta_t^2(x + K_1) \geq M^2 \quad \text{for all } x \in A \quad \text{and} \quad \zeta_t^1(K_{5/2}) \leq c_+ M \quad \text{for all} \quad t \in (0, s)$$

on the event $\mathbf{K}^c$.Therefore,

on the event $\{\xi_t(x) = 1\} \cap \mathbf{K}^c$,

$$\phi(x, \xi_t) \geq a_{12}\, M^2 \quad \text{for all} \quad x \in A$$

on the event $\{\xi_t(x) = 2\} \cap \mathbf{K}^c$,

$$\phi(x, \xi_t) \leq a_{21}\, c_+ M + a_{22}\, (2M+1)^2 \leq 2^3\, a_{22}\, M^2 \quad \text{for all} \quad x \in K_{3/2}.$$

In particular, given $(\mathbf{A'} \cup \mathbf{K})^c$ where $\mathbf{A'}$ is the first event (4.11) for $a = 1$, each time a player in $B$ updates her strategy, she remains/becomes of type 1 with probability at least

$$\begin{aligned} q_1 \ &\geq \ (a_{12}\, M^2)(2^{-2}\, M)((a_{12}\, M^2)(2^{-2}\, M) + (2^3\, a_{22}\, M^2)(2M+1)^2)^{-1} \\ &\geq \ (a_{12}\, M^2)(2^{-2}\, M)((2^4\, a_{22}\, M^2)(2M+1)^2)^{-1} \\ &\geq \ 2^{-9}\, (a_{12}/a_{22})\, M^{-1}. \end{aligned} \tag{4.16}$$

Defining $X_u$ and $X_1$ as in (4.14), observing that, by (4.5) and (4.16),

$$2^{-5}\, M^2\, q_1 \geq 2^{-14}\, (a_{12}/a_{22})\, M \geq c_+ M$$

and using (4.8), we deduce that, for all $M$ sufficiently large,

$$\begin{aligned} & P\,(X_1 \leq c_+ M \mid (\mathbf{A'} \cup \mathbf{K})^c) \\ & \quad \leq \ P\,(X_u \leq 2^{-4}\, M^2) + P\,(X_1 \leq c_+ M \mid (\mathbf{A'} \cup \mathbf{K})^c \text{ and } X_u > 2^{-4}\, M^2) \\ & \quad \leq \ P\,(\text{Binomial}\,(M^2/4, 1/2) \leq 2^{-4}\, M^2) + P\,(\text{Binomial}\,(2^{-4}\, M^2, q_1) \leq c_+ M) \\ & \quad \leq \ P\,(\text{Binomial}\,(M^2/4, 1/2) \leq 2^{-4}\, M^2) + P\,(\text{Binomial}\,(2^{-4}\, M^2, q_1) \leq 2^{-5}\, M^2\, q_1) \\ & \quad \leq \ \exp(-2^{-6}\, M^2) + \exp(-2^{-7}\, M^2\, q_1) \leq (1/4)\exp(-M^{1/2}). \end{aligned} \tag{4.17}$$

Using again $X_1 \leq \zeta_s^1(B)$ and combining (4.12) with $a = 1$ and (4.17), we get

$$\begin{aligned} P\,(\mathbf{K}^c \mid \zeta_0^1(A) \geq M) \ &= \ P\,(\zeta_s^1(B) \leq c_+ M \text{ and } \mathbf{K}^c \mid \zeta_0^1(A) \geq M) \\ &\leq \ P\,(\mathbf{A'} \mid \zeta_0^1(A) \geq M) + P\,(\zeta_s^1(B) \leq c_+ M \mid (\mathbf{A'} \cup \mathbf{K})^c) \\ &\leq \ \exp(-2^{-4}\, M) + (1/4)\exp(-M^{1/2}) \\ &\leq \ (1/2)\exp(-M^{1/2}) \end{aligned} \tag{4.18}$$

for all dispersal range $M$ sufficiently large. Note that, for the first inequality above, we also use the fact that the death-birth updating process $\xi_t$ is Markov.

**Step 2** – Now, given $\mathbf{K}$, there is a box with at least $5^{-2}\,c_+ M$ type 1 players. One can move a fraction of these players to the target set $B$ in at most five steps, applying Lemma 13. To begin with, we observe that some basic geometry implies that, given $\mathbf{K}$, there exist

$$B_0, \ldots, B_5 \subset K_{5/2} \quad \text{and} \quad t_0 \in (0, s)$$

such that the following three conditions hold:

(a) We have $\zeta^1_{t_0}(B_0) \geq 5^{-2}\,c_+ M$.

(b) For $k = 0, \ldots, 5$, we have $\mathrm{card}(B_k) = 2^{-2}\,M^2$ with $B_5 = B$.

(c) For $k = 0, \ldots, 4$, we have $\max_{j=1,2} |x_j - y_j| \leq M$ for all $(x, y) \in B_k \times B_{k+1}$.

We refer to Figure 13 for an illustration of the worst case scenario where all the type 1 players are located in one of the corners of $K_{5/2}$. Under these conditions, we can bring type 1 players to our target set in at most five steps applying repeatedly Lemma 13. Indeed,

$$\begin{aligned} & P\,(\zeta^1_t(B) \leq 2^7 M \text{ for all } t \in (0, 6s) \mid \mathbf{K}) \\ & \quad \leq \ P\,(\zeta^1_{t_0+5s}(B_5) \leq 2^7 M \mid \zeta^1_{t_0}(B_0) \geq 5^{-2}\,c_+ M = 2^7 (c_-)^{-5}\,M) \\ & \quad \leq \ 1 - \textstyle\prod_{k=0,1,2,3,4} P\,(\zeta^1_s(B_{k+1}) \geq 2^7 (c_-)^{k-4}\,M \mid \zeta^1_0(B_k) \geq 2^7 (c_-)^{k-5}\,M) \\ & \quad \leq \ 1 - \textstyle\prod_{k=0,1,2,3,4} (1 - \exp(-2^{7/2}\,(c_-)^{(k-5)/2}\,M^{1/2})) \\ & \quad \leq \ 1 - (1 - \exp(-2^{7/2}\,M^{1/2}))^5 \\ & \quad \leq \ 5 \times \exp(-2^{7/2}\,M^{1/2}) \leq (1/4) \exp(-M^{1/2}) \end{aligned} \tag{4.19}$$

for all dispersal range $M$ sufficiently large.

**Conclusion** – Combining (4.18)–(4.19), we deduce that

$$\begin{aligned} & P\,(\zeta^1_t(B) \leq 2^7 M \text{ for all } t \in (0, 6s) \mid \zeta^1_0(A) \geq M) \\ & \quad \leq \ P\,(\mathbf{K}^c \mid \zeta^1_0(A) \geq M) + P\,(\zeta^1_t(B) \leq 2^7 M \text{ for all } t \in (0, 6s) \mid \mathbf{K}) \\ & \quad \leq \ (1/2) \exp(-M^{1/2}) + (1/4) \exp(-M^{1/2}) = (3/4) \exp(-M^{1/2}) \end{aligned}$$

which, applying Lemma 12 with $n = 6$ and $K = 2^7 M$, implies that

$$\begin{aligned} & P\,(\zeta^1_{6s}(B) \leq M \mid \zeta^1_0(A) \geq M) \\ & \quad \leq \ P\,(\zeta^1_t(B) \leq 2^7 M \text{ for all } t \in (0, 6s) \mid \zeta^1_0(A) \geq M) \\ & \qquad\qquad + P\,(\zeta^1_{6s}(B) \leq 2^{-7}\,2^7 M \mid \zeta^1_t(B) \geq 2^7 M \text{ for some } t \in (0, 6s)) \\ & \quad \leq \ (3/4) \exp(-M^{1/2}) + \exp(-2^{-2}\,M) \end{aligned}$$

Therefore, for all $M$ large,

$$P\left(\zeta_{6s}^1(B) \leq M \mid \zeta_0^1(A) \geq M\right) \leq \exp(-M^{1/2}).$$

This completes the proof. $\square$

**Block construction** – To deduce coexistence, we use Lemma 14 in combination with some obvious symmetry and a block construction, a technique that has been introduced in Bramson and Durrett 1988 and is reviewed in Durrett 1995. The idea of the block construction is to define a coupling between the process under consideration properly rescaled in space and time and supercritical percolation. More precisely, let $\mathscr{H}$ be the directed graph with vertex set

$$H := \{(z,n) \in \mathbb{Z}^2 \times \mathbb{Z}_+ : z_1 + z_2 + n \text{ is even}\}$$

and in which there is an oriented edge

$$(z,n) \to (z',n') \quad \text{if and only if} \quad (z' = z \pm e_1 \text{ or } z' = z \pm e_2) \ \text{ and } \ n' = n+1$$

where $e_j$ is the $j$th unit vector. Then, we consider the 14 dependent oriented site percolation process with density equal to $1 - \epsilon$ on this directed graph, i.e., we assume that

$$P\left((z_i, n_i) \text{ is closed for } i = 1, 2, \ldots, m\right) = \epsilon^m$$

whenever $|z_i - z_j| \vee |n_i - n_j| > 14$ for $i \neq j$. Then, we set

$$B_z := (M/2)\, z + [-M/4, M/4)^2 \ \text{ for all } \ z \in \mathbb{Z}^2 \quad \text{and} \quad T := 6s = 6 \ln(2)$$

and declare site $(z,n) \in H$ to be good whenever

$$\zeta_{nT}^i(B_z) = \text{card}\,\{x \in B_z : \xi_{nT}(x) = i\} \geq M \quad \text{for} \quad i = 1, 2.$$

Finally, for all $n \in \mathbb{N}$, we define

$$W_n^\epsilon := \{z : (z,n) \text{ is wet}\} \quad \text{and} \quad X_n := \{z : (z,n) \text{ is good}\}$$

where a site is said to be wet if it can be reached from a directed path of open sites starting at level zero. The next lemma shows that, for all $\epsilon > 0$, one can find a sufficiently large dispersal range such that the set of good sites dominates stochastically the set of wet sites. In view of the definition of a good site, this will imply coexistence of both strategies.

**Lemma 15** – Assume (4.5) and fix $\epsilon > 0$. Then, for all $M$ large,

$$P(z \in W_n^\epsilon) \leq P(z \in X_n) \quad \text{for all} \quad (z,n) \in H \quad \text{whenever} \quad W_0^\epsilon \subset X_0.$$

*Proof.* First, we define the collection of events

$$\mathbf{B}_i(z,n) := \{\zeta_{nT}^i(B_z) \geq M\} \quad \text{for all} \quad (z,n) \in H \text{ and } i = 1,2.$$

Then, since for $j = 1,2$,

$$B_z, B_{z \pm e_j} \subset ((M/2)\,z \pm (M/4)\,e_j) + K_{1/2} \quad \text{and} \quad \text{card}(B_z) = \text{card}(B_{z \pm e_j}) = 2^{-2}\,M^2$$

we can apply Lemma 14 to get

$$\begin{aligned} & P(\mathbf{B}_1(z \pm e_j, n+1) \text{ for } j = 1,2 \mid \mathbf{B}_1(z,n)) \\ & \quad = P(\zeta_{(n+1)T}^1(B_{z \pm e_j}) \geq M \text{ for } j = 1,2 \mid \zeta_{nT}^1(B_z) \geq M) \\ & \quad \geq 1 - 4 \times P(\zeta_{(n+1)T}^1(B_{z+e_1}) \leq M \mid \zeta_{nT}^1(B_z) \geq M) \geq 1 - 4 \times \exp(-M^{1/2}) \\ & \quad \geq 1 - \epsilon/2 \end{aligned}$$

for all $M$ large. Since all the estimates in the proof of Lemma 14 hold uniformly in all possible initial configurations such that $\zeta_0^1(A) \geq M$, we also have

$$P(\mathbf{B}_1(z \pm e_j, n+1) \text{ for } j = 1,2 \,|\, \mathbf{B}_1(z,n) \cap \mathbf{B}_2(z,n)) \geq 1 - \epsilon/2.$$

By symmetry, the same holds for strategy 2, therefore

$$\begin{aligned} & P(z \pm e_j \in X_{n+1} \text{ for } j = 1,2 \mid z \in X_n) \\ & \quad = P(\mathbf{B}_1(z \pm e_j, n+1) \cap \mathbf{B}_2(z \pm e_j, n+1) \text{ for } j = 1,2 \mid \mathbf{B}_1(z,n) \cap \mathbf{B}_2(z,n)) \\ & \quad \geq -1 + P(\mathbf{B}_1(z \pm e_j, n+1) \text{ for } j = 1,2 \mid \mathbf{B}_1(z,n) \cap \mathbf{B}_2(z,n)) \\ & \qquad\qquad + P(\mathbf{B}_2(z \pm e_j, n+1) \text{ for } j = 1,2 \mid \mathbf{B}_1(z,n) \cap \mathbf{B}_2(z,n)) \\ & \quad \geq -1 + 2\,(1 - \epsilon/2) = 1 - \epsilon. \end{aligned} \tag{4.20}$$

Now, for every $(z,n) \in H$, we let $\mathbf{G}(z,n)$ be the set of realizations of the graphical representation restricted to the finite space-time box

$$R(z,n) := ((M/2)z, nT) + (K_{7/2} \times [0,T])$$

and such that $(z \pm e_j, n+1)$ are good whenever $(z,n)$ is good and $\mathbf{G}(z,n)$ occurs. Since all the estimates in Lemma 14 hold regardless of the configuration outside the spatial region $K_{5/2}$, it follows from (4.20) that the set of realizations $\mathbf{G}(z,n)$ has probability at least $1-\epsilon$. In summary, we have a collection of events that satisfy the following three properties:

(a) $\mathbf{G}(z,n)$ is measurable with respect to the graphical representation in $R(z,n)$.

(b) For all $M$ large, we have $P(\mathbf{G}(z,n)) \geq 1-\epsilon$.

(c) We have the inclusion $\mathbf{G}(z,n) \cap \{z \in X_n\} \subset \{z \pm e_j \in X_{n+1} \text{ for } j=1,2\}$.

Observing also that

$$R(z,n) \cap R(z',n') = \varnothing \quad \text{when} \quad |z-z'| \vee |n-n'| \geq 2 \times 7 = 14,$$

we deduce from Durrett 1995, Theorem 4.3 the existence of a coupling between the long range death-birth process and the oriented site percolation process such that

$$P(W_n^\epsilon \subset X_n) = 1 \quad \text{whenever} \quad W_0^\epsilon \subset X_0.$$

The lemma directly follows from the existence of this coupling. □

Using the previous lemma, we can now prove Theorem 7. Fix $\epsilon > 0$ small enough to make the percolation process supercritical and $M$ accordingly, and observe that

$$\begin{aligned} \liminf_{t\to\infty} P(\xi_t(x) \neq \xi_t(y)) &= \liminf_{t\to\infty} P(\xi_t(0) \neq \xi_t(y-x)) \\ &\geq \liminf_{t\to\infty} P(\xi_t(0) \neq \xi_t(y-x) \,|\, 0 \in W_{2\lfloor t/T \rfloor}^\epsilon)\, P(0 \in W_{2\lfloor t/T\rfloor}^\epsilon \,|\, W_0^\epsilon = X_0) \end{aligned} \tag{4.21}$$

for all $x,y \in \mathbb{Z}^2$. In view of the definition of a good site and the fact that, for the coupling defined in the proof of Lemma 15, wet sites are also good, the range of the interactions $M$ being fixed, there exists a positive constant $p > 0$ that depends on $M$ but not on time $t$ such that

$$P(\xi_t(0) \neq \xi_t(y-x) \,|\, 0 \in W_{2\lfloor t/T\rfloor}^\epsilon) \geq p > 0 \quad \text{for all} \quad x \neq y. \tag{4.22}$$

Now, starting from a product measure with a positive density of both strategies, the number of good sites at level zero is almost surely infinite. Since in addition $\epsilon$ has been fixed so that the percolation process is supercritical, we deduce that

$$\liminf_{n\to\infty} P(0 \in W_{2n}^\epsilon \,|\, W_0^\epsilon = X_0) > 0. \tag{4.23}$$

Combining (4.21)–(4.23), we conclude that, for all $M$ large,

$$\liminf_{t\to\infty} P(\xi_t(x) \neq \xi_t(y)) \geq p \liminf_{n\to\infty} P(0 \in W_{2n}^\epsilon \,|\, W_0^\epsilon = X_0) > 0$$

for all $x \neq y$. This completes the proof of Theorem 7.

### 4.3 Coupling With Modified Voter Models

This section is devoted to the proof of Theorem 8. The common ingredient to prove both parts of the theorem is to couple the process with the modified voter models $\zeta_t^1$ and $\zeta_t^2$ whose transitions at vertex $x$ are given by the following expressions

$$i \ \to \ j \quad \text{at rate} \quad c_{i\to j}(x,\zeta^i) := (1-\epsilon)\, f_j(x,\zeta^i) + \epsilon\, \mathbf{1}\{f_i(x,\zeta^i) = 0\}$$

$$j \ \to \ i \quad \text{at rate} \quad c_{j\to i}(x,\zeta^i) := (1-\epsilon)\, f_i(x,\zeta^i) + \epsilon\, \mathbf{1}\{f_i(x,\zeta^i) \neq 0\}.$$

for $\{i,j\} = \{1,2\}$ and where

$$f_j(x,\zeta^i) = \operatorname{card}\{y \in N_x : \zeta^i(x) = j\} / \operatorname{card} N_x = (1/N)\, N_j(x,\zeta^i)$$

denotes the fraction of neighbors of vertex $x$ in state $j$. In words, the transition rates indicate that particles are updated at rate one and that, at the time of an update,

- with probability $1-\epsilon > 0$, the new type is chosen uniformly at random from the interaction neighborhood just like in the voter model,
- with probability $\epsilon > 0$, the new type is $i$ unless all the neighbors are of type $j$.

The results of Lanchier 2013, section 3 show using duality that type $i$ particles win for this process. In particular, to prove that strategy 1 wins, it suffices to prove that the set of type 1 players in the death-birth process dominates stochastically its counterpart in $\zeta_t^1$, which follows from

$$\xi \leq \zeta^1 \ \text{ and } \ \xi(x) = \zeta^1(x) \quad \text{implies that}$$
$$p_{1\to 2}(x,\xi) \leq c_{1\to 2}(x,\zeta^1) \ \text{ and } \ p_{2\to 1}(x,\xi) \geq c_{2\to 1}(x,\zeta^1)$$

according to Theorem III.1.5 in Liggett 1985. Since in addition the transition rates of the modified voter models are monotone with respect to the number of neighbors of each type, in order to show that strategy 1 wins, it suffices to prove that the simplified implication

$$\begin{array}{l} N_1(x,\xi) = N_1(x,\zeta^1) \ \text{ and } \ \xi(x) = \zeta^1(x) \quad \text{implies that} \\ \qquad p_{1\to 2}(x,\xi) \leq c_{1\to 2}(x,\zeta^1) \ \text{ and } \ p_{2\to 1}(x,\xi) \geq c_{2\to 1}(x,\zeta^1) \end{array} \tag{4.24}$$

holds for some $\epsilon > 0$. By symmetry, strategy 2 wins if the implication

$$\begin{array}{l} N_2(x,\xi) = N_2(x,\zeta^2) \ \text{ and } \ \xi(x) = \zeta^2(x) \quad \text{implies that} \\ \qquad p_{1\to 2}(x,\xi) \geq c_{1\to 2}(x,\zeta^2) \ \text{ and } \ p_{2\to 1}(x,\xi) \leq c_{2\to 1}(x,\zeta^2) \end{array} \tag{4.25}$$

holds for some $\epsilon > 0$. Using (4.24), we now prove Theorem 8.a.

**Lemma 16** – Recall that $a_{21} > a_{12}$. Then, strategy 1 wins when

$$\min(a_{12} - a_{22}, a_{11} - a_{21}) > (N-1)(a_{21} - a_{12}).$$

*Proof.* In view of the discussion above, it suffices to prove that (4.24) holds. First, we observe that, when the fraction of type 1 neighbors of vertex $x$ is equal to either zero or one, the transition rates are the same for both processes so it remains to prove (4.24) under the assumption

$$N_1 N_2 \neq 0 \quad \text{where} \quad N_j := N_j(x,\xi) = N_j(x,\zeta^1) \quad \text{for} \quad j = 1,2. \tag{4.26}$$

The transition rate at vertex $x$ depends on the payoff of its neighbors, and the main idea is to express the transition rates by distinguishing between the part of the payoff coming from $x$ and the part of the payoff coming from the other neighbors' neighbors. In order to make this distinction, we introduce the following four weighting factors:

$$w_{ij} := \textstyle\sum_{y \sim x} \big(\mathbf{1}\{\xi(y) = i\} \sum_{z \sim y, z \neq x} \mathbf{1}\{\xi(z) = j\}\big) \quad \text{for} \quad i,j = 1,2.$$

That is, $w_{ij}$ is the number of type $j$ neighbors (excluding vertex $x$) of a type $i$ neighbor of vertex $x$ counted with order of multiplicity. Note that, for $i = 1,2$, we have

$$N_i + \textstyle\sum_{j=1,2} w_{ij} = N N_i. \tag{4.27}$$

In addition, for $i \neq j$, the transition rates can be expressed as

$$p_{i \to j}(x,\xi) = \frac{(N_j + w_{ji})\, a_{ji} + w_{jj}\, a_{jj}}{(N_i + w_{ii})\, a_{ii} + w_{ij}\, a_{ij} + (N_j + w_{ji})\, a_{ji} + w_{jj}\, a_{jj}}. \tag{4.28}$$

Using (4.27)–(4.28), we now prove that (4.24) holds in the nontrivial case (4.26).

**Transition** $1 \to 2$ – Using (4.27) and $a_{22} < a_{12} < a_{21}$, we get

$$\begin{aligned}
(N_2 + w_{21})\, a_{21} + w_{22}\, a_{22} \;&\leq\; (N_2 + w_{21} + w_{22})\, a_{21} = N N_2\, a_{21} \\
(N_1 + w_{11})\, a_{11} + w_{12}\, a_{12} \;&=\; (N_1 + w_{11} + w_{12})\, a_{11} + w_{12}\,(a_{12} - a_{11}) \\
&=\; N N_1\, a_{11} + w_{12}\,(a_{12} - a_{11}).
\end{aligned}$$

This, together with (4.28) for $i = 1$ and $j = 2$, implies that

$$\begin{aligned}
p_{1 \to 2}(x,\xi) \;&\leq\; \frac{N N_2\, a_{21}}{N N_1\, a_{11} + w_{12}\,(a_{12} - a_{11}) + N N_2\, a_{21}} \\
&=\; \frac{N N_2\, a_{21}}{N N_1\,(a_{11} - a_{21}) + w_{12}\,(a_{12} - a_{11}) + N^2\, a_{21}} = \frac{N N_2\, a_{21}}{N^2 a_{21} + \rho_1}
\end{aligned} \tag{4.29}$$

where, since $a_{11} - a_{21} > (N-1)(a_{21} - a_{12})$,

$$\begin{aligned} \rho_1 \;&:=\; N N_1 \, (a_{11} - a_{21}) + w_{12} \, (a_{12} - a_{11}) \\ &=\; (N N_1 - w_{12})(a_{11} - a_{21}) + w_{12} \, (a_{12} - a_{21}) \\ &>\; (N-1)(N N_1 - w_{12})(a_{21} - a_{12}) + w_{12} \, (a_{12} - a_{21}) \\ &=\; N \, ((N-1) \, N_1 - w_{12})(a_{21} - a_{12}) = N \, w_{11} \, (a_{21} - a_{12}) \geq 0. \end{aligned} \tag{4.30}$$

Note that the strict inequality holds because (4.26)–(4.27) imply that

$$N N_1 - w_{12} = (N_1 + w_{11} + w_{12}) - w_{12} = N_1 + w_{11} \geq N_1 > 0.$$

In view of (4.29)–(4.30), there exists $\epsilon_1 > 0$ small such that

$$\begin{aligned} p_{1 \to 2}(x, \xi) \;&\leq\; N N_2 \, a_{21} \, (N^2 a_{21} + \rho_1)^{-1} = N_2 \, (N + (\rho_1 / N a_{21}))^{-1} \\ &\leq\; (1 - \epsilon) \, f_2(x, \xi) = (1 - \epsilon) \, f_2(x, \zeta^1) = c_{1 \to 2}(x, \zeta^1) \end{aligned}$$

whenever $\epsilon < \epsilon_1$ and (4.26) holds.

**Transition** $2 \to 1$ – Since $a_{11} > a_{21} > a_{12}$ and $a_{22} - a_{12} < (N-1)(a_{12} - a_{21})$, using the previous estimates and obvious symmetry, we show that

$$p_{2 \to 1}(x, \xi) \geq N N_1 \, a_{12} \, (N^2 a_{12} + \rho_2)^{-1} \quad \text{where} \quad \rho_2 < N \, w_{22} \, (a_{12} - a_{21}) \leq 0.$$

In particular, there exists $\epsilon_2 > 0$ small such that

$$\begin{aligned} p_{2 \to 1}(x, \xi) \;&\geq\; N_1 \, (N + (\rho_2 / N a_{12}))^{-1} \\ &\geq\; (1 - \epsilon) \, f_1(x, \xi) + \epsilon = (1 - \epsilon) \, f_1(x, \zeta^1) + \epsilon = c_{2 \to 1}(x, \zeta^1) \end{aligned}$$

whenever $\epsilon < \epsilon_2$ and (4.26) holds.

In conclusion, the implication (4.24) holds for $\epsilon$ smaller than $\min(\epsilon_1, \epsilon_2) > 0$, which shows that strategy 1 wins under the assumptions of the lemma. □

Repeating the proof of Lemma 16 step by step but exchanging the role of both strategies only shows that strategy 2 wins under the strong assumption $a_{11} < a_{12} < a_{21} < a_{22}$. In fact, this sufficient condition for strategy 2 to win can be easily improved to

$$\max(a_{11}, a_{12}) < \min(a_{21}, a_{22})$$

by using a coupling between the death-birth process and a biased voter model with a selective advantage for type 2 particles to show that the former dominates the latter. To prove that strategy 2 wins in the larger region stated in Theorem 8.b, we couple the death-birth process with the second modified voter model $\zeta_t^2$ but using techniques different from the ones used to show the first part of the theorem. To explain the assumptions of the theorem, we note that our approach requires the interaction neighborhood to have a certain connectivity property which does not hold in the one-dimensional nearest neighbor case. First, we introduce the payoff functions

$$\phi_1(z) \ := \ a_{11}\,(z/N) + a_{12}\,(1 - z/N) = (a_{11} - a_{12})(z/N) + a_{12}$$
$$\phi_2(z) \ := \ a_{22}\,(z/N) + a_{21}\,(1 - z/N) = (a_{22} - a_{21})(z/N) + a_{21}$$

for all $z = 0, 1, \ldots, N$, and let $a := \max\,(a_{11}, a_{12}, a_{21}, a_{22})$ and

$$M_+ := \max_z \ \phi_1(z) \ \geq \ \max_{z \neq N} \ \phi_1(z) =: M_-$$
$$m_- := \min_z \ \phi_2(z) \ \leq \ \min_{z \neq N} \ \phi_2(z) =: m_+.$$

The following lemma gives a sufficient condition on these minimum and maximum payoffs for strategy 2 to win. This condition is made more explicit in the subsequent lemma.

**Lemma 17** – Strategy 2 wins whenever $(M, d) \neq (1, 1)$ and

$$(N-1)\,m_+ > (N-2)\,M_+ + M_- \quad \text{and} \quad (N-1)\,M_- < (N-2)\,m_- + m_+. \tag{4.31}$$

*Proof.* Following as in the proof of Lemma 16, it suffices to show that (4.25) holds. This is again trivial when the fraction of type 1 neighbors of vertex $x$ is equal to either zero or one so we focus from now on on the nontrivial case where

$$N_1 N_2 \neq 0 \quad \text{where} \quad N_j := N_j(x, \xi) = N_j(x, \zeta^2) \quad \text{for} \quad j = 1, 2, \tag{4.32}$$

indicating that $x$ has two neighbors $y^*$ and $z^*$ with different strategies. Except in the one-dimensional nearest neighbor case $M = d = 1$, one can find two vertices $y_*$ and $z_*$ such that

$$y_*, z_* \neq x \quad \text{and} \quad y_* \in N_x \cap N_{y^*} \quad \text{and} \quad z_* \in N_x \cap N_{z^*} \quad \text{and} \quad y_* \in N_{z_*}$$

and we may assume that $y^*$ and $z^*$ are neighbors of each other:

$$\xi(y^*) = 1 \quad \text{and} \quad \xi(z^*) = 2 \quad \text{and} \quad y^*, z^* \in N_x \quad \text{and} \quad y^* \in N_{z^*}.$$

The rest of the proof is divided into two steps depending on the transition.

**Transition** $1 \to 2$ – Assume that $\xi(x) = \zeta^2(x) = 1$. Then for all $y \in N_x$,

$$\phi(y, \xi) \leq M_+ \text{ when } \xi(y) = 1 \quad \text{while} \quad \phi(y, \xi) \geq m_+ \text{ when } \xi(y) = 2.$$

In addition, we have $\phi(y^*, \xi) \le M_-$ therefore

$$\begin{aligned} p_{1\to 2}(x,\xi) \ &\ge \ \frac{m_+\, N_2}{(N_1 - 1)\, M_+ + M_- + m_+\, N_2} \\ &= \ \frac{m_+\, N_2}{(m_+ - M_+)\, N_2 + (N-1)\, M_+ + M_-}. \end{aligned} \tag{4.33}$$

Now, for all $z \in [0, N]$, we define the functions

$$g_1(z) := \frac{m_+\, z}{(m_+ - M_+)\, z + (N-1)\, M_+ + M_-} \quad \text{and} \quad h_1(z) := \frac{(1-\epsilon)\, z}{N} + \epsilon.$$

Since $(N-1)\, m_+ > (N-2)\, M_+ + M_-$,

$$g_1(1) = \frac{m_+}{m_+ + (N-2)\, M_+ + M_-} > \frac{1}{N}. \tag{4.34}$$

We then distinguish two cases.

**Case 1** – When $M_+ > m_+$, it follows from (4.34) that

$$g_1(z) \ge g_1(1)\, z > z/N \quad \text{for all} \quad z \in [1, N-1]$$

therefore $g_1(z) \ge h_1(z)$ for all $\epsilon > 0$ small by continuity.

**Case 2** – When $M_+ \le m_+$, the function $g_1$ is concave. Moreover,

$$g_1(1) \ge \frac{1}{N} + \left(1 - \frac{1}{N}\right) \epsilon = h_1(1)$$

for all $\epsilon > 0$ small according to (4.34), while

$$g_1(N) = \frac{N m_+}{N m_+ - M_+ + M_-} \ge 1 = h_1(N).$$

The previous two inequalities together with the fact that the function $g_1$ is concave imply that $g_1$ dominates $h_1$ for all $\epsilon > 0$ small (see Figure 14 for a picture).

Recalling (4.33), we deduce that, in both cases and when (4.32) holds,

$$p_{1\to 2}(x,\xi) \ge g_1(N_2) \ge h_1(N_2) = c_{1\to 2}(x, \zeta^2)$$

which proves the first inequality in (4.25).

**Transition** $2 \to 1$ – Assume that $\xi(x) = \zeta^2(x) = 2$. Then for all $y \in N_x$,

$$\phi(y,\xi) \le M_- \text{ when } \xi(y) = 1 \quad \text{while} \quad \phi(y,\xi) \ge m_- \text{ when } \xi(y) = 2.$$

In addition, we have $\phi(z^*, \xi) \geq m_+$ therefore

$$
\begin{aligned}
p_{2\to1}(x,\xi) \ &\leq \ \frac{M_-\,N_1}{M_-\,N_1 + (N_2 - 1)\,m_- + m_+} \\
&= \ \frac{M_-\,N_1}{(M_- - m_-)\,N_1 + (N-1)\,m_- + m_+}.
\end{aligned}
\tag{4.35}
$$

Now, for all $z \in [0, N]$, we define the functions

$$
g_2(z) := \frac{M_-\,z}{(M_- - m_-)\,z + (N-1)\,m_- + m_+} \quad \text{and} \quad h_2(z) := \frac{(1-\epsilon)\,z}{N}.
$$

Since $(N-1)\,M_- < (N-2)\,m_- + m_+ \leq (N-1)\,m_+$,

$$
\begin{aligned}
g_2(1) \ &= \ \frac{M_-}{M_- + (N-2)\,m_- + m_+} < \frac{1}{N} \\
g_2(N-1) \ &= \ \frac{(N-1)\,M_-}{(N-1)\,M_- + m_+} < 1 - \frac{1}{N}.
\end{aligned}
\tag{4.36}
$$

As before, we distinguish two cases.

**Case 1** – When $M_- > m_-$, it follows from (4.36) that

$$
g_2(z) \leq g_2(1)\,z < z/N \quad \text{for all} \quad z \in [1, N-1]
$$

therefore $g_2(z) \leq h_2(z)$ for all $\epsilon > 0$ small by continuity.

**Case 2** – When $M_- \leq m_-$, the function $g_2$ is convex. Moreover,

$$
\begin{aligned}
g_2(N-1) \ &\geq \ (1 - 1/N)(1-\epsilon) = h_2(N-1) \quad \text{for} \quad \epsilon > 0 \text{ small} \\
g_2(0) \ &= \ h_2(0) = 0
\end{aligned}
$$

where the first inequality follows from (4.36). This again implies that $h_2$ dominates $g_2$ for all $\epsilon > 0$ small, and we refer to the right-hand side of Figure 14 for a picture.

Recalling (4.35), we deduce that, in both cases and when (4.32) holds,

$$
p_{2\to1}(x,\xi) \leq g_2(N_1) \leq h_2(N_1) = c_{2\to1}(x, \zeta^2)
$$

which proves the second inequality in (4.25). This completes the proof.

□

To complete the proof of Theorem 8, the last step is to re-express the condition in the previous lemma using the payoff coefficients.

**Lemma 18** – Let $a_{12} < a_{21}$ and $(M, D) \neq (1, 1)$. Then (4.31) holds whenever

$$
(N^2 - N - 1)\,\max\,(a_{11} - a_{21}, a_{12} - a_{22}, a_{11} - a_{22}) < a_{21} - a_{12}. \tag{4.37}
$$

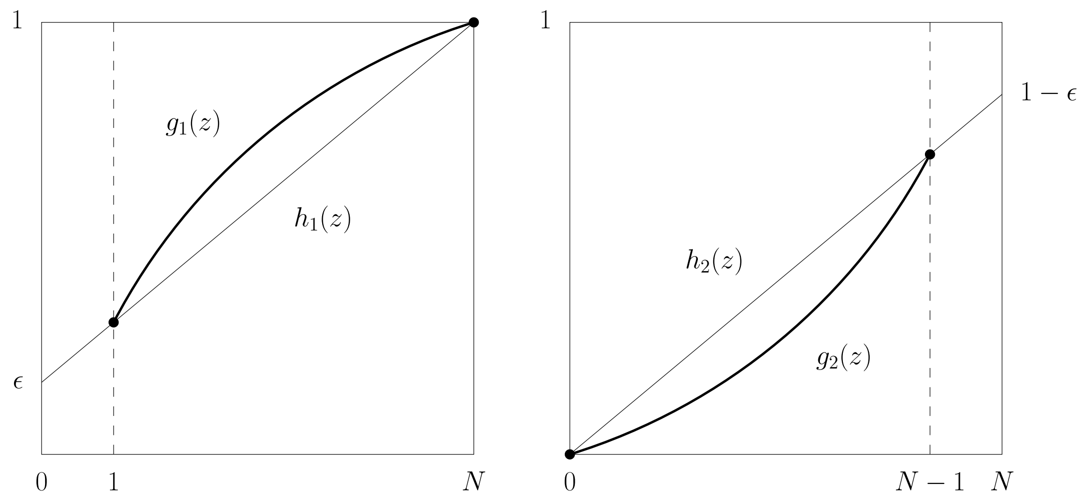



Figure 14: *Picture related to the proof of Lemma 17 illustrating the concavity arguments*

*Proof.* We distinguish four cases depending on the sign of $a_{11} - a_{12}$ and $a_{22} - a_{21}$.

**Case 1** – When $a_{11} < a_{12}$ and $a_{22} > a_{21}$, we have

$$M_+ = M_- = a_{12} \quad \text{and} \quad m_- = m_+ = a_{21}$$

therefore (4.31) holds if and only if $a_{12} < a_{21}$, which is true by assumption.

**Case 2** – When $a_{11} > a_{12}$ and $a_{22} > a_{21}$, we have

$$M_+ = a_{11}, \quad M_- = a_{11} + (1/N)(a_{12} - a_{11}), \quad m_- = m_+ = a_{21}.$$

Using some basic algebra, we deduce that (4.31) holds if and only if

$$(N^2 - N - 1)(a_{11} - a_{21}) < a_{21} - a_{12} \quad \text{and} \quad (N - 1)(a_{11} - a_{21}) < a_{21} - a_{12}$$

therefore (4.31) holds if and only if $(N^2 - N - 1)(a_{11} - a_{21}) < a_{21} - a_{12}$.

**Case 3** – Assume that $a_{11} < a_{12}$ and $a_{22} < a_{21}$. This case can be deduced from the previous one by symmetry exchanging the role of the two strategies, and we find that

$$\text{(4.31) holds} \quad \text{if and only if} \quad (N^2 - N - 1)(a_{12} - a_{22}) < a_{21} - a_{12}.$$

**Case 4** – When $a_{11} > a_{12}$ and $a_{22} < a_{21}$, it is easier to prove the result graphically and we refer to the phase diagram of Figure 11 for an illustration of some of the arguments of the proof. In this case, the minimum and maximum payoffs are given by

$$M_+ = a_{11} \quad \text{and} \quad M_- = a_{11} + (1/N)(a_{12} - a_{11})$$

$$m_- = a_{22} \quad \text{and} \quad m_+ = a_{22} + (1/N)(a_{21} - a_{22})$$

so the two inequalities in (4.31) are respectively equivalent to

$$\begin{aligned}(N-1)^2\, a_{22} \ &> \ (N^2-N-1)\, a_{11} + a_{12} - (N-1)\, a_{21} \\ (N-1)^2\, a_{11} \ &< \ (N^2-N-1)\, a_{22} + a_{21} - (N-1)\, a_{12}.\end{aligned} \tag{4.38}$$

Since in addition $a_{11} > a_{12}$ and $a_{22} < a_{21}$, this specifies two triangles with two common sides, one vertical side and one horizontal side that intersect at point

$$p := (a_{12}, a_{21}) \quad \text{in the } a_{11} - a_{22} \text{ plane.}$$

For the first inequality in (4.38), the third side of the triangle is the segment line going through point $p_+$ and with slope $s_+$ where

$$\begin{aligned}p_+ \ &:= \ (a_{21} + (N^2-N-1)^{-1}\,(a_{21}-a_{12}), a_{21}) \\ s_+ \ &:= \ (N^2-N-1)(N-1)^{-2} = 1 + (N-2)(N-1)^{-2} > 1.\end{aligned} \tag{4.39}$$

Using some obvious symmetry, one finds that the third side of the triangle specified by the second inequality in (4.38) is characterized by the point and slope

$$p_- := (a_{12}, a_{12} - (N^2-N-1)^{-1}\,(a_{21}-a_{12})) \quad \text{and} \quad s_- = 1/s_+ < 1. \tag{4.40}$$

Since the segment line connecting $p_-$ and $p_+$ has slope one, the triangle $(p, p_-, p_+)$ is contained in the intersection of the two triangles specified by (4.39)–(4.40). In particular, whenever the payoff coefficients are in this triangle, which is equivalent to

$$a_{11} > a_{12} \quad \text{and} \quad a_{22} < a_{21} \quad \text{and} \quad (N^2-N-1)(a_{11}-a_{22}) < a_{21} - a_{12},$$

the two inequalities in (4.38) hold.

Since all four cases hold simultaneously when (4.37) holds, the proof is complete. □

Theorem 8.b directly follows from Lemmas 17 and 18.

## 4.4 Coupling With a Pure Growth Process

This section is devoted to the proof of Theorem 9 which states that, the other payoffs being fixed, strategy 1 wins whenever the payoff coefficient $a_{11}$ is sufficiently large. The intuition behind this result, which is also the first step of the proof, is to observe that, in the limit as $a_{11}$ goes to infinity, type 1 players with at least one neighbor of their own type never change their strategy. In particular, the set of type 1 players dominates a Richardson model Richardson 1973, i.e., a contact process with no death, which obviously implies that strategy 1 wins. Using a block construction and the fact that the transition rates are continuous functions of the payoffs, we deduce that the process reaches an equilibrium with a density of type 1 close to one when $a_{11}$ is finite but large. The rest of the proof consists in showing that we can indeed convert the remaining type 2 players, which directly follows from percolation results already established in Durrett 1992; Lanchier 2013.

To turn our sketch into a rigorous proof, we let $\zeta_t$ be the $d$-dimensional Richardson model with parameter $\mu$. In this spin system, each vertex of the $d$-dimensional integer lattice is either empty or occupied by a particle. Each particle produces a new particle which is then sent to a neighbor chosen uniformly at random at rate $\mu$. This results in either an empty site becoming occupied or two particles coalescing in case the target site is already occupied. In addition, occupied sites remain occupied forever. More formally, using the usual notation 0 for empty and 1 for occupied, the transition rates of the process at vertex $x$ are given by

$$c_{0\to 1}(x,\zeta) = \mu\, f_1(x,\zeta) \quad \text{and} \quad c_{1\to 0}(x,\zeta) = 0.$$

In order to compare the process properly rescaled in space and time with oriented site percolation, we also introduce the space-time regions

$$B_K := [-K,K]^d \quad \text{and} \quad B_K(z) := Kz + B_K \quad \text{for all} \quad z \in \mathbb{Z}^d.$$

Then, we have the following lemma, where the processes under consideration have been identified to the set of vertices in state 1 to lighten the expressions.

**Lemma 19** – For all $a, \epsilon > 0$ there exist $A, K, c < \infty$ such that

$$P\left(B_{2K} \not\subset \xi_t \text{ for some } t \in (cK, 2cK) \,|\, B_K \subset \xi_0\right) \le \epsilon$$

whenever $\max(a_{21}, a_{22}) \le a$ and $a_{11} > A$.

*Proof.* Note that, even in the limit as $a_{11} \to \infty$, a type 1 player can change her strategy if all her neighbors follow strategy 2. However, since we have

$$\begin{aligned} \phi(x,\xi) &\ge a_{11}\, N_1(x,\xi) \ge a_{11} && \text{when} \quad x \in \xi \text{ and } f_1(x,\xi) \neq 0 \\ \phi(x,\xi) &\le (2M+1)^d \max(a_{21}, a_{22}) && \text{when} \quad x \notin \xi, \end{aligned}$$

the rate at which each of two type 1 players located in the same interaction neighborhood changes her strategy goes to zero in the limit as $a_{11} \to \infty$. In addition, each type 2 player in the neighborhood of one of these two type 1 players changes her strategy with probability converging to one at the next update, which gives the following limits:

$$\begin{aligned} p_{1\to 2}(x,\xi) &\to 0 \quad \text{when} \quad f_1(x,\xi) \neq 0 \\ p_{2\to 1}(x,\xi) &\to 1 \quad \text{when} \quad f_1(x,\xi)\, f_1(y,\xi) \neq 0 \text{ for some } y \in N_x. \end{aligned} \tag{4.41}$$

Also, since a type 2 player can change her strategy only if she has a type 1 neighbor,

$$x \in \xi_t \quad \text{implies that} \quad f_1(x,\xi_t) \neq 0 \quad \text{for all times } t$$

provided this holds at time zero. In particular, the transition rates in (4.41) indeed describe the death-birth process in the limit as $a_{11} \to \infty$ whenever each type 1 player has initially at least one type 1 neighbor. Note that the same property holds for the Richardson model: because there is no death and because each particle newly created must be in the neighborhood of its parent, the set of occupied sites satisfies the following connectivity property:

$$x \in \zeta_t \quad \text{implies that} \quad f_1(x,\zeta_t) \neq 0 \quad \text{for all times } t \tag{4.42}$$

provided this holds at time zero. Combining (4.41)–(4.42) and using that the set $B_K$ is connected, we deduce that the death-birth process can be coupled with the Richardson model with parameter one in such a way that, for all fixed $K, c > 0$,

$$P\,(B_{2K} \cap \zeta_t \not\subset B_{2K} \cap \xi_t \text{ for some } t \in (cK, 2cK) \mid \zeta_0 = \xi_0 = B_K) \to 0 \tag{4.43}$$

as $a_{11} \to \infty$.Note that the death-birth process dominates the Richardson model with a probability that goes to one only in finite space-time regions. In particular, the limit in (4.43) only holds for finite $c$ and $K$. In other respects, it directly follows from the shape theorem Richardson 1973 for the Richardson model that there exists a positive constant $c > 0$ such that

$$\begin{aligned} & P\,(B_{2K} \not\subset \zeta_t \text{ for some } t \in (cK, 2cK) \mid B_K \subset \zeta_0) \\ & \quad = \; P\,(B_{2K} \not\subset \zeta_{cK} \mid B_K \subset \zeta_0) \leq P\,(B_{2K} \not\subset \zeta_{cK} \mid \zeta_0 = \{0\}) \leq \epsilon/2 \end{aligned} \tag{4.44}$$

for all $K$ large. Now, fix $K, c > 0$ such that (4.44) holds. Since the transition rates of the death-birth updating process are continuous with respect to the payoff coefficients and since the space-time region in the event in (4.43) is finite, there is $A < \infty$ such that

$$P\,(B_{2K} \cap \zeta_t \not\subset B_{2K} \cap \xi_t \text{ for some } t \in (cK, 2cK) \mid \zeta_0 = \xi_0 = B_K) \leq \epsilon/2 \tag{4.45}$$

for all $a_{11} > A$. Combining (4.44)–(4.45), we conclude that

$$\begin{aligned}
& P\,(B_{2K} \not\subset \xi_t \text{ for some } t \in (cK, 2cK) \,|\, B_K \subset \xi_0) \\
& \quad \leq \; P\,(B_{2K} \not\subset \zeta_t \text{ for some } t \in (cK, 2cK) \,|\, B_K \subset \zeta_0) \\
& \qquad + P\,(B_{2K} \cap \zeta_t \not\subset B_{2K} \cap \xi_t \text{ for some } t \in (cK, 2cK) \,|\, \zeta_0 = \xi_0 = B_K) \\
& \quad \leq \; \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}$$

for all $a_{11} > A$. This completes the proof. □

From the lemma, we deduce that, starting from a product measure with a positive density of type 1 players, the density of type 1 at equilibrium is close to one when $a_{11}$ is large. To prove this, we consider as previously the directed graph $\mathscr{H}$ with vertex set

$$H := \{(z, n) \in \mathbb{Z}^d \times \mathbb{Z}_+ : z_1 + z_2 + \cdots + z_d + n \text{ is even}\}$$

and in which there is an edge $(z, n) \to (z', n')$ if and only if

$$z' = z \pm e_j \text{ for some } j = 1, 2, \ldots, d \quad \text{and} \quad n' = n + 1.$$

Then, calling $(z, n) \in H$ an occupied site when

$$B_K(z) \subset \xi_t \quad \text{for all} \quad t \in cnK + (0, cK)$$

it follows from Lemma 19 that, for all $\epsilon > 0$, one can choose $a_{11}$ large enough so that the set of occupied sites dominates stochastically the set of wet sites in the percolation process where sites are closed with probability $\epsilon$. Since the probability $\epsilon$ can be made arbitrarily small, the density of type 1 players at equilibrium can be made arbitrarily close to one.

The last step is to turn the remaining type 2 players into type 1 players. To do this, the basic idea is to rely on the lack of percolation of the dry (not wet) sites for a certain oriented site percolation process where sites are closed with a sufficiently small probability $\epsilon$. The fact that the set of dry sites does not percolate for small positive $\epsilon$ is proved in Durrett 1992, section 3 for the percolation process described above. This result, however, is not sufficient to conclude because the lack of percolation of the dry sites for this percolation process does not imply extinction of strategy 2. To solve the problem, we consider oriented site percolation on a directed graph $\mathscr{H}_+$ that has the same vertex set as before but additional arrows, namely

$$(z,n) \to (z',n') \quad \text{if and only if}$$
$$z' = z \pm e_j \text{ for some } j = 1,2,\ldots,d \quad \text{and} \quad n' = n+1$$
$$\text{or} \quad z' = z \pm 2e_j \text{ for some } j = 1,2,\ldots,d \quad \text{and} \quad n' = n.$$

The process on $\mathscr{H}_+$ has the following two key properties:

1. As for the process on $\mathscr{H}$, the dry sites do not percolate if sites are closed with a small enough probability $\epsilon > 0$. This is proved in Lanchier 2013, section 3 following the ideas in Durrett 1992.

2. Recalling that the death-birth process and the percolation process on $\mathscr{H}$ are coupled in such a way that the set of occupied sites dominates the set of wet sites, if

$$\xi_t(x) = 2 \quad \text{for some} \quad (x,t) \in B_K(z) \times (cnK, c(n+1)K)$$

then site $(z,n)$ can be reached by a directed path of dry sites embedded in $\mathscr{H}_+$. This second property is also established in Lanchier 2013, section 3. Even though the proof applies to another model, it easily extends to the death-birth process because it only relies on the fact that a type 2 player can only appear in the neighborhood of a type 2 player.

To deduce extinction of strategy 2, we first fix $\epsilon > 0$ small such that the set of dry sites does not percolate for the percolation process on the directed graph $\mathscr{H}_+$. Then, we take $a_{11}$ large enough so that the set of occupied sites dominates the set of wet sites in the percolation process on the smaller directed graph $\mathscr{H}$. Finally, it follows from the second property above that, because the dry sites do not percolate, the type 2 players do not survive.

### 4.5 The Prisoner's Dilemma in One Dimension

This section is devoted to the proof of Theorem 11 which focuses on the one-dimensional death-birth process with nearest neighbor interactions. First, we explain where the mysterious expressions for $D_3$ and $D_4$ in the statement of the theorem come from. To do so, we let

$$\begin{aligned} p_i(n_1,n_2) \ &:= \ \text{the rate at which a type } i \text{ player with} \\ & \quad \ \text{one type 1 neighbor that has } n_1 \text{ type 1 neighbors and} \\ & \quad \ \text{one type 2 neighbor that has } n_2 \text{ type 2 neighbors} \\ & \quad \ \text{update her strategy} \end{aligned}$$

for $i = 1, 2$ and $n_1, n_2 = 0, 1, 2$, and note that

$$\begin{aligned} p_1(n_1,n_2) \ &= \ \frac{(2-n_2)\,a_{21} + n_2\,a_{22}}{n_1\,a_{11} + (2-n_1)\,a_{12} + (2-n_2)\,a_{21} + n_2\,a_{22}} \\ p_2(n_1,n_2) \ &= \ \frac{n_1\,a_{11} + (2-n_1)\,a_{12}}{n_1\,a_{11} + (2-n_1)\,a_{12} + (2-n_2)\,a_{21} + n_2\,a_{22}}. \end{aligned}$$

Now, for the process starting with only 1s to the left of the origin, we let

$$X_t := \inf\{x \in \mathbb{Z} : \xi_t(x) = 2\} - 1 \quad \text{and} \quad K_t := \inf\{x > 0 : \xi_t(x + X_t) = 1\}$$

be the position of the rightmost type 1 player with only type 1 players to her left and the distance between this type 1 player and the closest type 1 player to her right. Then, letting

$$D_j(\xi_t) := \lim_{h \to 0} h^{-1}\,E\,(X_{t+h} - X_t \,|\, \xi_t \text{ and } K_t = j)$$

we have the following almost sure estimates

$$\begin{array}{cccccccccl} & 1 & 1 & 1 & 2 & 1 & & & & \\ \cdots & \bullet & \bullet & \bullet & \circ & \bullet & \times & \times & & D_2(\xi_t) = D_2 = 2 - p_1(2,0) \\ & 1 & 1 & 1 & 2 & 2 & 1 & & & \\ \cdots & \bullet & \bullet & \bullet & \circ & \circ & \bullet & \times & & D_3(\xi_t) = D_3 = p_2(1,1) - p_1(2,1) \\ & 1 & 1 & 1 & 2 & 2 & 2 & & & \\ \cdots & \bullet & \bullet & \bullet & \circ & \circ & \circ & \times & & D_j(\xi_t) = D_4 = p_2(1,2) - p_1(2,1) \end{array}$$

for all $j > 3$ and where $\times$ means type 1 or type 2. In particular, $D_3$ and $D_4$ are possible drifts of the interface at $X_t$ depending on its distance to the next type 1 player. Also, plugging the expression of the rates $p_i(n_1,n_2)$ above into $D_3$ and $D_4$, we obtain

$$\begin{aligned} D_3 \ &= \ \frac{a_{11} + a_{12}}{a_{11} + a_{12} + a_{21} + a_{22}} - \frac{a_{21} + a_{22}}{2a_{11} + a_{21} + a_{22}} \\ D_4 \ &= \ \frac{a_{11} + a_{12}}{a_{11} + a_{12} + 2a_{22}} - \frac{a_{21} + a_{22}}{2a_{11} + a_{21} + a_{22}}. \end{aligned}$$

which are exactly the expressions given before the statement of Theorem 11.

Before studying the process starting from general initial configurations, we look at the process starting from configurations that have a finite interval of type 1 players, only type 2 players to the left of this interval and infinitely many players of each type to the right. In picture, this looks like

$$\begin{array}{cccccccccccccc} & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & & & \\ \cdots & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \times & \times & \times & \cdots \end{array}$$

For the process starting from this configuration, we let

$$Z_t^- := \inf \{x \in \mathbb{Z} : \xi_t(x) = 1\} \quad \text{and} \quad M_t := \inf \{x > 0 : \xi_t(x + Z_t^-) = 2\}$$

be the position of the leftmost type 1 player and the distance between this type 1 player and the closest type 2 player to her right. Then, we have the following result.

**Lemma 20** – Assume that $a_{22} < a_{21}$ and $D_3 + D_4 > 0$. Then,

$$P\,(M_t > 3 \text{ for all } t > 0 \text{ and } M_t \to \infty \,|\, M_0 > 3) \ \geq \ c \quad \text{for some} \quad c > 0.$$

*Proof.* To begin with, we let $Z_t := e^{-aM_t}$ where $a > 0$ and

$$Z_t^+ := Z_t^- + M_t - 1 \quad \text{and} \quad K_t := \inf \{x > 0 : \xi_t(x + Z_t^+) = 1\}$$

be the right boundary of the type 1 interval starting at $Z_t^-$ and the distance between this type 1 player and the closest type 1 player to her right. Then, we have

$$\begin{array}{rcl} \lim_{h \to 0} \ h^{-1}\, P\,(Z_{t+h}^+ = Z_t^+ + 1 \,|\, \xi_t \text{ and } M_t > 3 \text{ and } K_t = 3) & = & p_2(1,1) \\ \lim_{h \to 0} \ h^{-1}\, P\,(Z_{t+h}^+ = Z_t^+ - 1 \,|\, \xi_t \text{ and } M_t > 3 \text{ and } K_t = 3) & = & p_1(2,1) \end{array} \qquad (4.46)$$

with probability one, while for the left boundary $Z_t^-$,

$$\begin{array}{rcl} \lim_{h \to 0} \ h^{-1}\, P\,(Z_{t+h}^- = Z_t^- + 1 \,|\, \xi_t \text{ and } M_t > 3) & = & p_1(2,1) \\ \lim_{h \to 0} \ h^{-1}\, P\,(Z_{t+h}^- = Z_t^- - 1 \,|\, \xi_t \text{ and } M_t > 3) & = & p_2(1,2). \end{array} \qquad (4.47)$$

Now, for every integer $j > 1$, we define

$$\Phi_j(a) := \lim_{h \to 0} \ h^{-1}\, E\,(Z_{t+h} - Z_t \,|\, \xi_t \text{ and } M_t > 3 \text{ and } K_t = j).$$

Since $M_t = Z_t^+ - Z_t^- + 1$, it follows from (4.46)–(4.47) that

$$\begin{array}{rcl} \Phi_3(a) & := & \lim_{h \to 0} \ h^{-1}\, E\,(Z_{t+h} - Z_t \,|\, \xi_t \text{ and } M_t > 3 \text{ and } K_t = 3) \\ & = & Z_t\,(e^{-a} - 1) \lim_{h \to 0} \ h^{-1}\, P\,(M_{t+h} - M_t = 1 \,|\, M_t > 3 \text{ and } K_t = 3) \\ & & \quad + \ Z_t\,(e^{a} - 1) \lim_{h \to 0} \ h^{-1}\, P\,(M_{t+h} - M_t = -1 \,|\, M_t > 3 \text{ and } K_t = 3) \\ & = & (e^{-a} - 1)(p_2(1,1) + p_2(1,2))\, Z_t + 2\,(e^{a} - 1)\, p_1(2,1)\, Z_t \end{array}$$

therefore, taking the derivative at $a = 0$, we get

$$\Phi_3'(0) = -(p_2(1,1) + p_2(1,2))\, Z_t + 2\, p_1(2,1)\, Z_t = -(D_3 + D_4)\, Z_t.$$

Using the same approach, we prove in general that

$$\Phi_j'(0) = -(D_{j \wedge 4} + D_4)\, Z_t \leq -(D_3 + D_4)\, Z_t < 0 \quad \text{for all} \quad j > 1$$

since $D_3 < D_4 < D_2$ when $a_{22} < a_{21}$. In particular,

$$\Phi_j(b) \leq \Phi_j(0) = 0 \quad \text{for some } b > 0 \text{ fixed from now on}$$

showing that, as long as $M_t > 3$, the process $Z_t$ is a supermartingale with respect to the natural filtration of the death-birth process. To conclude, we now apply the optional stopping theorem to this supermartingale using the stopping times

$$\tau_3 := \inf \{t : M_t \leq 3\} \quad \text{and} \quad \tau_n := \inf \{t : M_t \geq n\} \quad \text{for all} \quad n > 3.$$

Since $T_n := \min(\tau_3, \tau_n)$ is finite, whenever $M_0 > 3$,

$$\begin{aligned} e^{-4b} \ &\geq \ E\,(Z_0) \geq E\,(Z_{T_n}) \\ &\geq \ E\,(Z_{T_n} \,|\, T_n = \tau_3)\, P\,(T_n = \tau_3) + E\,(Z_{T_n} \,|\, T_n = \tau_n)\, P\,(T_n = \tau_n) \\ &\geq \ e^{-3b}\,(1 - P\,(T_n = \tau_n)) + e^{-nb}\, P\,(T_n = \tau_n). \end{aligned} \tag{4.48}$$

Since the event $\{T_n = \tau_n\}$ is nonincreasing with respect to $n$ for the inclusion, we also deduce from the monotone convergence theorem that

$$\begin{aligned} & P\,(M_t > 3 \text{ for all } t > 0 \text{ and } M_t \to \infty) \\ & \qquad \geq \ P\,(T_n = \tau_n \text{ for all } n > 3) = \lim_{n \to \infty} P\,(T_n = \tau_n). \end{aligned} \tag{4.49}$$

Combining (4.48)–(4.49), we deduce that

$$\begin{aligned} & P\,(M_t > 3 \text{ for all } t > 0 \text{ and } M_t \to \infty \,|\, M_0 > 3) \\ & \qquad \geq \ \lim_{n \to \infty} P\,(T_n = \tau_n) \geq \lim_{n \to \infty} (e^{-3b} - e^{-4b})(e^{-3b} - e^{-nb})^{-1} \\ & \qquad \geq \ (e^{-3b} - e^{-4b})\, e^{3b} = 1 - e^{-b} \end{aligned}$$

therefore the lemma holds for $c := 1 - e^{-b} > 0$. □

To deal with the process starting from product measures, we note that every realization induces a partition of the space-time universe into type 1 and type 2 connected components. More precisely,

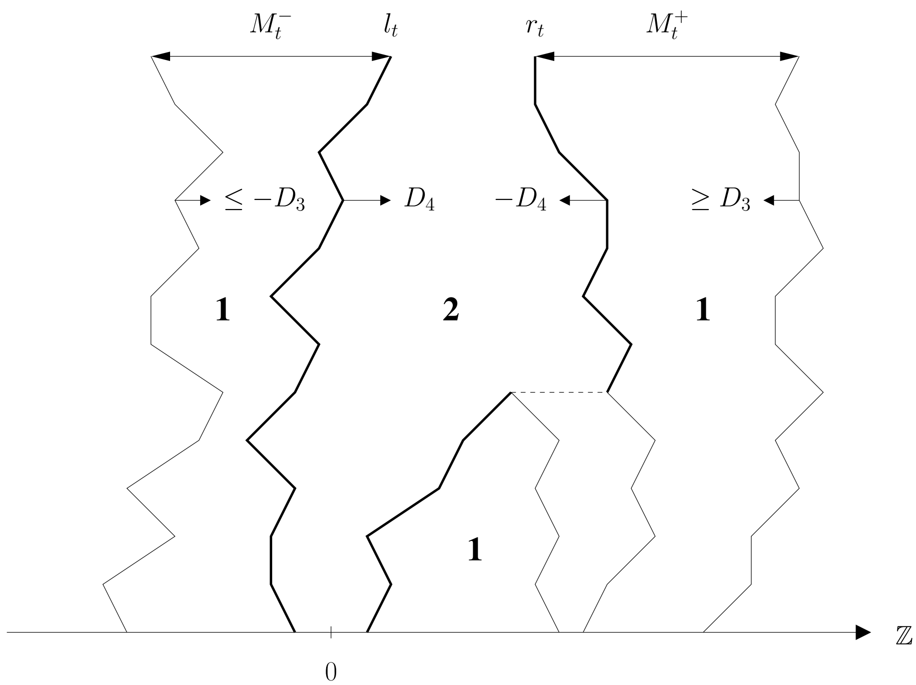



Figure 15: *Picture related to the proof of Lemma 21 illustrating the collapsing (finite) clusters of type 2 in the Prisoner's Dilemma*

assuming that there is initially a type 2 at the origin, we define the type 2 connected component starting at the origin as

$$C_0 := \{(x,t) \in \mathbb{Z} \times \mathbb{R}_+ : \text{there is a path } (0,0) \to_2 (x,t) \text{ going forward}\}$$

where $(0,0) \to_2 (x,t)$ means that there are times and vertices

$$0 = t_1 < t_2 < \cdots < t_{n+1} = t \quad \text{and} \quad 0 = x_1, x_2, \ldots, x_n = x$$

such that the following condition is satisfied:

$$(\xi_s(x_j) = 2 \ \text{ for all } \ t_j \leq s \leq t_{j+1}) \ \text{ holds for } \ j = 1, 2, \ldots, n.$$

Then, we have the following lemma.

**Lemma 21** – Assume that $a_{22} < a_{21}$ and $D_3 + D_4 > 0$. Then,

$$T := \inf\{t > 0 : C_0 \cap (\mathbb{Z} \times (t, \infty)) = \varnothing\} < \infty \quad \text{with probability one.}$$

*Proof.* We proceed by contradiction, showing that when $A := \{T = \infty\}$ occurs, its complement occurs with probability one. To begin with, note that

- on the event $A$, the type 2 connected component $C_0$ is unbounded and
- since there are infinitely many type 1 players on both sides of the origin at time zero, this property remains true at all times.

From these two observations, we deduce that

$$0 < \text{card}\,\{x \in \mathbb{Z} : (x,t) \in C_0\} < \infty \quad \text{for all} \quad t \in (0,\infty)$$

which, in turn, implies that the left boundary $l_t$ and right boundary $r_t$ of the type 2 connected component satisfy the following properties at all times:

$$\begin{aligned} -\infty < l_t &:= \inf\,\{x \in \mathbb{Z} : (x,t) \in C_0\} \\ &\leq \sup\,\{x \in \mathbb{Z} : (x,t) \in C_0\} =: r_t < \infty. \end{aligned}$$

We also observe that

$$\begin{aligned} M_t^- &:= \inf\,\{x > 0 : \xi_t(l_t - x) = 2\} > 1 \\ M_t^+ &:= \inf\,\{x > 0 : \xi_t(r_t + x) = 2\} > 1. \end{aligned} \tag{4.50}$$

Figure 15 gives an illustration of these processes. Now, the evolution rules of the death-birth updating process clearly imply that there exist $c_1, c_2 > 0$ such that

$$\begin{aligned} P\,(\min\,(M_{t+1}^-, M_{t+1}^+) > 3 \mid \min\,(M_t^-, M_t^+) > 1) &\geq c_1 \\ P\,(T < t+1 \mid r_t - l_t \leq 3) &\geq c_2 \end{aligned} \tag{4.51}$$

while it follows from Lemma 20 that

$$\begin{aligned} &P\,(\inf\,\{t > s : \min\,(M_t^-, M_t^+) \leq 3\} \\ &\qquad \geq \inf\,\{t > s : r_t - l_t \leq 3\} \mid \min\,(M_s^-, M_s^+) > 3) \geq c^2 > 0. \end{aligned} \tag{4.52}$$

Combining (4.50)–(4.52), we conclude that

$$\begin{aligned} A = \{T = \infty\} \text{ occurs} \quad &\text{implies that} \quad \min\,(M_t^-, M_t^+) > 1 \text{ at all times} \\ &\text{implies that} \quad \min\,(M_t^-, M_t^+) > 3 \text{ infinitely often} \\ &\text{implies that} \quad r_t - l_t \leq 3 \text{ infinitely often} \\ &\text{implies that} \quad T < \infty \text{ with probability one.} \end{aligned}$$

This completes the proof. □

**Lemma 22** – Assume that $a_{22} > a_{21}$ and $D_4 > 0$. Then,

$$T := \inf\,\{t > 0 : C_0 \cap (\mathbb{Z} \times (t,\infty)) = \varnothing\} < \infty \quad \text{with probability one.}$$

*Proof.* First, we note that the conclusion of Lemma 20 holds as well under the assumptions of the present lemma, since whenever $a_{22} > a_{21}$ we have

$$D_4 < D_3 < D_2 \quad \text{and} \quad D_{j \wedge 4} + D_4 \geq 2D_4 \ \text{ for all } \ j > 1.$$

In particular, the lemma follows by repeating the proof of Lemma 21. □

Finally, under the assumptions of Theorem 11 and conditional on the origin being initially occupied by a type 2 player, it follows from Lemmas 21–22 that the type 2 connected starting at the origin is bounded with probability one. Since in addition the number of players is countable, for every vertex $x \in \mathbb{Z}$, we have

$$\begin{aligned} \liminf_{t \to \infty} P(\xi_t(x) = 1) \ &= \ 1 - \limsup_{t \to \infty} P(\xi_t(x) = 2) \\ &\geq \ 1 - P(\xi_0(y) = 2 \text{ and } T_y = \infty \text{ for some } y \in \mathbb{Z}) \\ &\geq \ 1 - \textstyle\sum_{y : \xi_0(y) = 2} P(T_y = \infty) = 1 \end{aligned}$$

where $T_y$ denotes the time at which the connected component starting at vertex $y$ dies out, as defined in the statement of Lemmas 21–22. This proves Theorem 11.

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# APPENDIX A

# PAYOFFS AFFECTING BIRTH AND DEATH RATES

This appendix gives a summary of the proofs and results given in Lanchier 2015 analyzing a particular evolutionary game in the framework of interacting particle systems. The process studied here, in which payoffs affect birth and death rates, is an evolutionary game where players either die or give birth depending on the sign of their payoff, and at an exponential rate equal to the magnitude of their payoff. If the player dies, she is replaced by a neighbor chosen uniformly at random, and if the player gives birth, then a neighbor is chosen uniformly at random to be replaced by the player's offspring.

## A.1 Model Description

More specifically, the model studied in Lanchier 2015 is a continuous-time Markov chain $\eta_t$ whose state at time $t$ is a spatial configuration

$$\eta_t : \mathbb{Z}^d \longrightarrow \{1, 2\} := \text{the set of strategies.}$$

If the player at vertex $x$ has a positive payoff $\phi > 0$, then at rate $\phi$, a neighbor of $x$ is chosen uniformly at random to be replaced by (adopt the strategy of) $x$, simulating the player at $x$ giving birth. If the player at vertex $x$ has a negative payoff, then at rate $|\phi|$, she is replaced by (adopts the strategy of) one of her neighbors chosen uniformly at random, simulating the player at $x$ dying and being replaced. Formally, this can be described by the Markov generator

$$\begin{aligned} Lf(\eta) = \quad & N^{-1} \sum_x \sum_{y \in \mathcal{N}_x} \phi(y,\eta) \mathbf{1}\{\phi_1(y,\eta) > 0\} \ \mathbf{1}\{\eta(x) \neq \eta(y)\} \ [f(\eta_x) - f(\eta)] \\ & - N^{-1} \sum_x \sum_{y \in \mathcal{N}_x} \phi(y,\eta) \mathbf{1}\{\phi_1(x,\eta) < 0\} \ \mathbf{1}\{\eta(x) \neq \eta(y)\} \ [f(\eta_x) - f(\eta)] \end{aligned} \tag{A.1}$$

where $N$ is the number of neighbors, the configuration $\eta_x$ is obtained from $\eta$ by changing the strategy at vertex $x$ and leaving the strategies at all other vertices unchanged.

The first theorem shows that local interactions induce a reduction in the coexistence and bistable regions that are implied by replicator dynamics.

## A.2 Results

The results proved in Lanchier 2015 are given next, except that the roles of strategy 1 and strategy 2 are exchanged here in order to maintain consistency with the following chapters where we prescribe $a_{21} > a_{12}$.

**Theorem 23** – Assume that $a_{21} > a_{12}$. Then strategy 2 wins whenever

$$\max(a_{11}, a_{12}) - \min(a_{22}, a_{21}) < (N-1)^{-1}(a_{21} - a_{12}).$$

The proof of Theorem 23 relies on demonstrating a coupling between the spatial game and a biased voter model that favors strategy 2. The main idea here is to write the transition rates by distinguishing between transitions due to births from those due to deaths. Viewing the transition rates in this way, it can be shown that for certain choices of payoff parameters, the set of type 2 individuals in the spatial game is stochastically dominated by the set of type 2 individuals in the biased voter model. Since type 2 wins in the biased voter model, it also wins in the spatial game.

The next theorem serves two purposes. First, as the theorem is stated, it shows that the parameter region in which the replicator equation is bistable but the spatial game has a unique ESS is much larger than the parameter region given by Theorem 23 alone. Second, by symmetry, the role of strategies 1 and 2 can be exchanged in the theorem to show that regardless of any advantage given to strategy 2 by the payoff ordering $a_{21} > a_{12}$ or by the choice of $a_{22}$, there are always values of $a_{11}$ sufficiently large that strategy 1 will win in the spatial model. Let $\bar{a}_{22} := (a_{21}, a_{12}, a_{11})$.

**Theorem 24** – For all $\bar{a}_{22}$ there exists $m(\bar{a}_{22}) < \infty$ such that

$$\text{strategy 2 wins whenever } a_{22} > m(\bar{a}_{22}).$$

The proof of Theorem 24 as well as Theorems 25 and 27 below all rely on the use of a block construction (see Durrett 1995), by which we mean a coupling between a particular auxiliary process that tracks certain "good" events and a supercritical percolation process after appropriate rescaling in time and space. What we mean by "good" events changes according to the theorem, as do the methods required to exhibit the coupling. Common to all three theorems is the use of a so-called Harris Graphical Representation, in which the process is shown to be identical to a particular process on a space-time graph constructed by a collection of Poisson point processes. In the proof of Theorem 24, this graphical representation is first used to prove that strategy 2 survives by a comparison with the Richarson model. Then, it is shown that strategy 1 goes extinct by a block construction, in which a "good" site is one for which there is some type 1 individual nearby. By showing that the associated percolation process can be made subcritcal by an appropriate choice of the function $m(\bar{a}_{22})$, it is concluded that strategy 1 goes exctinct, so strategy 2 wins.

Theorem 25 demonstrates a further reduction in the coexistence region provided that $a_{21} < 0$, and/or a similar reduction by symmetry if $a_{12} < 0$.

**Theorem 25** – For all $\bar{a}_{22}$ such that $a_{21} < 0$, there exists $m(\bar{a}_{22}) < \infty$ such that

$$\text{strategy 1 wins whenever } a_{22} < -m(\bar{a}_{22}).$$

The proof of Theorem 25 again relies on a graphical representation and block construction. In this case, a "good" site is chosen to be one in which there are no type 2 individuals nearby. The very small (negative) value of $a_{22}$ assures that the process quickly becomes sparse, in the sense that the probability to two neighbors both being type 2 is very low. The added condition that $a_{21} < 0$ assures that lone type 2 individuals will also be replaced by type 1, which enables a coupling to a supercritical percolation and shows that strategy 1 wins.

Theorem 26 replaces Theorem 23 in the case of nearest-neighbor interactions on a one-dimensional lattice. By dropping the assumption that $a_{21} > a_{12}$, the symmetric form of this theorem demonstrates a specific region where type 1 wins (even when $a_{21} > a_{12}$), thereby showing agreement with the replicator dynamics for a large parameter region, and also further reducing the region of bistability. The second part of the theorem shows that coexistence does not occur except possibly in a measure zero parameter region corresponding to $A \notin \mathcal{M}_2^*$, where

$$\mathcal{M}_2^* := \{A = (a_{ij}) \text{ such that } a_{11}a_{12}a_{21}a_{22}(a_{11} + a_{12})(a_{22} + a_{21}) \neq 0 \text{ and } a_{11} + a_{12} \neq a_{22} + a_{21}\}.$$

**Theorem 26** – Assume that $M = d = 1$. Then:

- strategy 2 wins for all $a_{22} > \max(a_{11}, a_{12}) - (a_{21} - a_{12})$, and
- the system starting from any translation invariant distribution clusters for all $A \in \mathcal{M}_2^*$.

The proof of the first part of Theorem 26 relies on studying an auxiliary process that keeps track of the boundaries between clusters of each strategy. This requires the restriction to nearest-neighbor interactions and the 1-dimensional lattice, but it enables some stronger results. The idea of the proof is to first consider the process where the initial configuration consists of all type-1 particles to the left of some fixed value $\alpha > 0$. Then the leftmost particle can be defined. Using some bounds on transition rates, a particular auxiliary process is shown to be a supermartingale. By then applying the optional stopping theorem, the process tracking the leftmost particle is shown to tend to infinity with a high probability. To get the result for the more general initial condition (any product measure) in which we are interested, we need only to be able to find a large cluster of type 1 particles, and

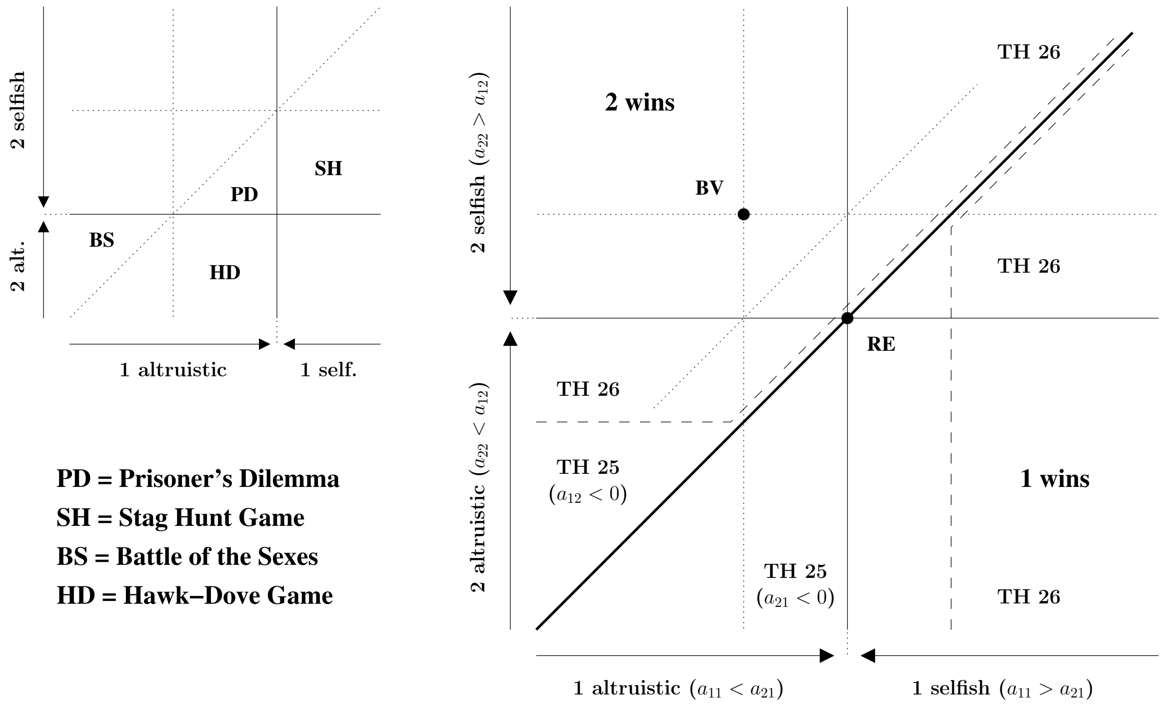


Figure 16: *List of the most popular* $2 \times 2$ *games on left. On right: phase diagram in the* $a_{11} - a_{22}$ *plane summarizing the results as applied to the one-dimensional, nearest-neighbor spatial game. The thick, solid line indicates the phase transition suggested by simulations. Dashed lines represent boundaries given by the results of this paper. BV = Biased Voter Model, RE = Replicatior Equation.*

then use some symmetry and the result just mentioned for the restricted initial condition. The second part of the theorem is proved by showing that the boundary process mentioned above tends to extinction. This is done by viewing the boundary process as an annihilating random walk and using some estimates on transition rates to show that in every group of four consecutive particles, at least one particle is killed after an almost surely finite time.

While the previous theorem showed that coexistence does not occur in the case of nearest-neighbor interactions on a one-dimensional lattice, the final theorem shows that for any other interaction range, $M$, or lattice dimension, $d$, coexistence does occur.

**Theorem 27** – There is $m := m(a_{12}, a_{21}) > 0$ such that coexistence occurs when

$$c(M,d)a_{11} < a_{22} < -m \quad \text{and} \quad c(M,d)a_{22} < a_{11} < -m,$$

where, for each range $M$ and spatial dimension $d$,

$$c(M,d) := \frac{2M((2M+1)^d - 2)}{(M+1)(2M(2M+1)^{d-1} - 1)}.$$

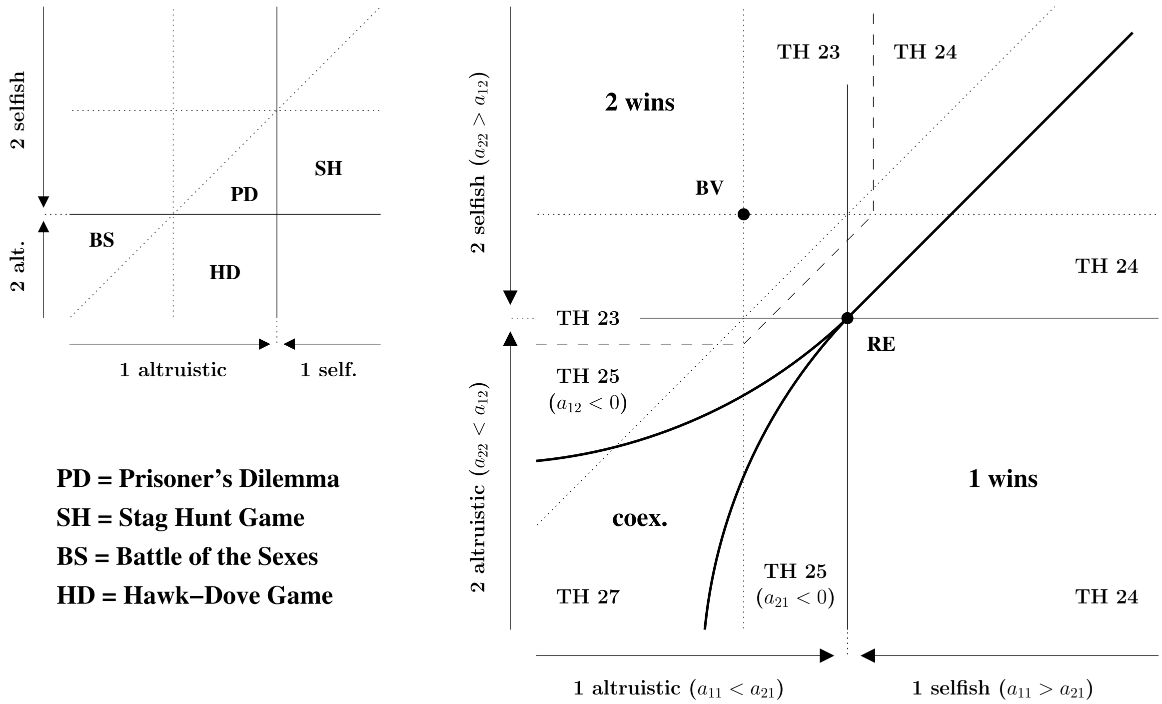


Figure 17: *List of the most popular* $2 \times 2$ *games on left. On right: summary of the results for the spatial game. The thick, solid line indicates the phase transition suggested by simulations. Dashed lines represent boundaries given by the results of this paper. BV = Biased Voter Model, RE = Replicatior Equation.*

As in Theorems 24 and 25, the proof of Theorem 27 also relies on a graphical representation and block construction. First, focusing on the case $a_{12} = a_{21} = 0$, and an initial condition with just one type 1 individual, it is shown that the process tracking the rightmost type 1 individual has positive drift. Using some trigonometry and the geometry of the square lattice, it is then shown that the Euclidean distance between a certain "tagged" individual and a given target site has negative drift. These results are used to show that the tagged player hits a subset of a target region in a short time, does not leave a certain larger region, and stays in the target region for a long time. This is done by identifying an auxiliary process that is a supermartingale, and using the optional stopping theorem. Finally, a coupling with a supercritical oriented site percolating is exhibited where the open sites in the percolation process are identified with regions of coexistence in the spatial game. Some continuity arguments and time rescaling extend the result to the parameter region described by the theorem, where the function $m(a_{12}, a_{21})$ arises from the time rescaling.

# APPENDIX B

## ACKNOWLEDGMENT OF CO-AUTHORSHIP ON INCLUDED WORK

Chapters 3 and 4 of this dissertation appeared in a somewhat modified form in Evilsizor and Lanchier 2014 and Evilsizor and Lanchier 2016, respectively. These papers were co-authored by the author of this dissertation and Nicolas Lanchier. Lanchier has given his approval that these works be included in this dissertation.